Quantum Hamiltonian Reduction
of the Schwinger Model

Kazunori Itakura\textsuperscript{1} and Koichi Ohta

\textit{Institute of Physics, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan}

Abstract

We reexamine a unitary-transformation method of extracting a physical Hamiltonian from a gauge field theory after quantizing all degrees of freedom including redundant variables. We show that this quantum Hamiltonian reduction method suffers from crucial modifications arising from regularization of composite operators. We assess the effects of regularization in the simplest gauge field theory, the Schwinger model. Without regularization, the quantum reduction yields the identical Hamiltonian with the classically reduced one. On the other hand, with regularization incorporated, the resulting Hamiltonian of the quantum reduction disagrees with that of the classical reduction. However, we find that the discrepancy is resolved by redefinitions of fermion currents and that the results are again consistent with those of the classical reduction.

\textsuperscript{1}e-mail address: itakura@hep1.c.u-tokyo.ac.jp
I. INTRODUCTION

Gauge theories are constructed to have physically irrelevant variables and therefore gauge fields need to be constrained to avoid having too many degrees of freedom. Identification of physical degrees of freedom is indispensable for extracting physical information from them. Although this identification is not difficult in Abelian gauge theories, we know that it is a highly non-trivial problem in non-Abelian gauge theories such as quantum chromodynamics (QCD): The gauge-fixing ambiguities prevent us from constructing the theory nonperturbatively.

The Hamiltonian formulation of a constrained system was founded by Dirac [1]. In his method, all variables including unphysical ones are treated on the equal footing. Constraints of a first-class system, $\Phi^i = 0$, commute with each other weakly in the Poisson brackets, (i.e., they commute on the constraint surface) and generate gauge transformations. Therefore a first-class constraint system is always a gauge system. Gauge degrees of freedom are eliminated by imposing gauge-fixing conditions so that the system becomes second class, whereby the Poisson brackets between constraints do not vanish any more. For a second-class system, we define Dirac's brackets and replace them with the commutators to quantize the system. If the system is a gauge theory, we can also proceed as follows: Ignoring constraints, we first quantize all variables except for Lagrange multipliers, and then reduce the extended Hilbert space to the physical subspace by imposing

$$\Phi^j|\text{phys} = 0.$$  

In this canonical Weyl- or temporal-gauge formulation, quantization precedes reduction of the Hilbert space. In this point, this is different from schemes such as the Faddeev-Jackiw formalism [8,9], in which quantization is performed only for physical variables in the classically reduced phase space. Various arguments have been made about the equivalence of these two approaches, for which there seems to be no a priori justification (see [2,3] and references therein). Furthermore, since the Hamiltonian in the canonical temporal-gauge formulation explicitly contains unphysical variables and constraints are complicated in general, it is hard to calculate various physical quantities.

Recently, Lenz et al. [4] have revamped a unitary-transformation method that is based on the canonical temporal-gauge formulation and gives without gauge-fixing procedure a physical Hamiltonian, i.e., a Hamiltonian described only by physical variables [5-7]. We shall call this method the quantum Hamiltonian reduction (QHR). This method makes it easy to identify unphysical variables. By performing a suitable unitary transformation $U$, we can eliminate unphysical coordinates from the Hamiltonian,

$$UHU^\dagger = H'(q^i, p^i, P^j),$$  

(1.2)
where $q^i$ and $p^i$ are physical coordinates and their conjugate momenta, and $P^j$ are unphysical momenta, respectively. Simultaneously the unphysical momenta are excluded by the transformed subsidiary conditions,

$$U\Phi^jU^\dagger = P^j, \quad P^j|\text{phys}\rangle' = 0,$$

(1.3)

where $|\text{phys}\rangle' = U|\text{phys}\rangle$ is a transformed physical state. Roughly speaking, this unitary transformation is interpreted as the change of variables so that some of the new variables parametrize gauge orbits themselves. In other words, it separates all degrees of freedom into those of gauge orbits and the rest. We will see this method later in more detail by using a simple field theoretical model. One of the advantages of the quantum Hamiltonian reduction is that we can go to other gauges once we choose other unitary transformations. The transition from one gauge to another is also implemented by a unitary transformation and the equivalence between two gauges is shown straightforwardly. To avoid confusion arising from the term quantum reduction, we emphasize that the reduction of the phase space is completed in (1.1) and quantum reduction refers not to this procedure but to extraction of a physical Hamiltonian by the unitary-transformation method.

Although QHR has been applied to various gauge field theories [5-7,4], the effects of regularization of composite operators have never been discussed. The primary purpose of this paper is to examine the influence of regularization on the reduction transformation and compare the results with those obtained without regularization and with those of the classical reduction. The Schwinger model - quantum electrodynamics in 1+1 dimensions with massless fermions - is the simplest gauge field theory and moreover is exactly solvable [11-13]. It has exhaustively been investigated mainly because, besides being solvable, it is interpreted as a toy model for QCD in four dimensional spacetime in that this model possesses a confining potential and a topological structure of the vacuum (the θ-vacuum). In the Hamiltonian formalism, we can explicitly construct the Hilbert space and implement regularization without resorting to any perturbative method [14-17]. Therefore the effects of a unitary transformation can be estimated unambiguously. From these reasons, we employ the Schwinger model as a laboratory to examine the effects of regularization.

This paper is organized as follows: In Sec. II, basic ingredients of the Schwinger model are reviewed briefly and the physical Hamiltonian is derived by classical reduction. In Sec. III, the conventional method of QHR is applied to the Schwinger model without taking into account any regularization. In Sec. IV, the fermion Fock space is constructed explicitly and regularization is incorporated completely to show that regularization indeed causes a modification to the Hamiltonian. In Sec. V the results obtained in Sec. IV are compared with those in Sec. III and those of the classical reduction, and consistency of these results is shown. Finally in Sec. VI, a brief summary and discussion are given.
II. CLASSICAL REDUCTION OF THE SCHWINGER MODEL

The Schwinger model is described by the Lagrangian density

\[ \mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \bar{\psi} i \gamma^\mu D_\mu \psi, \]  

(2.1)

where the field strength tensor is written in terms of the gauge field \( A_\mu \) as \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and the covariant derivative is \( D_\mu = \partial_\mu + i e A_\mu \), with \( e \) being the coupling constant. The conjugate momenta of \( A_1 \) and \( \psi \) are \( \Pi^1 = F_{01} \) and \( \pi_\psi = i \psi \), respectively. Gauss’ law, defined as a constraint imposed by a Lagrange multiplier \( A_0 \), is

\[ G(x) \equiv \partial_1 \Pi_1 + e \rho = 0, \]  

(2.2)

with \( \rho = \psi^\dagger \psi \) being the electric charge density. Throughout this paper, we work in the temporal gauge, i.e., we set \( A_0 = 0 \) and consider a fixed time, so that time dependence is suppressed. The Hamiltonian density in this gauge is

\[ \mathcal{H} = \frac{1}{2} (\Pi_1)^2 + \bar{\psi} i \gamma_5 ( -i \partial_1 + e A_1 ) \psi. \]  

(2.3)

We suppose that the space is compactified to a circle with a circumference \( L \) so that there exists a physical degree of freedom in the gauge field [14-16]. We specify boundary conditions on the field variables as \( A_1(x + L) = A_1(x) \) and \( \psi(x + L) = e^{i\varphi} \psi(x) \), with \( \varphi \) being an arbitrary constant.

In the temporal gauge, there remains a local gauge symmetry. The Hamiltonian is classically invariant under the gauge transformations,

\[ \psi(x) \to \psi'(x) = e^{i \beta(x)} \psi(x), \]  

(2.4)

\[ A_1(x) \to A'_1(x) = A_1(x) - \partial_1 \beta(x), \]  

(2.5)

where \( \beta \) is an arbitrary time-independent function. Demanding that the transformed fields should also satisfy the given boundary conditions, we find the gauge parameter \( \beta \) to be of the form

\[ \beta(x) = \beta_p(x) + \beta_l(x), \]  

(2.6)

where a periodic function \( \beta_p(x) \) \((\beta_p(x + L) = \beta_p(x))\) gives a small gauge transformation and a linear function \( \beta_l \) with a discrete coefficient, \( \beta_l(x) = 2\pi k x / e L \) \((k \in \mathbb{Z})\) gives a large gauge transformation. The large gauge transformation is characterized by an arbitrary integer \( k \) representing the winding number of the homotopy group, \( \pi_1(U(1)) = \mathbb{Z} \).

