Entropy in dilatonic black hole background

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Abstract

The entropy of a scalar field is calculated semiclassically in the background of a dilatonic black hole. In general it diverges linearly when the cutoff distance from the horizon vanishes and is proportional to the area of the horizon. However for extremal black holes, where this area vanishes, there is a logarithmic singularity.

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The concept of entropy has been a puzzle in the theory of black holes. It is well known that the area of the horizon of a black hole can be interpreted as an entropy [4] and satisfies all the expected laws, though there is no understanding of this fact in terms of the usual formulation of entropy as a measure of the number of states available. But a naïve Lagrangian path integral leads to a partition function from which the area formula for entropy can be obtained [6] by neglecting quantum fluctuations.

There have been some attempts at calculating the entropy of quantum fields in black hole backgrounds [1, 2]. These contributions can be interpreted as quantum corrections to the area entropy. The calculations have produced divergences, but the area of the horizon has in fact appeared as a factor in the expression for the entropy. These results have been interpreted to mean that the gravitational constant gets renormalized in the presence of the quantum fields [2]. We shall pursue this line further by investigating the case of dilatonic black holes [5, 3], where it is possible to have a vanishing horizon area.

The low-energy limit of string theory with unbroken supersymmetry contains a massless dilaton field. Models where these dilatons are coupled with gravity may be used for studying black holes with small Compton wavelengths. The simplest four-dimensional model is

$$ S = \int d^4x \sqrt{-g} e^{-2\phi} (R + 4g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi) $$

(1)

where $\phi$ is the massless dilaton field, $R$ is the scalar curvature and $g_{\mu\nu}$ is the metric. As the curvature term contains an extra exponential factor, this is often removed by the conformal transformation

$$ \tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu}. $$

(2)

This transforms the action to

$$ S = \int d^4x \sqrt{-\tilde{g}} (\tilde{R} - 2\tilde{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi), $$

(3)

This is standard Einstein gravity coupled with a massless scalar field. Thus $\tilde{g}_{\mu\nu}$ is the appropriate metric for gravitational studies and in fact all theorems of general relativity are applicable in this metric. This is not the case with the original string metric $g_{\mu\nu}$, which however is the metric seen by the string. In this note we shall confine ourselves mostly to the metric $\tilde{g}_{\mu\nu}$. 

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The model can be extended to have electromagnetic interactions by including the term

$$-\frac{1}{2} \int d^4x \sqrt{-g} e^{-2\phi} g^{\mu \nu} \tilde{F}_{\mu \nu} F_{\nu \rho}$$

in the action (3). Exact black hole solutions of this model have been found with non-zero charge and angular momentum.

The black hole solution with zero angular momentum strongly resembles the Schwarzschild solution of standard general relativity.

$$d\tilde{s} = \tilde{g}_{\mu \nu} dx^\mu dx^\nu$$

$$= -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r(r - a) d\Omega^2$$

$$e^{-2\phi} = e^{-2\phi_0} \left(1 - \frac{a}{r}\right)$$

$$F_{\mu \nu} = Q \sin \theta$$

(5)

where $M$ is the mass of the black hole and the parameter $a = \frac{Q^2 e^{-2\phi_0}}{2M^2}$. $\phi_0$ is an arbitrary constant. This black hole has as usual a horizon at $r = 2M$. An interesting feature is that a curvature singularity occurs at $r = a$. The so-called extremal solution corresponds to the coincidence of these two regions and thus has $a = 2M$. This extremal limit is interesting also because the area $4\pi 2M(2M - a)$ of the horizon vanishes. All this is from the point of view of the gravitational metric. However from the string theory point of view the geometry in the extremal limit is perfectly non-singular. In the string metric the horizon disappears and as $r \to 2M$, the spacetime splits into a $(1+1)$ dimensional Minkowski spacetime times a sphere of constant radius $2M$ (the throat).

To calculate the entropy of a scalar field in the background of such a black hole we employ the brick-wall boundary condition [1]. In this model the wave function is cut off just outside the horizon. Mathematically,

$$\varphi(x) = 0 \quad \text{at } r = 2M + \epsilon$$

(6)

where $\epsilon$ is a small, positive, quantity and signifies an ultraviolet cut-off. There is also an infrared cut-off

$$\varphi(x) = 0 \quad \text{at } r = L$$

(7)

with $L \gg 2M$. 

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The wave equation for a scalar field in this spacetime reads
\[ \partial_{\nu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\varphi) - m^2 \varphi = 0. \] (8)

A solution of the form
\[ \varphi = e^{-iEt}f_{Ei}Y_{im} \] (9)
satisfies the radial equation
\[ (1 - \frac{2M}{r})^{-1}E^2f_{Ei} + \frac{1}{r(r-a)} \frac{\partial}{\partial r}[r(r-a)(r-2M)\frac{\partial f_{Ei}}{\partial r} - \frac{l(l+1)}{r(r-a)} + m^2]f_{Ei} = 0. \] (10)

An \( r \)-dependent radial wave number can be introduced from this equation by
\[ k^2(r, l, E) = (1 - \frac{2M}{r})^{-1}[(1 - \frac{2M}{r})^{-1}E^2 - \frac{l(l+1)}{r(r-a)} - m^2]. \] (11)

Only such values of \( E \) are to be considered here that the above expression is nonnegative. The values are further restricted by the semiclassical quantization condition
\[ n_r \pi = \int_{2M+\epsilon}^{L} dr \ k(r, l, E), \] (12)

where \( n_r \) has to be a positive integer.

