Abstract

We consider two-dimensional $QED$ with several fermion flavors on a finite spatial circle. We discuss a modified version of the model with \textit{flavor-dependent} boundary conditions $\psi_p(L) = e^{2\pi i p/N} \psi_p(0)$, $p = 1, \ldots, N$ where $N$ is the number of flavors. In this case the Euclidean path integral acquires the contribution from the gauge field configurations with fractional topological charge being an integer multiple of $1/N$. The configuration with $\nu = 1/N$ is responsible for the formation of the fermion condensate $\langle \bar{\psi}_p \psi_p \rangle_0$. The condensate dies out as a power of $L^{-1}$ when the length $L$ of the spatial box is sent to infinity. Implications of this result for non-abelian gauge field theories are discussed.
1 Motivation.

Since the pioneering work [1], it is known that the Euclidean path integrals in the gauge field theories get contributions from sectors with nonzero topological charge. In the nonabelian 4-dimensional gauge theories the topological charge coincides with the so called Pontryagin class and is given by the integral

\[ \nu_4 = \frac{g^2}{32\pi^2} \int d^4 x G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a. \]  

(1.1)

It is generally assumed that the field densities \( G_{\mu\nu} \) contributing to the path integral are not singular and fall off fast enough when \( x^2 = \tau^2 + x^2 \) is sent to infinity. In this case, the topological charge (1.1) must be an integer which describes the mapping \( S^3 \to \text{gauge group} \).

However, since the beginning of the eighties, different indications have been cropping up that the restriction \( \nu_4 = \text{integer} \) is too rigid, and in some cases configurations with the fractional topological charge may be relevant.

The most explicit indication has been unraveled while studying supersymmetric (SUSY) gauge theories. The simplest version of such theory (called supersymmetric Yang-Mills, SYM) has the lagrangian

\[ \mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + \frac{i}{2} \bar{\lambda} \gamma^\mu \lambda, \]  

(1.2)

where the adjoint Majorana massless field \( \lambda \) is the superpartner of the gluon field and is called gluino. Using supersymmetric Ward identities one can show [2] that, say, in the SU(2) SUSY Yang-Mills theory the Euclidean correlator

\[ C(x) = \langle \bar{\lambda}_L \lambda_R(x) \bar{\lambda}_L \lambda_R(0) \rangle, \quad \lambda_{L,R} = \frac{1}{2} (1 \pm \gamma_5) \lambda, \]  

(1.3)

does not depend on \( x \). On the other hand, it can be explicitly evaluated at small \( x \) using the instanton calculus. As a result, one gets

\[ C(x) = A \Lambda_{SYM}^6 \]  

(1.4)

at all \( x \); here \( A \) is a calculable number [2], \( \Lambda_{SYM} \) is a scale parameter of SYM. The cluster decomposition then implies that the gluino condensate is generated,

\[ \langle \bar{\lambda} \lambda \rangle = \pm A^{1/2} \Lambda_{SYM}^3 \]  

(1.5)

(in Eq. (1.5) we assumed the vacuum angle \( \theta \) to be zero; for a more detailed discussion of the issue of the \( \theta \) dependence and the double-valuedness of \( \langle \bar{\lambda} \lambda \rangle \) see below).

And here we run into a paradox. The point is that we cannot generate a non-vanishing gluino condensate

\[ \langle \bar{\lambda} \lambda \rangle = - \frac{\partial}{\partial m} \ln Z_m \big|_{m=0} \]  

(1.6)
\( Z_m \) is the partition function of the theory involving a small Majorana mass term which is treated as a source) directly from the path integral if we integrate only over the field configurations with the integer topological charge. Due to the index theorem [3], the configuration with a given \( \nu_4 \) involves \( N_c \) pairs of gluino zero modes (this number is just the Dynkin index of the adjoint representation) which it “too much”. The corresponding contribution to the partition function involves a suppression factor

\[
Z_m(\nu_4) \propto m^{\nu_4|N_c}
\]

and the contribution of all nonzero integer \( \nu_4 \) to the fermion condensate (1.6) is suppressed.