For an Abelian gauge theory on a circle there is precisely one physical degree of freedom in the gauge field, namely its zero mode

\[ e = \frac{1}{L} \int_0^L dx A_1(x). \]  

(2.7)
Physically this is almost trivial because the gauge invariant quantity constructed from the gauge field is only the phase of the Wilson loop variable, i.e., \( \exp\{-ie \int A_1(x) dx\} \). If the space is not compact, this contribution vanishes and the gauge field has no physical degrees of freedom. Corresponding to the variable \( c \), \( \Pi^1 \) has the zero mode
\[
p = \frac{1}{L} \int_0^L dx \Pi^1(x). \tag{2.8}
\]
Note that the momentum conjugate to \( c \) is \( Lp \).

Let us now consider the classical reduction of the Schwinger model. We decompose the gauge field and its conjugate into the physical and unphysical variables, i.e., the zero modes and remainders,
\[
A_1(x) = c + \tilde{A}_1(x), \tag{2.9}
\]
\[
\Pi^1(x) = p + \tilde{\Pi}^1(x). \tag{2.10}
\]
These variables are not independent because of the Gauss’ law constraint (2.2). Elimination of redundant variables is partly achieved by substitution of the solution to the constraint. It is easy to resolve Gauss’ law, (2.2),
\[
Q = \int_0^L dx \rho(x) = 0, \tag{2.11}
\]
\[
\tilde{\Pi}^1(x) = e \int_0^L dy \rho(y) \tilde{\theta}(x - y), \tag{2.12}
\]
where \( Q \) is the total fermionic charge and \( \tilde{\theta}(x - y) \) is the periodic step function
\[
\tilde{\theta}(x - y) \equiv \sum_{n \neq 0} \frac{1}{2\pi i n} e^{i\frac{2\pi n}{2 \pi}}(x - y), \quad \frac{\partial}{\partial x} \tilde{\theta}(x - y) = \delta(x - y) - \frac{1}{L}, \tag{2.13}
\]
with \( \delta(x - y) \) being the periodic delta function
\[
\delta(x - y) \equiv \frac{1}{L} \sum_{n \in \mathbb{Z}} e^{i\frac{2\pi n}{2 \pi}}(x - y). \tag{2.14}
\]
These solutions (2.11) and (2.12) correspond, respectively, to the zero mode and non-zero mode parts of the Gauss’ law operator. Substituting (2.12) into the Hamiltonian and making a field redefinition,
\[
\psi(x) \rightarrow e^{-i\alpha(x)} \psi(x), \tag{2.15}
\]
with
\[
\alpha(x) = e \int_0^L dy \tilde{\theta}(x - y) A_1(y), \tag{2.16}
\]
we obtain the reduced Hamiltonian
\[
H_{cl} = \int dx \left\{ \frac{1}{2} p^2 + \frac{1}{2} \eta^2 + \psi^\dagger \gamma_1 (-i\partial_t + \epsilon c) \psi \right\}, \tag{2.17}
\]
where
\[ \eta(x) = e^{\int_0^L dy \phi(y)} \hat{\phi}(y - x). \] (2.18)
Eq. (2.15) corresponds to the Darboux transformation in the Faddeev-Jackiw formalism [8]. All the variables describing the Hamiltonian \( H_{cl} \) are invariant under small gauge transformations. In this sense \( H_{cl} \) is described only by physical variables.

There still remains a gauge symmetry associated with \( Q \). This gauge symmetry is global and cannot be eliminated. Indeed it has a physical meaning: The electric flux always closes in the compact space and therefore the total charge must be zero. In addition to the global gauge symmetry, the reduced Hamiltonian (2.17) is invariant under a local gauge transformation,
\[
\begin{align*}
\psi &\rightarrow e^{2\pi ik/L} \psi, \\
c &\rightarrow c - \frac{2\pi k}{eL}.
\end{align*}
\] (2.19)
This symmetry under a large gauge transformation was also present in the unreduced Hamiltonian. Although we have stated that \( \psi \) and \( c \) are physical when they are invariant under small gauge transformations, they are not physical variables in the strict sense because they are subject to local gauge transformations (2.19) and (2.20). However, we stick to this terminology for the reasons put forth in the next section.

We have reduced the Schwinger model in the classical level. We defer the quantization of this reduced system until Sec. IV where we come back to this problem in comparison with the quantum Hamiltonian reduction.

### III. QUANTUM REDUCTION WITHOUT REGULARIZATION

In this section, we perform a quantum-mechanical reduction of the Schwinger model without regularization. As mentioned in the introduction, QHR is based on the canonical temporal-gauge formulation. Except for the Lagrange multiplier \( A_0 \), the fields are quantized
\[
\{ \psi_\alpha(x), \psi^\dagger_\beta(y) \} = \delta_{\alpha\beta} \delta_\varphi(x - y), \quad [A_1(x), \Pi^1(y)] = i\delta(x - y),
\] (3.1)
where \( \alpha \) and \( \beta \) are the spinor indices, \( \delta_\varphi(x - y) = e^{i\hat{\varphi}(x-y)}\delta(x - y) \) and \( \delta(x - y) \) is the periodic delta function defined by (2.14). The residual gauge transformation is effected by the unitary operator,
\[
\mathcal{G}[\beta] = \exp \left\{ -i \int_0^L dx \left( -\Pi_1 \frac{\partial}{\partial \hat{x}} + e\rho \right) \beta(\hat{x}) \right\}.
\] (3.2)
The space index with a tilde, \( \tilde{x} \), is defined by

\[
x = \left[ \frac{x}{L} \right] L + \tilde{x},
\]

where the square brackets stand for the Gauss notation, i.e., \( \lfloor x/L \rfloor \) is the integer part of \( x/L \). By definition \( 0 \leq \tilde{x} < L \). The use of \( \tilde{x} \) in place of \( x \) in the integrand is required to avoid undesired contributions from the boundaries and to reproduce the gauge transformations (2.4) and (2.5). Note that for \( \beta(x) \) given by (2.6), \( e^{i \beta(\tilde{x})} = e^{i \beta(x)} \). For periodic field operators, e.g., \( \Pi_1(\tilde{x}) = \Pi_1(x) \) and \( \rho(\tilde{x}) = \rho(x) \). With the decomposition (2.6), \( \mathcal{G}[\beta] = \mathcal{G}[\beta_p] \mathcal{G}[\beta_l] \). In

\[
\mathcal{G}[\beta_p] = \exp \left\{ -i \int_0^L dx G(x) \beta_p(x) \right\},
\]

the Gauss' law operator is the generator of small gauge transformations. On the other hand,

\[
\mathcal{G}[\beta_l] = \exp \left\{ -\frac{2\pi ik}{e L} \int_0^L dx (-\Pi_1(x) + e \tilde{x} \rho(x)) \right\}
\]

generates large gauge transformations. The gauge invariance of the Hamiltonian is equivalent with the invariance under the unitary transformation \( \mathcal{G}[\beta] \),

\[
\mathcal{G}[\beta] \mathcal{H} \mathcal{G}^\dagger[\beta] = \mathcal{H}.
\]

To eliminate this gauge symmetry from the total Hilbert space constructed with superfluous degrees of freedom, we impose a subsidiary condition,

\[
G(x)_{\text{physics}} = 0.
\]

In the canonical temporal-gauge formulation, the redundancies associated with the large gauge transformations cannot be eliminated because Gauss’ law (3.7) removes only the small gauge symmetries. Therefore within the framework of the canonical temporal-gauge formulation, physical variables are variables invariant under the small gauge transformations. In this sense, we call \( \psi \) and \( \epsilon \) physical and \( \tilde{A}_1 \) unphysical.