Accordingly, the free energy \( F \) at inverse temperature \( \beta \) is given by the formula
\[ \beta F = \sum_{n_r, l, m_1} \log(1 - e^{-\beta E}) \]
\[ \approx \int dl \ (2l+1) \int d\beta \log(1 - e^{-\beta E}) \]
\[ = -\int dl \ (2l+1) \int dE \ (e^{\beta E} - 1)^{-1}n_r \]
\[ = -\frac{\beta}{\pi} \int dl \ (2l+1) \int dE \ (e^{\beta E} - 1)^{-1} \int_{2M+\epsilon}^{L} dr \ (1 - \frac{2M}{r})^{-1} \]
\[ \sqrt{E^2 - (1 - \frac{2M}{r}) \frac{l(l+1)}{r(r-a)} + m^2} \]
\[ = -\frac{2\beta}{3\pi} \int_{2M+\epsilon}^{L} dr \ (1 - \frac{2M}{r})^{-2}r(r-a) \]
\[ \int dE \ (e^{\beta E} - 1)^{-1} [E^2 - (1 - \frac{2M}{r})m^2]^{3/2}. \] (13)
Here the limits of integration for $l, E$ are such that the arguments of the square roots are nonnegative. The $l$ integration is straightforward and has been explicitly carried out. The $E$ integral can be evaluated only approximately.

The contribution to the $r$ integral from large values of $r$ yields the expression for the free energy valid in flat spacetime ($M = 0$):

$$F_0 = -\frac{2}{9\pi} L^3 \int_{m}^{\infty} dE \frac{(E^2 - m^2)^{3/2}}{e^{\beta E} - 1}. \quad (14)$$

We ignore this part $[6, 1]$. The contribution of a nonzero $M$ is singular in the limit $\epsilon \to 0$. The leading singularity is linear:

$$F_{\text{lin}} \approx -\frac{2\pi^3}{45\epsilon} \left( 1 - \frac{\alpha}{2M} \right) \left( \frac{2M}{\beta} \right)^4, \quad (15)$$

where the lower limit of the $E$ integral has been approximately set equal to zero. If the proper value is taken, there are corrections involving $m^2 \beta^2$ which will be ignored here. This result reduces to the formula $[1]$ for the Schwarzschild black hole when $a = 0$. In general, there is simply a multiplicative factor $(1 - \frac{a}{2M})$.

There is a logarithmic singularity as well, but it is in general ignored because of the presence of the linearly divergent term $F_{\text{lin}}$. However, the linear term vanishes when $a = 2M$, i.e., when the black hole becomes extremal. In this case, the logarithmic term is the dominant one. It is

$$F_{\log} \approx -\frac{\pi^3}{45M} \log \left( \frac{2M}{\epsilon} \right) \left( \frac{2M}{\beta} \right)^4 \quad (16)$$

in the same approximation as above.

The entropy due to a nonzero $M$ can be obtained from the formula

$$S = \beta^2 \frac{\partial F}{\partial \beta}. \quad (17)$$

This gives

$$S = \frac{8\pi^3}{45} \left( \frac{2M}{\beta} \right)^3 \left( 1 - \frac{\alpha}{2M} \right) \left( \frac{2M}{\epsilon} \right)^2 \text{for } a \neq 2M \quad (18)$$

and

$$S = \frac{8\pi^3}{45} \left( \frac{2M}{\beta} \right)^3 \log \left( \frac{2M}{\epsilon} \right)^2 \text{for } a = 2M. \quad (19)$$
Thus, for $a \neq 2M$, namely for nonextremal dilatonic black holes, the Schwarzschild expression is valid, but with the area factor $(2M)^2$ corrected by the appropriate coefficient $(1 - \frac{a}{2M})$. Note that the factor $(2M^2)$ is a constant if the Hawking temperature is used, because of its inverse dependence on the mass, while the quantity $2M_a$ may be regarded as giving an invariant measure of the distance of the brick wall from the horizon [1]. If we assume, following [6], that the entropy of the dilatonic black hole is still $S = (\text{Area})/(4G_0)$, the above divergent contribution may be understood as a renormalization of the gravitational coupling constant $G_0$ [2]. However, quantum gravity being non-renormalizable, this interpretation cannot be extended to include quantum fluctuations of the gravitational fields.

In the case of extremal dilatonic black holes a logarithmic formula appears, in which the usual factor $\frac{2M^2}{2\pi}$ is replaced by its logarithm. For these black holes, where the area of the horizon vanishes, one might have expected the entropy to vanish altogether. What does happen is that the linear divergence vanishes, but the logarithmic divergence, which is of course weaker, stays on. A similar logarithmic divergence is known to occur if the theory is truncated to $(1+1)$ dimensions [2]. Our calculation shows that this is already present in $(3+1)$ dimensions.

To summarize, we have calculated semiclassically the entropy of a scalar field in the background of a dilatonic black hole. The results are similar to the case of a Schwarzschild black hole, and involve a linear divergence in general. But the area of the horizon may vanish here, and in that case the singularity becomes logarithmic.

References