To be more precise, one should regularize the theory by putting it in a large but finite Euclidean volume \( V \gg \Lambda_{SYM}^{-3} \), and take the limit \( m \to 0 \) while keeping the volume fixed. Due to the absence of massless states in the physical spectrum of the supersymmetric Yang-Mills theory, the chiral limit is smooth here, see Refs. [4, 5] for a detailed discussion. As a matter of fact, one can show [6] that in SYM the value of the gluino condensate does not depend on the volume at all. When considered in a small three-dimensional volume, \( V_3 < \Lambda_{SYM}^{-3} \), the system becomes quasiclassical, and the path integral must be saturated by saddle points. Then the fact that \( \langle \bar{\lambda} \lambda \rangle \neq 0 \) proves that the field configurations with fractional topological charge must contribute to the path integral.

The partition function in the topologically trivial sector is expanded in even powers of \( m \). Indeed, the spectrum of the Euclidean Dirac operator on a topologically trivial background involves only nonzero eigenvalues which come in pairs \((\mu_n, -\mu_n)\) so that the determinant of the Dirac operator \( \propto \prod_n (\mu_n^2 + m^2) \) does not produce a linear in \( m \) term in the expansion. Hence, it also does not contribute to the condensate.

The only way out is to admit the existence of field configurations with a fractional topological charge which contribute to the path integral. If \( \nu_4 = \pm 1/N_c \), \( Z_m(\nu_4) \propto m \), i.e. just what is required to generate the fermion condensate (1.6).

But what are these configurations? For the SYM theory with the unitary gauge group, one can give at least a partial answer to this question. As has been observed by ’t Hooft long ago [7], such configurations arise, indeed, if we define the theory not on a 4-dimensional large sphere and not on a cylinder \( S^3 \otimes R^1 \) (the theorem that \( \nu_4 \) should be integer applies only to these cases) but on a 4-dimensional torus, and allow for non-standard (twisted) boundary conditions for the gauge field, the so called torons. Witten further studied the twisted construction on \( T^3 \otimes R^1 \) [8] and revealed the field configurations between which the torons interpolate. Cohen and Gomez showed that torons do indeed contribute to the gluino condensate [9]. In Ref. [4] the volume and mass dependence of the partition function in the toron sector has been determined. Zhitnitsky [10] suggested another type of the field configurations with the fractional topological charge. They are defined on \( S^4 \) but are singular (topological classification is valid only for smooth fields). The singularity is not too
strong, however, so that the action of such singular fracton is still finite.

The existence of all these solutions crucially depends, however, on the presence of the center subgroup in the gauge group. For $SU(N)$ the center is $Z_N$. It is even better to say that the proper gauge group of the theory is not $SU(N)$ but rather $SU(N)/Z_N$ (the elements of the center do not transform the fields in the adjoint color representation). The factorization over $Z_N$ makes $\pi_1(\text{gauge group})$ nontrivial which brings about new topological possibilities [11, 5].

The real problem appears when we consider the SYM theories with orthogonal gauge groups $O(N), \ N \geq 5$ [12, 5]. The Ward identities + instanton calculus imply a non-vanishing gluino condensate exactly by the same token as in the unitary case. The index theorem dictates the presence of $2(N-2)$ gluino zero modes in the instanton background – too much to generate the fermion condensate. On the other hand, the center is either absent here (odd $N$) or is too small ($Z_2$ for even $N$), and we cannot construct a configuration which carries just 2 fermion zero modes and do not understand the mechanism for generating the fermion condensate. The same situation takes place for exceptional groups [13].

A similar problem appears also in two-dimensional QCD with the adjoint Majorana fermions [5]. The paradox arises here already for the unitary gauge groups if the number of colors is 3 or more. The instantons (they appear in the adjoint QCD$_2$ due to the presence of nontrivial $\pi_1(SU(N)/Z_N) = Z_N$) involve here $2(N_c-1)$ fermion zero modes and cannot generate the fermion condensate for $N_c \geq 3$. On the other hand, independent arguments (based on the bosonized representation of the theory) do imply the presence of the fermion condensate. There are no massless states in the spectrum, the chiral limit $m \to 0$ is smooth, and the only way to generate the nonzero condensate is to admit the existence of some field configurations carrying just 2 zero modes and contributing to the Euclidean path integral. Such configuration are not known so far, however.