Physical predictions can be made by the constrained Hamiltonian (2.3) under the subsidiary condition (3.7), but by exploiting an appropriate unitary transformation we can go to a more transparent representation, where the subsidiary condition becomes trivial and the Hamiltonian is decomposed definitely into physical and unphysical parts. Consider a unitary operator \( U \) defined by

\[
U = e^{i \int dx \rho(x) \alpha(x)},
\]
where \( \alpha(x) \) was given by (2.16). The fundamental field operators are transformed as

\[
U\psi(x)U^\dagger = e^{-i\alpha(x)}\psi(x),
\]

\[
UA_1(x)U^\dagger = A_1(x),
\]

\[
U\Pi_1(x)U^\dagger = \Pi_1(x) + \eta(x).
\]

To prove (3.9) we have used the fact that \( \alpha(x) \) is a periodic function. From (3.9)-(3.11), we obtain the transformed Hamiltonian,

\[
U\mathcal{H}U^\dagger = \frac{1}{2}(\Pi_1 + \eta)^2 + \psi^\dagger\gamma_5(-i\partial_1 + e\epsilon)\psi
\]

\[
= \mathcal{H}_C + \mathcal{H}'_C,
\]

with

\[
\mathcal{H}_C \equiv \frac{1}{2}p^2 + \frac{1}{2}\eta^2 + \psi^\dagger\gamma_5(-i\partial_1 + e\epsilon)\psi,
\]

\[
\mathcal{H}'_C \equiv \frac{1}{2}\bar{\Pi}_1(\bar{\Pi}_1 + 2\eta),
\]

where the suffix C means that the transformed quantities belong to the Coulomb-gauge representation. The Gauss’ law operator becomes

\[
G_\text{C}(x) \equiv UG(x)U^\dagger = \partial_1(\Pi_1 + \eta) + e\rho
\]

\[
= \partial_1\Pi_1 + e\frac{Q}{L}.
\]

This picks up the physical states,

\[
\left(\partial_1\Pi_1 + e\frac{Q}{L}\right)|\text{phys}\rangle' = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l}
Q|\text{phys}\rangle' = 0 \\
\bar{\Pi}_1|\text{phys}\rangle' = 0
\end{array} \right.,
\]

where \(|\text{phys}\rangle' = U|\text{phys}\rangle\) is the transformed physical state. The charge \( Q \) was present in the classical reduction as the generator of the global gauge transformation.

The transformed unitary operator for gauge transformations is

\[
\mathcal{G}_\text{C}[\beta] \equiv U\mathcal{G}[\beta]U^\dagger = e^{-i\int_0^L dy G_\text{C}(x)\beta_5(y)}\mathcal{G}[\beta],
\]

where \( \mathcal{G}[\beta] \), defined by (3.5), is not modified by the unitary transformation owing to the identity \( \int_0^L \eta(x)dx = 0 \). \( \mathcal{G}_\text{C}[\beta] \) brings about the transformations,

\[
\mathcal{G}_\text{C}[\beta]|\psi(x)\rangle\mathcal{G}_\text{C}[\beta] = e^{i\int_0^L \beta_5(y)dy + 2\pi ik_5/L}\psi(x),
\]

\[
\mathcal{G}_\text{C}[\beta]c\mathcal{G}_\text{C}[\beta] = c - \frac{2\pi k}{eL},
\]

\[
\mathcal{G}_\text{C}[\beta]A_1(x)\mathcal{G}_\text{C}[\beta] = \tilde{A}_1(x) - \partial_1\beta_5(x),
\]

whereas \( p \) and \( \bar{\Pi}_1 \) are invariant. Note that for small gauge transformations \( (\mathcal{G}[\beta] = 1) \), only \( \tilde{A}_1 \) changes locally.
Furthermore, we obtain in the physical subspace

$$(\mathcal{H}_C + \mathcal{H}_C) |\text{phys}\rangle' = \mathcal{H}_C |\text{phys}\rangle'$$.

When limited to the physical states, this result is the same as that of the classical reduction (2.17). The transformed Hamiltonian (3.12) is invariant under gauge transformations generated by the transformed Gauss’ law operator (3.15).

We have been able to extract the physical Hamiltonian $\mathcal{H}_C$ from the constrained one (2.3) by the unitary transformation (3.8). The key of QHR is that the following two things occur simultaneously; elimination of redundant variables from a Hamiltonian and simplification of constraints. While the redundant coordinates vanish from the Hamiltonian, the momenta conjugate to them are excluded by the transformed constraints. Hence all the redundant variables are removed from the Hamiltonian. In the classical reduction in Sec. II, the above two things were performed step by step, resolution of Gauss’ law, (2.11)-(2.12) and redefinition of the fermion field (2.15).

Before leaving this section it would be interesting to comment on another meaning of the unitary transformation $U$ [6]. In the Schrödinger representation in which $A_1$ is diagonal, i.e., $\Pi_1(x) = i\delta/\delta A_1(x)$, (3.7) is rewritten as a functional-derivative equation,

$$\left(i\partial_1 + \frac{\delta}{\delta A_1} + \epsilon \rho\right) \Psi[A_1] = 0, \quad (3.22)$$

where $\Psi[A_1] = \langle A_1|\text{phys}\rangle$ is a wavefunctional of a state $|\text{phys}\rangle$. Eq. (3.22) is solved for

$$\Psi[c, \hat{A}_1] = U[\hat{A}_1] \Phi[c], \quad (3.23)$$

where $U$ is identical with the unitary operator (3.8) and $\Phi[c]$ depends only on $c$. Note that the unitary operator $U$ is independent of $c$ because $\alpha(x)$ defined in (2.16) does not contain $c$ (remember that $\hat{\theta}(x)$ is the periodic step function). The condition that $\Phi[c]$ should include only $c$ can be expressed as

$$\frac{\delta}{\delta A_1} \Phi[c] = 0. \quad (3.24)$$

This is nothing but the transformed subsidiary condition $\hat{\Pi}_1|\text{phys}\rangle' = 0$ with the identification $\Phi[c] = \langle A_1|\text{phys}\rangle'$. Therefore the unitary transformation $U$ is also interpreted as the wavefunctional for the unphysical gauge field.

**IV. QUANTUM REDUCTION WITH REGULARIZATION**

When we deal with a quantum field theory, we must regularize composite operators. The reflection of regularization can be observed, for example, in the current algebra as an anomalous term called the ‘Schwinger term’ [10]. This extra term must have influence
on the reduction transformations. Consider a transformation of the fermion current $J_\mu = \psi^\dagger \gamma_\mu \psi$. If we perform the unitary transformation $U$ defined by (3.8) without regularization, we have

$$U J_\mu U^\dagger = U \psi^\dagger \gamma_\mu U \psi U^\dagger = J_\mu.$$  

On the other hand, for regularized currents, we have

$$U J_\mu^{reg} U^\dagger = J_\mu^{reg} + i \int dy [J_\mu^{reg} (y), J_\mu^{reg} (x)] \a(y) + \cdots.$$ 

Therefore $U J_\mu^{reg} U^\dagger = J_\mu^{reg}$ but $U J_\mu^{reg} U^\dagger \neq J_\mu^{reg}$ because of the Schwinger term. Since the Hamiltonian density (2.3) contains the fermion current in the interaction term with the gauge field $(-e J_1 A_1)$, it also transforms non-trivially at least for the interaction term. Therefore QHR should be carefully performed with regularization of composite operators such as fermion currents, charge and Hamiltonian.

### A. Construction of the fermion Fock space and regularization

We construct the fermion Fock space to evaluate regularization explicitly. We essentially follow Iso and Murayama [16]. The fermionic part of the Hamiltonian density (2.3) can be expressed as

$$\mathcal{H}_F = \psi^\dagger \begin{pmatrix} h_F & 0 \\ 0 & -h_F \end{pmatrix} \psi,$$

where $h_F = -i \partial_t + e A_1$ and we have used the chiral representation for the $\gamma$-matrices:

$$\gamma^0 = \sigma_1, \quad \gamma^1 = -i \sigma_2, \quad \text{and} \quad \gamma^3 = \gamma^0 \gamma^1 = \sigma_3,$$

where $\sigma_i$ is the Pauli matrix. Solving the eigen-equations $h_F \chi_n(x) = \varepsilon_n \chi_n(x)$, we obtain

$$\chi_n(x) = \frac{1}{\sqrt{L}} e^{-ie \int_0^x A_1(y) dy} e^{i \varepsilon_n x}, \quad \varepsilon_n = \frac{2 \pi}{L} \left( n + \frac{\varphi}{2 \pi} + \frac{e c L}{2 \pi} \right)$$

for $n \in \mathbb{Z}$. Note that the energy spectrum depends only on $c$.