In this paper we discuss instead the simplest possible model where a similar paradox can be formulated and successfully resolved. We hasten to add that the results reported here literally speaking add little to this issue in the non-abelian theories. The paradox is still there. Some parallels, however, seem to be instructive, and may provide insights in the non-abelian case in future.

The model to be considered below is basically the Schwinger model (QED in 1 + 1 dimensions) treated in a finite spatial box of dimension $L$. The lagrangian includes not one (as in the original Schwinger model) but several fermion flavors:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + i \sum_{p=1}^{N} \bar{\psi}_p D_{\mu} \psi_p - m \sum_{p=1}^{N} \bar{\psi}_p \psi_p$$

(1.8)

where $D_{\mu} = \partial_{\mu} - ig A_{\mu}$, $N$ is the number of flavors, and the charge $g$ has dimension of mass and defines the characteristic mass scale of the theory (it plays the same role as $\Lambda_{QCD}$ and $\Lambda_{SYM}$). The fermion mass $m$ is chosen to be much less than $g$ and will be treated as a perturbation, to be put equal to zero at the very end. The nontrivial picture which resembles the situation in the non-abelian gauge theories
outlined above arises if we impose \textit{flavor-dependent} spatial boundary conditions on the fermion fields. Namely,
\begin{equation}
\psi_p(x = L) = e^{2\pi ip/N} \psi_p(x = 0), \quad p = 1, 2, ..., N, \tag{1.9}
\end{equation}
where \(L\) is the length of the circle where the theory is defined. This non-standard boundary condition, a heart of our construction, will lead to a dramatic deviation from the standard description of the Schwinger model.\footnote{Twisted boundary conditions on fermions conceptually resembling ours were discussed previously in QCD with \(N_c = N_f\) in Ref. [14].}

Let us remind that the theory involves instantons, and the usual topological arguments require
\begin{equation}
\nu_2 = \frac{g}{4\pi} \int \epsilon_{\mu\nu} F_{\mu\nu} d^2 x \tag{1.10}
\end{equation}
(the two-dimensional analog of the Pontryagin class) to be an integer. The index theorem implies then the existence of a complex zero mode in the instanton background for each fermion flavor. Thus, the partition function is
\begin{equation}
Z_m(\text{inst}) \propto m^N, \tag{1.11}
\end{equation}
and the fermion condensate seemingly cannot be generated.

Independent arguments presented below show, however, that the condensate \textit{is} generated under the boundary conditions (1.9). This fact can be explained only if the path integral gets a contribution from the field configurations with a fractional topological charge.

We will show that in the case at hand such fractons are, indeed, present. We display their origin and calculate explicitly the path integral on the fracton background (since the model in the small mass limit is exactly soluble, all calculations can be done analytically and the results are \textit{exact}).

The main difference between this toy model and complicated non-abelian models discussed above is that here the fermion condensate is essentially a finite-volume effect. It dies out when the length of the box is sent to infinity. (This is, of course, natural. If the box is large, the nonstandard boundary conditions (1.9) should not affect local physics. For large \(L\), the theory has the same properties as the standard multiflavor Schwinger model where no condensate appears). Apart from this distinction other qualitative features of our construction will hopefully serve as a prototype for the SYM theories.

The paper is organized as follows. In Sect. 2 we remind in brief the solution of the standard multiflavor Schwinger model (with the trivial boundary conditions). In Sect. 3 we start our discussion of the twisted model and consider first the case of a small spatial circle, \(gL \ll 1\), where the situation is especially simple and transparent. In Sect. 4 we calculate the fermion condensate at arbitrary \(L\) via the correlator \(\langle \bar{\psi} \psi(\tau) \bar{\psi} \psi(0) \rangle\) at large Euclidean time \(\tau\). In Sect. 5 we rederive this result defining and calculating the corresponding path integral in the fracton background. General discussion is given in Sect. 6.
2 Multiflavor Schwinger model

Consider first the theory with the lagrangian (1.8) in the two-dimensional space-time without boundaries. In the massless limit, the tree lagrangian has the symmetry 

\[ SU_L(N) \otimes SU_R(f) \otimes U(1) \otimes U(1). \]

As well-known, the \( U_A(1) \) part of it is anomalous, and is absent in the full quantum theory (for a review see, e. g. [15]). The \( SU_A(N) \) part of the symmetry is broken explicitly by the mass term in the lagrangian. But if the mass is small, \( m \ll g \), the breaking is weak.