Next we expand $\psi$ in the orthonormal complete set $\{ \chi_n \}$,

$$\psi(x) = \sum_{n \in \mathbb{Z}} \left\{ a_n \chi_n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b_n \chi_n(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

and impose anti-commutation relations,

$$\{a_n, a_m^\dagger \} = \delta_{nm}, \quad \{b_n, b_m^\dagger \} = \delta_{nm}.$$ 

The fermion Hamiltonian $H_F = \int_0^L \mathcal{H}_F dx$ is diagonalized in terms of these operators and decomposed into the positive- and negative-chirality pieces as

$$H_F = H_+ + H_-,$$
\[ H_+ = \sum_{n \in \mathbb{Z}} \varepsilon_n a_n^\dagger a_n, \quad H_- = \sum_{n \in \mathbb{Z}} (-\varepsilon_n) b_n^\dagger b_n. \] (4.6)

We define \( N_\pm \)-vacua by

\[ |\text{vac}; N_+ \rangle_+ \equiv \prod_{n=-\infty}^{N_+-1} a_n^\dagger |0\rangle, \quad N_+ \in \mathbb{Z}, \] (4.7)

\[ |\text{vac}; N_- \rangle_- \equiv \prod_{n=N_-}^{\infty} b_n^\dagger |0\rangle, \quad N_- \in \mathbb{Z}, \] (4.8)

where \( |0\rangle \) is the state of nothing defined by \( a_n|0\rangle = b_n|0\rangle = 0 \) for \( \forall n \in \mathbb{Z} \). Since the positive- and negative-chirality sectors are decoupled, the total fermion vacua are given by

\[ |\text{vac}; N_+, N_- \rangle = |\text{vac}; N_+ \rangle_+ |\text{vac}; N_- \rangle_- . \] (4.9)

For some fixed \( N_\pm \), we obtain the fermion Fock space by acting creation or annihilation operators on the \( N_\pm \)-vacua.

Since the composite operators such as \( H_\pm \) become divergent on the \( N_\pm \)-vacua, we regularize them by the \( \zeta \)-function regularization,

\[ H_+^{\zeta r} = \lim_{s \to 0} \sum_{n \in \mathbb{Z}} \varepsilon_n a_n^\dagger a_n \frac{1}{(\lambda \varepsilon_n)^s}, \] (4.10)

\[ H_-^{\zeta r} = \lim_{s \to 0} \sum_{n \in \mathbb{Z}} (-\varepsilon_n) b_n^\dagger b_n \frac{1}{(\lambda \varepsilon_n)^s}, \] (4.11)

where \( \lambda \) is an arbitrary constant with the dimension of length. It is easy to check that this regularization respects gauge symmetry. We also regularize the current operators \( J_\pm(x) \equiv \frac{1}{2} (J_0(x) \pm J_1(x)) \) by

\[ j_+^n = \sum_{y \in \mathbb{Z}} a_y^\dagger a_{y+n} \to (j_+^n)^{\zeta r} = \lim_{s \to 0} \sum_{y \in \mathbb{Z}} a_y^\dagger a_{y+n} \frac{1}{(\lambda \varepsilon_y)^s}; \] (4.12)

\[ j_-^n = \sum_{y \in \mathbb{Z}} b_{y+n}^\dagger b_y \to (j_-^n)^{\zeta r} = \lim_{s \to 0} \sum_{y \in \mathbb{Z}} b_{y+n}^\dagger b_y \frac{1}{(\lambda \varepsilon_y)^s}; \] (4.13)

where \( j_\pm^a \) are the Fourier transforms defined by \( J_\pm(x) = L^{-1} \sum_{y \in \mathbb{Z}} j_\pm^y \exp(\mp i \frac{2\pi}{L} x) \). Here we have adopted the \( \zeta \)-function method, but we can resort to any other regularizations, e.g., the heat-kernel or the point-splitting regularization [14,10], as long as they respect the gauge symmetry. Using the formulae for the \( \zeta \)-function, \( \zeta(s,a) = \sum_{n=0}^{\infty} (n + a)^{-s} \),

\[ \zeta(0, a) = \frac{1}{2} - a, \quad \zeta(-1, a) = -\frac{1}{2} \left( \frac{1}{2} - a \right)^2 + \frac{1}{24}, \] (4.14)

we can evaluate the eigen-values of the regularized operators. For example, we obtain

\[ Q_\pm^{\zeta r}|\text{vac}; N_\pm \rangle_+ = \pm \left( N_\pm + \frac{e \epsilon L}{2 \pi} \right) |\text{vac}; N_\pm \rangle_+, \] (4.15)
where \( Q_{\pm}^{reg} = (j_{n=0}^{\pm})^{reg} \) are the fermionic charge operators. As previously mentioned, the current algebra is modified by the regularization (see Appendix),

\[
[(j_{\pm}^{\pm})^{reg}, (j_{\pm}^{\pm\dagger})^{reg}] = n\delta_{nm},
\]

or in the coordinate space,

\[
[J_{\pm}^{reg}(x), J_{\pm}^{reg}(y)] = \pm \frac{i}{2\pi} \frac{\partial}{\partial x} \delta(x - y).
\]

From (4.18), we have

\[
[J_{1}^{reg}(x), J_{1}^{reg}(y)] = \frac{i}{\pi} \frac{\partial}{\partial x} \delta(x - y),
\]

\[
[J_{0}^{reg}(x), J_{0}^{reg}(y)] = [J_{1}^{reg}(x), J_{1}^{reg}(y)] = 0.
\]

Note that only the first relation (4.19) has the extra term originating from regularization.

By commuting \( H_{\pm}^{reg} \) with \( (j_{\pm}^{n})^{reg\dagger} \), we have

\[
[H_{\pm}^{reg}, (j_{\pm}^{\pm})^{reg\dagger}] = \frac{2\pi n}{L} (j_{\pm}^{n})^{reg\dagger},
\]

and this leads to

\[
[H_{\pm}^{reg}, J_{\pm}^{reg}(x)] = \mp i\partial_{x} J_{\pm}^{reg}(x).
\]

This relation and the current algebra (4.18) are the indispensable tools for the calculation of the reduction transformation.

\[\text{B. Reduction transformation}\]

Now we are ready to make a unitary transformation of regularized operators. As a unitary operator, we employ \( U \) defined in (3.8). It is divided into positive- and negative-chirality parts

\[
U = e^{i \int dx J_{+}(x) \alpha(x)} e^{i \int dx J_{-}(x) \alpha(x)} \equiv U_{+} U_{-}.
\]

For the positive-chirality sector, its fermion Hamiltonian is transformed as

\[
U_{+} H_{\pm}^{reg} U_{+}^{\dagger} = H_{\pm}^{reg} + i \int dx [J_{\pm}^{reg}(x), H_{\pm}^{reg}] \alpha(x) + \frac{1}{24} i^{2} \int dx dy [J_{\pm}^{reg}(y), [J_{\pm}^{reg}(x), H_{\pm}^{reg}]] \alpha(x) \alpha(y) + \cdots
\]

\[
= H_{\pm}^{reg} + \int dx J_{\pm}^{reg}(x) \partial_{x} \alpha(x) + \frac{1}{4\pi} \int dx \partial_{x} \alpha(x) \partial_{y} \alpha(x).
\]
where we have used the relations (4.22) and (4.17). All composite operators must be regularized but regularization of the gauge fields does not affect our results. Note that, because of the Schwinger term, the third term turns out not to vanish in contrast to the reduction without regularization. Likewise, the negative-chirality fermion Hamiltonian is transformed as

\[ U^- H^{reg} U^\dagger = H_F^{reg} - \int dx J_F^{reg}(x) \partial_1 \alpha(x) + \frac{1}{4\pi} \int dx \partial_1 \alpha(x) \partial_1 \alpha(x). \] (4.25)

Combining (4.24) and (4.25), we obtain for the total fermion Hamiltonian,

\[ U^- H_F^{reg} U^\dagger = H_F^{reg} + \int dx \left( J_F^{reg} \partial_1 \alpha + \frac{1}{2\pi} (\partial_1 \alpha)^2 \right) \]

\[ = H_F^{reg} + \int dx \left( e J_F^{reg} \tilde{A}_1 + \frac{1}{2} m^2 \tilde{A}_1^2 \right), \] (4.26)

where \( m = e/\sqrt{\pi} \) and we have used \( \partial_1 \alpha = e \tilde{A}_1 \). To get a transformation law for the total Hamiltonian, we have to know a commutator between the kinetic term of the gauge field, \( \frac{1}{2} \Pi_1^2 \), and the regularized fermion charge density \( J_F^{reg} \), but the commutator does not change by regularization. (Although Gauss’ law relates \( \Pi_1 \) with \( \partial_1 \alpha = e \tilde{A}_1 \), the commutator between \( J_0 \)’s does not get modified. See Eq. (4.20).) Eventually the result of the unitary transformation for the total Hamiltonian is

\[ H_C \equiv U^- H_F^{reg} U^\dagger = H_F^{reg} + \int dx \left\{ \frac{1}{2} (\Pi_1 + \eta)^2 + e J_F^{reg} \tilde{A}_1 + \frac{1}{2} m^2 \tilde{A}_1^2 \right\}. \] (4.27)

We have introduced the unitary transformation \( U \) to eliminate non-zero modes \( \tilde{A}_1 \) from the Hamiltonian, as is seen in Eq. (3.12), but it has turned out that with regularization \( H_C \) contains \( \tilde{A}_1 \).

Since the Gauss’ law operator is not modified by regularization, the physical state is specified by the same conditions as those of the classical reduction:

\[ \tilde{H}_1 |phys \rangle' = 0, \quad Q |phys \rangle' = 0. \] (4.28)

As stated in Sec. III, in the Schrödinger representation where \( A_1 \) is diagonal, (4.28) requires that the physical states should not depend on \( \tilde{A}_1 \). If we are allowed to write the Hamiltonian \( H_C \) as

\[ H_C = \int dx \left\{ \frac{1}{2} (\Pi_1 + \eta)^2 + \psi^\dagger \gamma_5 (-i \partial_1 + ec) \psi + \frac{1}{2} m^2 \tilde{A}_1^2 \right\}^{reg}, \] (4.29)

it is evident that only the last term is an extra additional to the result (3.12).
C. Hamiltonian in terms of the currents

For later convenience we rewrite the Hamiltonian in terms of the fermion currents. To this end, we notice that operators $B_n^\alpha, B_n^{\alpha\dagger}$ satisfy the bosonic commutator

$$[B_n^\alpha, B_m^{\beta\dagger}] = \delta_{mn},$$

and from Eq. (4.21) we have

$$[H_F, B_n^{\alpha\dagger}] = \frac{2\pi n}{L} B_n^{\alpha\dagger}.$$  \hspace{1cm} (4.31)

These relations and the vacuum expectation value of $H_F$, (4.16), indicate that $H_F$ can be expressed in terms of the currents as [16]

$$H_F = \frac{2\pi}{L} \left\{ \frac{1}{2} (Q_+^n + Q_-^n) - \frac{1}{12} + \sum_{n>0} \left( j_+^{n\dagger} j_+^n + j_-^{n\dagger} j_-^n \right) \right\}.$$  \hspace{1cm} (4.32)

It is a direct consequence of regularization that we can express the fermion Hamiltonian in terms of the currents. Thus when we write $H_F$ in terms of the fermion currents without the suffix ‘reg,’ we take it as a matter of course that the currents are regularized. By acting the creation operators $B_n^{\alpha\dagger}$ on the $N_\pm$-vacua, we obtain the Fock space that is equivalent to the one constructed by $a_n^{(1)}$ and $b_n^{(1)}$. Observing that

$$\frac{1}{2} \int dx \eta^2(x) = \frac{1}{2L} \sum_{n \neq 0} \left( \frac{eL}{2\pi n} \right)^2 \left( j_+^{n\dagger} + j_-^{n\dagger} \right) \left( j_+^n + j_-^n \right),$$

we can rewrite the total Hamiltonian

$$H_C = U H U^\dagger = h_C + h'_C,$$  \hspace{1cm} (4.34)

$$h_C = \left. \frac{1}{2L} \frac{\partial^2}{\partial c^2} + \frac{2\pi}{L} \left\{ \frac{1}{2} (Q_+^2 + Q_-^2) - \frac{1}{12} \right\} \right|_{\gamma = 0}$$

$$+ \frac{1}{2L} \sum_{n \neq 0} \left( \frac{eL}{2\pi n} \right)^2 \left( j_+^{n\dagger} + j_-^{n\dagger} \right) \left( j_+^n + j_-^n \right)$$

$$+ \frac{2\pi}{L} \sum_{n>0} \left( j_+^{n\dagger} j_+^n + j_-^{n\dagger} j_-^n \right),$$  \hspace{1cm} (4.35)

$$h'_C = \int_0^L dx \left\{ \frac{1}{2} \hat{H}_1 (\hat{H}_1 + 2\eta) + e J_1 \hat{A}_1 + \frac{1}{2} m^2 \hat{A}_1^2 \right\},$$  \hspace{1cm} (4.36)

where $L_p = -i \partial / \partial c$ was used. The first part of the transformed Hamiltonian, $h_C$, has exactly the same form as that of Ref. [16], in which Gauss’ law is solved explicitly after quantization.

On the other hand, $h'_C$ is estimated on the physical state,

$$h'_C |_{\text{phys}} = \int dx \left( e J_1 \hat{A}_1 + \frac{1}{2} m^2 \hat{A}_1^2 \right) |_{\text{phys}}.$$  \hspace{1cm} (4.37)
Although the physical state does not depend on \( \tilde{A}_1 \), we cannot ignore \( h' \): If limited to the physical state, \( \tilde{A}_1 \) seems to serve as an auxiliary field generating current-current interaction. Therefore one might be tempted to conclude that our results disagree with those of Ref. [16]. However we should not compare them at this stage. In the next section one sees that our results turn out to be consistent.

The comparison should be made with the regularized Hamiltonian that was reduced classically. We regularize the classically reduced Hamiltonian \( H_{cl} \) (see (2.17)) following the same way as that of the unreduced case. The important point to be noted is that unlike the preceding case, the first quantization is performed for the fermion Hamiltonian of \( H_{cl} \), i.e., \( (H_{cl})_F = \int dx \psi \Gamma \gamma_5 (\imath \partial_1 + cc) \psi \), where only the zero mode of the gauge field is coupled to the fermion field. In this case, the regularization respects a residual gauge symmetry. Representing \( H_{cl} \) with the regularized currents, we obtain a Hamiltonian of the same form as that of Ref. [16]. For clarity, we give its expression again at the risk of being tedious,

\[
(H_{cl})^{reg} = -\frac{1}{2L} \frac{\partial^2}{\partial c^2} + \frac{2\pi}{L} \left( \frac{1}{2} (Q_+^2 + Q_-^2) - \frac{1}{12} \right) \\
+ \frac{1}{2L} \sum_{n \neq 0} \left( \frac{2\pi}{\zeta L} \right)^2 \left( n_+^2 + n_-^2 \right) \left( n_+ n_+^\dagger + n_- n_-^\dagger \right) \\
+ \frac{2\pi}{L} \sum_{n > 0} \left( n_+ n_+^\dagger + n_- n_-^\dagger \right),
\]

where we denote the currents as \( j_{\pm}^n \) to distinguish them from \( j_{\pm}^n \).