The spectrum of the theory involves a “massive photon” (a flavor-singlet scalar meson) and \( N^2 - 1 \) light mesons belonging to the \( SU(N) \) multiplet of the flavor group [16]. The photon mass is

\[ \mu^2 = \frac{Ng^2}{\pi} + \mathcal{O}(mg) \]  

while the mass of light mesons is

\[ m_{light}^2 \sim g^{1/(N+1)} m^{N/(N+1)}. \]  

The multiplet of light mesons superficially resembles a pseudogoldstone multiplet in the standard 4-dimensional QCD; there is an important distinction, however. In the chiral limit \( m \to 0 \), not only the mass of these “pseudo-pseudogoldstones” turns to zero but, also, their coupling to the massive sector vanishes (in contrast to pions which are coupled to massive hadrons). The fact that the massless states become sterile is specific for two dimensions – a consistent two-dimensional field theory with massless interacting bosons does not exist [17].

The same Coleman theorem prohibits the formation of the fermion condensate in the massless Schwinger model with several flavors. Indeed, the condensate would break spontaneously the symmetry \( SU_L(N) \otimes SU_R(N) \) down to \( SU_V(N) \) which would lead to appearance of non-sterile Goldstone particles in the spectrum not allowed in two dimensions.

If \( m \neq 0 \) and the symmetry \( SU_A(N) \) is already broken explicitly, the formation of the condensate is allowed. It has a funny non-perturbative dependence on the fermion mass [18],

\[ \langle \bar{\psi} \psi \rangle \sim g^{2/(N+1)} m^{(N-1)/(N+1)}. \]

It is instructive to consider the correlator

\[ C_N(x) = \langle T \{ \bar{\psi}_L \psi_R(x) \} \bar{\psi}_R \psi_L(0) \rangle_0 \]

in the massless theory. At large \( x, gx \gg 1 \), it falls off as a power,

\[ C_N(x) \sim g^{2/N} x^{2/N-2}. \]
One may ask how one can reconcile this power falloff of the correlator which may be only due to massless intermediate states with the fact that the massless states decouple in the chiral limit. The answer is as follows. The Coleman theorem dictates that all matrix elements \( \langle 0 | \bar{\psi} \gamma^\mu \sigma \psi | \eta_{ij} \rangle \) vanish in the chiral limit. And they do. Let us allow, however, for a small nonzero mass \( m \). The correlator (2.5) is saturated by the sum over all possible intermediate states involving light mesons. The smaller the mass, the larger the characteristic number \( n \) of "pseudo-pseudogoldstones" is in the intermediate state. As a result, the chiral limit of the whole sum is nonzero (see Eq. (5.6) in Ref. [18]).

Anyway, the correlator tends to zero in the limit \( x \to \infty \) and the fermion condensate vanishes. No contradiction with the instanton counting arises.

It is interesting, however, to derive the result (2.5) without using the bosonization approach (as was done in [18]) but directly from the Euclidean path integral involving fermion and gauge fields. One can then ask what characteristic field configurations are contributing to the path integral. In answering this question fractons popped out unexpectedly [19].

The picture outlined above refers to the infinite volume. Let us proceed now to the theory defined in a finite volume.

If the volume where the theory is defined is large the quantum fluctuations are strong and the quasiclassical picture does not work. The situation is much simpler and more clear when the theory is defined on a cylinder \( S^1 \otimes R^3 \) where \( R^3 \) is a small spatial circle with the length \( L \ll g^{-1} \).