From Eq. (4.15), the zero mode part of the Hamiltonian is estimated on the physical state as

\[
H^{zero} = -\frac{1}{2L} \frac{\partial^2}{\partial c^2} + \frac{2\pi}{L} \left\{ \frac{1}{2} (Q_+^2 + Q_-^2) \right\} \\
= -\frac{1}{2L} \frac{\partial^2}{\partial c^2} + \frac{1}{2} \left( \kappa + \frac{2\pi N}{\zeta L} \right)^2,
\]

where \( N = N_+ = N_- \) from the requirement that the total charge must be zero, \( Q = N_+ - N_- = 0 \). Note that this Hamiltonian contains the mass term for \( c \).

V. CONSISTENCY OF THE CLASSICAL AND QUANTUM REDUCTIONS

The unitary transformation \( U \) has yielded a new Hamiltonian \( H_C \), (4.27), and a new Gauss' law operator \( G_C \), (3.15). It is well known that the regularization produces a restoring force for the zero mode \( c \), but we have now discovered that the regularization has also produced a mass term for the non-zero mode \( \tilde{A}_1 \). These results apparently contradict with those without regularization. In this section we pin down the origin of
this apparent discrepancy and prove somewhat heuristically that these two approaches are nonetheless consistent.

To begin with, let us observe the strange behavior of $H_C$ under gauge transformations. Its accurate understanding gives a key to the problem of the equivalence between quantum and classical reductions.

The gauge transformations of the fundamental operators was given in (3.18)-(3.20). It follows immediately that our result (4.27) is not gauge invariant classically because the mass term is present for $\hat{A}_1$. Nevertheless the invariance of the unreduced Hamiltonian ($\mathcal{G}H\mathcal{G}^\dagger = H$) ensures that the new Hamiltonian (4.27) is also invariant:

$$\mathcal{G}_C[\beta] H_C \mathcal{G}_C^\dagger[\beta] = H_C.$$  \hspace{1cm} (5.1)

Indeed this can be shown explicitly as a regularized equation if we notice that

$$[\Pi_1(x), J_1^{reg}(y)] = -i\frac{e}{\pi} \left( \delta(x - y) - \frac{1}{\pi} \right).$$  \hspace{1cm} (5.2)

In Appendix we derive this within the framework of the $\zeta$-function regularization.

On the other hand, the result of the unitary transformation without regularization, (3.12)-(3.14), is gauge invariant classically, but is not invariant if we consider regularization. What caused the violation of the classical gauge symmetry? The reason is self-evident. It is true that our regularization respects the gauge symmetry generated by $G$, but after the unitary transformation, the Gauss’ law operator changes into $G_C$ and therefore the gauge transformation is different from the original one. On the contrary, the reduction was performed by the unitary transformation under the regularization that respects the original gauge symmetry $G$. Thus there is no reason why the transformed Hamiltonian possesses the classical gauge symmetry.

This can be viewed more transparently in the point-splitting regularization. Since our regularization preserves the original gauge symmetry, the currents are regularized as

$$J_\mu^{reg}(x) = \lim_{\varepsilon \to 0} \left\{ \bar{\psi}(x + \varepsilon) \gamma_\mu \exp \left( ie \int_{x}^{x+\varepsilon} dy A_1(y) \right) \psi(x) - v.e.v. \right\},$$  \hspace{1cm} (5.3)

where the line integral is inserted to preserve the gauge symmetry under $\mathcal{G}$. On the other hand, if we want to preserve the gauge symmetry $\mathcal{G}_C$, we should regularize the currents as

$$J_\mu^{reg}(x) \equiv \lim_{\varepsilon \to 0} \left\{ \bar{\psi}(x + \varepsilon) \gamma_\mu \exp \left( ie \int_{x}^{x+\varepsilon} dy c \right) \psi(x) - v.e.v. \right\}.$$  \hspace{1cm} (5.4)

Classically these currents are invariant under either of the gauge transformations $\mathcal{G}$ and $\mathcal{G}_C$. If we consider the regularization, however, the current $J_1^{reg}$ is transformed as

$$\mathcal{G}[\beta] J_1^{reg} \mathcal{G}^\dagger[\beta] = J_1^{reg},$$  \hspace{1cm} (5.5)

$$\mathcal{G}_C[\beta] J_1^{reg} \mathcal{G}_C^\dagger[\beta] = J_1^{reg} + \frac{e}{\pi} \partial_1 \beta_\mu.$$  \hspace{1cm} (5.6)
By contrast, \( \mathcal{J}_1^{reg} \) behaves as
\[
\mathcal{G}[\beta] \mathcal{J}_1^{reg} \mathcal{G}^\dagger[\beta] = \mathcal{J}_1^{reg} - \frac{e}{\pi} \partial_1 \beta_p, \quad (5.7)
\]
\[
\mathcal{G}_C[\beta] \mathcal{J}_1^{reg} \mathcal{G}_C^\dagger[\beta] = \mathcal{J}_1^{reg}. \quad (5.8)
\]
Therefore \( J_1^{reg} \) and \( \mathcal{J}_1^{reg} \) are invariant under \( \mathcal{G} \) and \( \mathcal{G}_C \), respectively. For definiteness, let us denote the regularization that respects the original gauge symmetry as \( \mathcal{R}[\mathcal{G}] \), and the one that respects the gauge symmetry generated by \( \mathcal{G}_C \) as \( \mathcal{R}[\mathcal{G}_C] \).

The regularization \( \mathcal{R}[\mathcal{G}_C] \) respects the residual global symmetry of the classically reduced Hamiltonian,
\[
\mathcal{G}_C[\beta] (H_\beta)_C^{reg} \mathcal{G}_C^\dagger[\beta] = (H_\beta)_C^{reg}. \quad (5.9)
\]
Therefore the Fourier components of \( \mathcal{J}_\pm^{reg} \) in the point-splitting regularization, (5.4), should be identified with \( l_n^\pm \) that appeared in \((H_\beta)_F^{reg}, (4.38)\),
\[
\mathcal{J}_\pm^{reg}(x) = \frac{1}{L} \sum_{n \in \mathbb{Z}} (l_n^\pm)^{reg} e^{i \frac{2\pi x n}{L}}. \quad (5.10)
\]
Here one must be aware that the conceptually different two currents \((l_\pm^\pm)^{reg}\) and \((j_\pm^\pm)^{reg}\) are identical in the \( \zeta \)-function regularization since the fermion energy \( \varepsilon \) depends only on the zero mode of the gauge field. The difference was absorbed into the basis function \( \chi_n \).

Now we can understand the coincidence of the Hamiltonian in Ref. [16] with our result (4.38). It is accidental that \( H_F \) and \((H_\beta)_F\) are represented in the same form. Indeed \( H_F \) does contain the unphysical field and \((H_\beta)_F\) does not. Therefore the result of Ref. [16] is insufficient in that all the unphysical degrees of freedom are not eliminated from the Hamiltonian.

We have learned that currents in different regularization schemes cannot be identified straightforwardly. Therefore it is not meaningful to discuss the equivalence of the Hamiltonians by comparing only their forms. We must find an appropriate current to be compared with that of the classically reduced case.

From (5.6), we see that the current \( J_1^{reg} \) (regularized by \( \mathcal{R}[\mathcal{G}] \)) is not invariant under the residual gauge transformation. By now, we have constructed this system in terms of gauge variant variables unintentionally. We need to reformulate it in terms of gauge invariant variables. Transforming Eq. (5.5) by \( U \), we obtain \( \mathcal{G}_C(U J_1^{reg} U^\dagger) \mathcal{G}_C^\dagger = U J_1^{reg} U^\dagger \). Thus \( U J_1^{reg} U^\dagger = J_1^{reg} + \frac{\varepsilon}{\pi} \hat{A}_1 \) is gauge invariant under \( \mathcal{G}_C \) and we can define a gauge-invariant current by
\[
\tilde{J}_1^{reg} = J_1^{reg} + \frac{\varepsilon}{\pi} \hat{A}_1. \quad (5.11)
\]
Since the physical state does not contain $\tilde{A}_1$, we are allowed to identify these two currents $\hat{J}_{reg}^r$ and $\hat{J}_{reg}^u$ in the physical subspace. Let us rewrite our result (4.34) in terms of the gauge-invariant current. The Fourier components of $\hat{J}_{reg}^r = \hat{J}_{reg}^u + \frac{e}{2\pi} \tilde{A}_1$ is written as