It will be shown in Sect. 4 that in the small \( gL \) limit the path integral for the correlator

\[
C(\tau_0) = \langle \{ \bar{\psi}_L \psi_R(\tau_0, 0) \} \rangle
\]

in the multiflavor Schwinger model defined on a circle (\( \tau_0 \gg g^{-1} \) is the Euclidean time) is saturated by the gauge field configuration

\[
A_1(\tau) = A_1^{fr}(\tau) - A_1^{fr}(\tau - \tau_0)
\]

where

\[
A_1^{fr}(\tau) = \left\{ \begin{array}{ll}
\frac{2}{N_\sigma L} \exp\{\mu \tau\}, & \tau \leq 0 \\
\frac{2}{N_\sigma L} \left[ 2 - \exp\{-\mu \tau\} \right], & \tau \geq 0
\end{array} \right.
\]

(the gauge \( A_0 = 0 \) is chosen; \( \mu \) is the photon mass defined in (2.2); note that the contribution (2.7) does not depend on \( x \) as it should be on a small circle). The quantum fluctuations around the stationary quasiclassical configuration (2.7) are suppressed with respect to the characteristic amplitude of the solution (2.7) by the factor \( \sim \sqrt{gL} \ll 1 \).

\[\text{Fractons also show up in the standard one-flavor Schwinger model in the expectation value} \langle \bar{\psi}(x) \exp\{ie \int_0^\tau A_\mu dx_\mu \} \psi(0) \rangle.\]
The function (2.7) is plotted in Fig. 1. We see that the total topological charge (1.10) of this configuration is zero:

$$\nu = \frac{gL}{2\pi} \int_{-\infty}^{\infty} \frac{A_1 d\tau}{A_1} = \frac{gL}{2\pi} [A_1(\infty) - A_1(-\infty)] = 0.$$  \hspace{1cm} (2.9)

That is what one could expect right from the beginning as the correlator (2.6) gets contributions only from the topologically trivial sector. However, the zero result is due to cancellation of two distinct nonzero contributions with the opposite signs coming from different regions in the integral (2.9); $\nu = 1/N$ due to the region $\tau \sim 0$ and $\nu = -1/N$ due to the region $\tau \sim \tau_0$ (it is assumed that $\tau_0 g \gg 1$).

If $N > 1$ the individual function (2.8) of which the full stationary solution (2.7) is composed presents a *fracton*, the configuration with a fractional topological charge.

The configuration (2.7) is the stationary point of the path integral both in the standard multilavor Schwinger model (with periodic boundary conditions for all flavors) and in the twisted Schwinger model (with the boundary conditions (1.9)).

The behavior of the correlator (2.6) is, however, completely different in these two cases. When boundary conditions are standard, the correlator falls off exponentially at large $\tau_0$, $\tau_0 \gg L$ (if $\tau_0 < L$ the fall-off is power-like, see above), the fermion condensate is zero, and, though fractons appear as an ingredient in the relevant field configuration (2.7), there is no need to consider an isolated fracton. The total topological charge of field configurations in the path integral is always integer.

For the twisted model, the situation is, however, different. The correlator (2.6) tends to a constant at large $\tau_0$ in this case. The cluster decomposition property then implies the appearance of the fermion condensate and we need an isolated fracton to generate it.

The picture in the twisted model is similar to what we had in the standard Schwinger model with one flavor, see Ref. [20]. In this work the Schwinger model at large temperature was considered, on the infinite circle. But the results can be easily translated to the theory at zero temperature and on a small circle after rotation in the Euclidean space by $\pi/2$. The correlator (2.6) in Ref. [20] was saturated by an instanton-antiinstanton configuration yielding a constant at large $\tau_0$. The fermion condensate itself was generated by an isolated instanton.

### 3 Theory on small circle: Hamiltonian approach

In this section we calculate the fermion condensate in the model defined on a small spatial circle. The case $gL \ll 1$ is especially simple because the higher Fourier harmonics decouple here (with reservations to be specified later) and we are left with a quantum-mechanical problem which is easily analyzed within the Hamiltonian approach. To make discussion self-contained and transparent we first describe the well-known situation in the standard Schwinger model with one fermion flavor (see e.g. [15]) and then pass to a more complicated twisted multilavor case.
3.1 The standard Schwinger model

The Hamiltonian of the theory is

\[ H = \int_0^L dx \left\{ \frac{E^2(x)}{2} - i \tilde{\psi}(x) \gamma_1 [\partial_1 - ig A_1(x)] \psi(x) \right\} \quad (3.1) \]

where \( E(x) = -i \partial/\partial A_1(x) \) is the electric field. Not all eigenstates of the Hamiltonian are admissible but only those which satisfy the Gauss law constraint

\[ G(x) = \partial_x E(x) - g \tilde{\psi}(x) \gamma_0 \psi(x) = 0. \quad (3.2) \]

Two-dimensional \( \gamma \) matrices may be chosen (in the Minkowski space) as \( \gamma_0 = \sigma_2, \gamma_1 = i \sigma_1; \gamma^5 = \sigma_3 \). Let us impose the periodic boundary conditions on all fields and expand them in the Fourier series

\[ A_1(x) = \sum_{k=-\infty}^{\infty} A_1^{(k)} e^{2\pi i k x / L}, \]

\[ \psi(x) = \sum_{k=-\infty}^{\infty} \psi^{(k)} e^{2\pi i k x / L}. \quad (3.3) \]

When \( gL \) is small, one can separate all dynamical variables in two classes: the slow variables

\[ A_1^{(0)} \equiv a \]

(we shall shortly see that its characteristic excitation energy is of order \( g \) and the fast variables \( \psi^{(k)} \) and \( A_1^{(k \neq 0)} \) with characteristic excitation energies \( \sim 1/L \gg g \).

Then we can treat the system in the spirit of the Born-Oppenheimer approximation – integrate over fast variables and solve the Schrödinger equation for the effective Hamiltonian thus obtained which depends only on the slow variable \( a \).

One could adopt a slightly different formulation of the Hamiltonian approach (see, e. g. [15]). Instead of imposing the Gauss law constraints on the states one could eliminate the gauge degrees of freedom of the photon field. Namely, all non-zero modes \( A_1^{(k \neq 0)} \) can be gauged away. Our system then would consist of the modes of the fermion field \( \psi^{(k)} \) coupled to \( a \) and interacting with each other via the Coulomb interaction. The latter is non-local in space, instantaneous in time and can be neglected altogether provided that \( gL \ll 1 \). After integrating out the fast variables we would arrive at the same description of the slow variable \( a \).

In the lowest order in the Born-Oppenheimer expansion, the induced potential \( V^{\text{eff}}(a) \) is given just by the sum of the zero-point energies of the fermion oscillators estimated in a constant (time-independent) background \( a \) (the higher harmonics \( A_1^{(k \neq 0)} \) are not coupled directly to \( a \) and can be neglected in this order). The structure of fermion levels as a function of \( a \) is shown in Fig. 2.

The fermion ground state corresponds to filling out all levels with the negative energies (the Dirac sea) and leaving empty all levels with the positive energy. From
the picture in Fig. 2 it is clear that the minimum of the induced potential \( V_{\text{eff}}(a) \)

is achieved at

\[
a = \frac{\pi}{gL}(2n + 1), \quad n \ \text{integer}.
\]  

As usual, the sum of the zero-point energies involves an infinite constant which should be subtracted. It is convenient to normalize \( V_{\text{eff}}(a) \) at its minimal value achieved in the points (3.4). Then the effective Born-Oppenheimer Hamiltonian calculated in the vicinity of the \( n \)-th minimum takes the form

\[
H_{\text{eff}}^{(n)}(a) = \frac{1}{2L}(\frac{-id}{da})^2 + \frac{\mu^2 L}{2} \left( a - \frac{\pi(2n + 1)}{gL} \right)^2
\]  

where \( \mu^2 = g^2 / \pi \). Characteristic excitation energies of this oscillator Hamiltonian are of order \( g \) as was mentioned earlier. In the region

\[
a \sim \frac{\pi(2n + 1)}{gL}
\]

the ground state wave function of the full Hamiltonian (3.1) is the product of the ground state wave function of the slow Hamiltonian (3.5) and the fast fermion wave functions corresponding to filling out all negative levels. We have

\[
|\text{vac}_n\rangle = \left( \frac{L\mu}{\pi} \right)^{1/4} e^{-\frac{g^2 L}{4}[a - \frac{\pi(2n+1)}{gL}]^2} \prod_k \psi^{(k)}_L \prod_k \psi^{(k)}_R,
\]  

where

\[
\psi_{L,R} = \left( \frac{1 \pm \gamma^3}{2} \right) \psi
\]