$$(\hat{j}_n^r)^{reg} = (\hat{j}_n^u)^{reg} + \frac{e}{2\pi} \tilde{A}_n,$$  \hspace{1cm} (5.12)$$

where $\tilde{A}_n$ is the Fourier component of $\tilde{A}_1(x)$,

$$\tilde{A}_1(x) = \frac{1}{L} \sum_{n \neq 0} \tilde{A}_n e^{2\pi i n x}.$$  

Inserting (5.12) into (4.34), we finally obtain the desired result:

$$H_C = -\frac{1}{2L} \frac{\partial^2}{\partial \phi^2} + \frac{2\pi}{L} \left\{ \frac{1}{2} \left( \frac{Q_+^2 + Q_-^2}{L} - \frac{1}{12} \right) + \frac{1}{2L} \sum_{n \neq 0} \left( \frac{\epsilon L}{2\pi n} \right)^2 (\hat{j}_n^+ + \hat{j}_n^-) (\hat{j}_n^+ + \hat{j}_n^-) + \frac{2\pi}{L} \sum_{n > 0} \left( \hat{j}_n^+ \hat{j}_n^- + \hat{j}_n^+ \hat{j}_n^- \right) + \frac{1}{2} \int_0^L dx \tilde{\Pi}_1 (\tilde{\Pi}_1 + 2\eta). \right\}$$  \hspace{1cm} (5.13)$$

Note that, from the definition (5.11), the charges are not modified, $Q_\pm = \hat{j}_n^{reg, \pm} = \hat{j}_n^{reg, \pm}$. Except for the last term that vanishes on the physical state, (5.13) is exactly equivalent to the classical result (4.38). We may be allowed to identify the gauge invariant current $\hat{J}_{reg}^r$ with $\hat{J}_{reg}^u$ without limiting to the physical state,

$$(\hat{J}_{reg}^r) \leftrightarrow (\hat{J}_{reg}^u),$$  \hspace{1cm} (5.14)$$

because both of these are invariant under the residual gauge transformation ($G_C$). (Classically $\hat{J}_{reg}^r$ is not invariant, but it does not matter.) This is almost trivial in the point-splitting regularization,

$$\hat{J}_{reg}^r (x) = \lim_{\varepsilon \to 0} \left\{ \tilde{\psi}(x + \varepsilon) \gamma_1 \exp \left( i e \int_{x}^{x+\varepsilon} dy A_1(y) \right) \tilde{\psi}(x) - v.e.v. \right\}$$

$$= \lim_{\varepsilon \to 0} \left\{ \tilde{\psi}(x) \gamma_1 \exp \left( i e \int_{x}^{x+\varepsilon} dy c \right) \tilde{\psi}(x) - v.e.v. \right\} - \frac{e}{\pi} \tilde{A}_1$$

$$= \hat{J}_{reg}^u = \frac{e}{\pi} \tilde{A}_1.$$  \hspace{1cm} (5.15)$$

At this final stage, we can say that QHR of the Schwinger model gives the same result as that of the classical reduction. While the unphysical field $\tilde{A}_1$ is absorbed into the regularized current and the mass term for $\tilde{A}_1$ vanishes from the Hamiltonian, the zero mode $c$ survives after the redefinition of the currents (see Eq. (4.39)). This is because
in (5.15) the zero mode should be retained as the line integral so that $J_{reg}$ should be invariant under the residual gauge transformation.

Since a further study of the Schwinger model lies outside the scope of this paper and it has already been investigated intensively, we restrict ourselves to comment on the following two points [16]. First we can show that, by using the Bogoliubov transformation, the Hamiltonian (5.13) is equivalent to that of the massive boson with mass $m = \epsilon / \sqrt{\pi}$. This corresponds to the bosonization of the Schwinger model. Next if we require the invariance of the vacuum under the large gauge transformation, we obtain the non-trivial vacuum, the $\theta$-vacuum that is also present in the four-dimensional SU($N$) Yang-Mills theories.

**VI. SUMMARY AND DISCUSSION**

We have investigated a quantum Hamiltonian reduction based on the unitary transformation method. We have seen in the Schwinger model that the regularization gives decisive effects on the unitary transformation, but finally we recovered those in the classical reduction. This was achieved by a careful treatment of the regularized currents. Note that the equivalence could not be shown until the Hamiltonian was written in terms of the currents.

Similar observation is expected to apply to other gauge theories. To define a quantum system in QFT, we have to specify a regularization in addition to a Lagrangian. If the system has a gauge symmetry, the regularization should be chosen to preserve it. Therefore in the canonical temporal-gauge formulation, a quantum gauge system is specified uniquely by a set $(H, G, \mathcal{R}[G])$, i.e., a Hamiltonian, constraints which induce gauge transformations, and a regularization that respects the gauge symmetries. The quantum reduction is implemented by some unitary transformations. Since the fields before and after unitary transformations are generally treated without distinction, the regularization of the operators after reduction is the same as that before reduction. On the other hand, the quantum reduction is intended to simplify the constraints, i.e., the generators of the gauge transformations. In other words, by the reduction transformation, the gauge transformations inevitably change into the ones that are not respected by the regularization. This discrepancy causes a classically gauge-variant Hamiltonian such as $H_C$ (see Fig. 1). Now the quantum gauge system is unitary transformed into a set $(H', G', \mathcal{R}[G'])$, where $G'$ represents residual constraints concerning the global gauge transformations, e.g., $Q$ in the Schwinger model on a circle. We should note that the equivalent quantum theory to the original set $(H, G, \mathcal{R}[G])$ is not $(H_{cl}, G', \mathcal{R}[G'])$ but $(H', G', \mathcal{R}[G'])$. The set $(H_{cl}, G', \mathcal{R}[G'])$ is a result of the classical reduction followed by the regularization preserving the gauge symmetries concerning $G'$. To fill the gap between these two results, we introduce new
composite operators which can be identified with those of the classical reduction. In the Schwinger model, because the current construction (the Sugawara construction) of the Hamiltonian can be done explicitly, the redefinition procedure was completed only for the currents. However generally in higher dimensional models such as the four dimensional QCD, the redefinition of other composite operators will be needed for the proof of their equivalence.

In non-Abelian gauge theories, we are confronted with another problem associated with the gauge-fixing ambiguities. We know that the Coulomb or Lorentz gauge is not a good gauge because of the Gribov ambiguities [18]. In the presence of these ambiguities, we cannot construct gauge invariant variables in the strictest sense. This situation is similar to those of the Schwinger model on a circle because it is subject to a local redundant gauge transformation, i.e., a large gauge transformation. As commented previously, elimination of this redundancy causes a non-trivial structure of the vacuum. In the recent papers on the Yang-Mills theories in a cylindrical spacetime, it has been reported that the Gribov ambiguities also give rise to such a non-trivial structure [19-21]. This effect should be investigated in higher dimensional models.

Lastly in relation to Fig. 1, we would like to comment on the chiral Schwinger model [22,23]. It is known as one of the anomalous gauge theories, which we cannot regularize while preserving gauge symmetry. In this paper, we have implicitly assumed that there always exists a regularization which respects gauge symmetry. The scheme in Fig. 1 is limited to such theories. In the case of the chiral Schwinger model, it is evident that the results of the classical and quantum Hamiltonian reductions disagree. In the classical reduction, we can reduce the system as a gauge theory, but in the quantum reduction, it becomes second class due to an anomalous term in the Gauss' law commutator [22]. Now that it is a second-class system, the canonical temporal-gauge formulation cannot be applied to it. However, if we further make an appropriate unitary transformation, we may be able to reach a similar Hamiltonian to that of the classical reduction. It might be worth while to see the differences between them.

ACKNOWLEDGEMENT

One of the authors (K.I.) would like to thank T. Ikehashi for various discussions.