are the upper and the lower components of the spinor field \( \psi \), correspondingly. The fermion part of the wave function (3.6) is written symbolically: which particular levels have negative energies depends on \( n \); for more details see Ref. [15]. The freedom in choosing \( n \) corresponds to the symmetry of the full Hamiltonian (3.1):

\[
S : \quad \begin{cases} 
A_1(x) \rightarrow A_1(x) + \frac{2\pi}{gL} \\
\psi(x) \rightarrow e^{2\pi i/(gL)} \psi(x)
\end{cases}
\]  

so that \( S|\text{vac}_n\rangle = |\text{vac}_{n+1}\rangle \). This symmetry presents the so called large gauge transformation. The Gauss law constraint (3.2) requires the invariance of the wave functions with respect to infinitesimal gauge transformations but does not say anything about transformation properties of the wave functions under the action of the transformation \( S \). We can impose, however, an additional restriction (called the superselection rule) and consider the sector of the theory with all wave functions satisfying the requirement

\[
\Psi( A^S, \psi^S) = e^{i\theta} \Psi(A, \psi).
\]
Then the action of any local operator on the wave function belonging to the class (3.8) leaves it within the same class. The ground state wave function in the sector with the given \( \theta \) has the form

\[
\Psi_\theta \equiv |vac\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta} |vac_n\rangle;
\]  

(3.9)

\( \theta \) has the same meaning as in the Yang-Mills theory and is called the vacuum angle.

Everything is ready now for calculating the fermion condensate. By convoluting the wave functions (3.6) we get 4

\[
\langle \bar{\psi}_R \psi_L \rangle_\theta = \frac{\langle vac | \bar{\psi}_R \psi_L | vac \rangle}{\langle vac | vac \rangle} = e^{-i\theta} \langle vac_n | \bar{\psi}_R \psi_L | vac_{n-1} \rangle
\]

\[
= e^{-i\theta} \frac{1}{L} \exp\left\{-\frac{\pi}{\mu L}\right\}
\]  

(3.10)

and, correspondingly,

\[
\langle \bar{\psi}_R \psi_L \rangle_\theta = e^{i\theta} \frac{1}{L} \exp\left\{-\frac{\pi}{\mu L}\right\}.
\]  

(3.11)

The exponent in Eqs. (3.10), (3.11) is essentially the action of the instanton – the Euclidean field configuration which interpolates between the adjacent minima (3.4) and minimizes the effective action [20]. The same result for the fermion condensate can be obtained through bosonization [21, 18].

### 3.2 Twisted model

Let us consider first, for simplicity, the model with two fermion flavors. The hamiltonian of the model is

\[
H = \int_0^L dx \left\{ \frac{E^2(x)}{2} - i\bar{\psi}(x)\gamma_1[\partial_1 - igA_1(x)]\psi(x) - i\bar{\chi}(x)\gamma_1[\partial_1 - igA_1(x)]\chi(x) \right\}
\]  

(3.12)

with the constraint

\[
G(x) = \partial_x E(x) - g\bar{\psi}(x)\gamma_0\psi(x) - g\bar{\chi}(x)\gamma_0\chi(x) = 0.
\]  

(3.13)

The twisted boundary conditions

\[
\psi(x = L) = \psi(x = 0),
\]

4 Here \( \bar{\psi} \) and \( \bar{\psi} \) stand for the Heisenberg field operators defined in a usual way so that \( \psi_R \) involves annihilation operators for the right-handed states and \( \psi_L \) creation operators for the left-handed states. The fields \( \psi \) and \( \bar{\psi} \) are not to be mixed up with the holomorphic field variables appearing in Eq. (3.6).
\[ \chi(x = L) = -\chi(x = 0) \]  

are chosen. As earlier, we assume \( gL \ll 1 \) which allows us to repeat the same Born-Oppenheimer type analysis as in the previous subsection. The structure of the fermion levels is shown in Fig. 3. We see that the minima of the effective potential \( V^{eff}(a) \) occur now at

\[ a = \frac{\pi}{2gL}(2n + 1), \quad n \text{ integer}. \]  