APPENDIX

In this Appendix, we shall derive the fermion current algebra (4.17) and the commutator (5.2) in the text within the framework of the $\zeta$-function regularization. In Refs. [14,16], the authors derived (4.17) without resorting to any regularization scheme. We shall prove this explicitly for the regularized currents on the fermion Fock space (see Ref.
[17] for the derivation of (4.17) by the heat-kernel method. For simplicity, we assume $n, m > 0$. Using the definitions (4.12) and (4.13), we have
\[
[(j_+^n)^{reg}, (j_+^m)^{reg}]|vac; N_+\rangle_+ = \lim_{s \to 0} \sum_{q \in \mathbb{Z}} a_{q+m}^+ a_{q+n}|vac; N_+\rangle_+ \frac{1}{(e_{q+m})^s(e_{q+n})^s} - \frac{1}{(e_q)^s(e_q)^s}, \tag{A1}
\]
where $e_q = q + \frac{\varphi}{2\pi} + \frac{eCL}{2\pi}$. The surviving terms in the sum are classified into two cases:

\begin{align*}
\text{case 1} & \quad \left\{ \begin{array}{l}
q + n \leq N_+ - 1 \\
q + m \geq N_+
\end{array} \right., \\
\text{case 2} & \quad \left\{ \begin{array}{l}
q + n \leq N_+ - 1 \\
q + m = q + n
\end{array} \right.. 
\end{align*}

In case 1, the integer $q$ is delimited to a finite number from $N_+ - m$ to $N_+ - 1 - n$. (We assume $m - 1 \geq n$, because otherwise there would be no contribution.) On the other hand, in case 2, an infinite number of $q$ are involved ($q \leq N_+ - 1 - n$). Combining these two contributions, we can estimate Eq. (A1) as
\[
[(j_+^n)^{reg}, (j_+^m)^{reg}]|vac; N_+\rangle_+ = \lim_{s \to 0} \sum_{q=N_+ - m}^{N_+ - 1-n} a_{q+m}^+ a_{q+n}|vac; N_+\rangle_+ \frac{1}{(e_{q+m})^s(e_{q+n})^s} - \frac{1}{(e_q)^s(e_q)^s} \\
+ \lim_{s \to 0} \sum_{q=-\infty}^{N_+ - 1-n} \delta_{nm} |vac; N_+\rangle_+ \frac{1}{(e_{q+n})^s(e_{q+n})^s} - \frac{1}{(e_q)^s(e_q)^s}. \tag{A2}
\]
Since the first term is a finite series, we can take the $s \to 0$ limit first. Then this term vanishes. Though we cannot change the order of the limit and the summation in the second term of (A2), we can estimate it using the $\zeta$-function:
\[
\lim_{s \to 0} \sum_{q=-\infty}^{N_+ - 1-n} \left\{ \frac{1}{(e_{q+n})^s(e_{q+n})^s} - \frac{1}{(e_q)^s(e_q)^s} \right\} = \lim_{s \to 0} \sum_{q=-\infty}^{N_+ - 1-n} \left\{ \frac{1}{(q + n + \frac{\varphi}{2\pi} + \frac{eCL}{2\pi})^s} - \frac{1}{(q + \frac{\varphi}{2\pi} + \frac{eCL}{2\pi})^s} \right\} = \lim_{s \to 0} \left\{ \zeta \left( 2s, 1 - \frac{\varphi}{2\pi} - \frac{eCL}{2\pi} - N_+ \right) - \zeta \left( 2s, 1 - \frac{\varphi}{2\pi} - \frac{eCL}{2\pi} - N_+ + n \right) \right\} = n. \tag{A3}
\]
Now we obtain Eq. (4.17) on the $N_+$-vacuum. We can also evaluate the summation of infinite series without using the $\zeta$-function regularization:
\[
\lim_{s \to 0} \sum_{q=-\infty}^{N_+ - 1-n} \left\{ \frac{1}{(e_{q+n})^s(e_{q+n})^s} - \frac{1}{(e_q)^s(e_q)^s} \right\} = \lim_{s \to 0} \left\{ \sum_{q=-\infty}^{N_+ - 1} - \sum_{q=-\infty}^{N_+ - 1-n} \right\} \frac{1}{e_q^{2s}} = \lim_{s \to 0} \frac{N_+ - 1}{e_q^{2s}} = \sum_{q=N_+ - 2-n}^{N_+ - 1} 1 = n. \tag{A4}
\]
where we have changed the order, because the infinite series can be made finite. The authors of Refs. [14,16] were able to derive Eq. (4.17) without resorting to any regularization because they proceeded in such a way that the number of non-vanishing terms is always finite. So far as the number of creation and annihilation operators acting on the \( N_4 \)-vacuum is finite, we can follow the same way and obtain the same results for arbitrary states. It is straightforward to extend the calculation to the negative-chirality sector.

It is not trivial to derive the commutator (5.2) by the \( \zeta \)-function regularization. To begin with, we have to calculate the commutator between the annihilation operator \( a_n \) and \( \Pi_1 \). The basis function \( \chi_n \), a functional of \( c \), does not commute with \( \Pi_1 \),

\[
[\chi_n(x), \Pi_1(y)] = -\epsilon \left( \delta(x-y) + \delta(y) \right) \chi_n(x), \tag{A5}
\]

but since \([\psi(x), \Pi_1(y)] = 0\), we obtain the commutator,

\[
[a_n, \Pi_1(y)] = \epsilon a_n \delta(y) + \sum_{i \neq 0} a_{n+i} \frac{\epsilon}{2\pi i q} e^{-\frac{2\pi i y}{L}}. \tag{A6}
\]

After some manipulations, we have

\[
[\Pi_1(x), (j^\mu_{+})^{\text{reg}}] = -\lim_{s \to 0} \sum_{p \in \mathbb{Z}} \sum_{q \neq 0} a_p^\dagger a_{p+q+n} \left\{ \frac{1}{(\lambda \varepsilon_p)^s} - \frac{1}{(\lambda \varepsilon_{p+q})^s} \right\} \frac{\epsilon}{2\pi i q} e^{-\frac{2\pi i y}{L}}. \tag{A7}
\]

If this is estimated on the \( N_4 \)-vacuum, the surviving terms are classified into a finite series and an infinite series. Since the finite series does not contribute, the commutator reduces to

\[
[\Pi_1(x), (j^\mu_{+})^{\text{reg}}]|_{\text{vac} \; N_+}^+ = -\lim_{s \to 0} \sum_{p \in \mathbb{Z}} \sum_{q \neq 0} a_p^\dagger a_{p+q+n} \left\{ \frac{1}{(\lambda \varepsilon_p)^s} - \frac{1}{(\lambda \varepsilon_{p+q})^s} \right\} \frac{\epsilon}{-2\pi i n} e^{\frac{2\pi i n y}{L}} \quad (n \neq 0)
\]

\[
= \frac{\epsilon}{2\pi i} e^{\frac{2\pi i n y}{L}} |\text{vac} \; N_+|^+ \quad (n \neq 0). \tag{A8}
\]

When \( n = 0 \), we have \([\Pi_1(x), (j^\mu_{+})^{\text{reg}}] = 0\). This leads to

\[
[\Pi_1(x), J^\mu_{+}(y)]|_{\text{vac} \; N_+}^+ = \frac{\epsilon}{2\pi i} \left( \delta(x-y) - \frac{1}{L} \right) |\text{vac} \; N_+|^+. \tag{A9}
\]

Combining this and the similar results for the negative-chirality sector, we finally obtain

\[
[\Pi_1(x), J^\mu_{+}(y)]|_{\text{vac} \; N_+, N_-} = -\frac{i\epsilon}{\pi} \left( \delta(x-y) - \frac{1}{L} \right) |\text{vac} \; N_+, N_-|^+. \tag{A10}
\]

The same relation holds for general states as long as the number of creation or annihilation operators acting on the vacuum is finite.
References


Figure Captions

FIG. 1. Starting from a classical unreduced system whose constraint is $G = 0$, the upper approach corresponds to a classical reduction, and the lower approach to a quantum reduction. In the classical reduction, the constraint becomes $G' = 0$ after the reduction of the phase space, and regularization ($\mathcal{R}[G']$) is performed to preserve the gauge symmetry generated by $G'$. On the other hand, in the quantum reduction, the regularization ($\mathcal{R}[G]$) respects the gauge symmetry generated by $G$. As in the Schwinger model, these two results are expected to be equivalent by some redefinitions of composite operators such as currents.