The distance between the adjacent minima is now only \( \pi/gL \), twice smaller than in the standard Schwinger model (or in the multiflavor Schwinger model with the standard boundary conditions). The appearance of the new minima is due to a new global symmetry of the hamiltonian (3.12):

\[ \tilde{S} : \begin{cases} A_1(x) \to A_1(x) + \frac{\pi}{gL} \\ \psi(x) \to e^{i\pi \bar{x}/L} \psi(x) \\ \chi(x) \to e^{i\pi \bar{x}/L} \chi(x) \end{cases} \]  

This new symmetry is a combination of two “non-symmetries”: a previously forbidden gauge transformation and a flavor SU(2) rotation \( \psi \leftrightarrow \chi \) forbidden by itself by the twisted boundary conditions. The double application of \( \tilde{S} \) is equivalent to \( S \). The symmetry

\[ S = \tilde{S}^2 = \begin{cases} A_1(x) \to A_1(x) + \frac{2\pi}{gL} \\ \psi(x) \to e^{2i\pi \bar{x}/L} \psi(x) \\ \chi(x) \to e^{2i\pi \bar{x}/L} \chi(x) \end{cases} \]  

persists in the theory with any choice of boundary conditions. The square root \( \tilde{S} \) can be extracted only under the choice (3.14). The symmetry \( \tilde{S} \) is not just a large gauge transformation, unlike \( S \).

The effective hamiltonian in the vicinity of one of the minima is

\[ H^{eff}_n(a) = \frac{1}{2L}(-id/da)^2 + \frac{\mu^2 L}{2} \left( a - \frac{\pi(2n + 1)}{2gL} \right)^2 \]  

where now \footnote{One and the same letter \( \mu \) denotes different mass terms, depending on the fermion content of the theory at hand. We hope that this fact will cause no confusion.} \[ \mu^2 = 2g^2/\pi. \]
The ground state wave function of the hamiltonian (3.18) is

$$|\text{vac}_n\rangle = \left(\frac{Lg}{\pi}\right)^{1/4} e^{-\frac{g^2}{2}\left[\sum_{n=0}^{n=2n+1}\right]^2} \prod_k \psi_L^{(k)} \prod_k \psi_R^{(k)} \prod_k \chi_L^{(k)} \prod_k \chi_R^{(k)}$$

(3.19)

where, again, the products involve only the states with the negative energies. We can impose now a modified superselection rule subdividing the whole Hilbert space of the hamiltonian (3.12) into the sectors with wave functions belonging to a particular irreducible representation of the symmetry $\tilde{S}$:

$$\Psi(A^S, \psi^S, \chi^S) = e^{i\tilde{\theta}} \Psi(A, \psi, \chi)$$

(3.20)

The physical vacuum state is given, as previously, by a superposition similar to (3.9),

$$\Psi_\tilde{\theta} = \sum_{n=-\infty}^{\infty} e^{in\tilde{\theta}} |\text{vac}_n\rangle \equiv \Psi_1 + \Psi_2$$

(3.21)

where

$$\Psi_1 = \sum_{n=\text{odd}} e^{in\tilde{\theta}} |\text{vac}_n\rangle, \quad \Psi_2 = \sum_{n=\text{even}} e^{in\tilde{\theta}} |\text{vac}_n\rangle$$

(3.22)

are the eigenfunctions of the large gauge transformation $S$. The standard instanton calculation of any quantity corresponds to averaging this quantity over $\Psi_1$ or $\Psi_2$, rather than over the true vacuum state $\Psi_{\tilde{\theta}}$. Notice that $\Psi_1$ and $\Psi_2$ are linear combinations of

$$\Psi_{\tilde{\theta}} \text{ and } \Psi_{\tilde{\theta}^\prime}, \quad \tilde{\theta}^\prime = \tilde{\theta} + \pi \ (\text{mod} \ 2\pi).$$

It is instructive to consider the action of the conserved non-anomalous charges on the vacuum wave function. The vector charges

$$\int dx \bar{\psi} \gamma_0 \psi, \quad \text{and} \quad \int dx \bar{\chi} \gamma_0 \chi$$

can be defined in such a way that the corresponding charge operators will annihilate $\Psi_{\tilde{\theta}}$ (i.e. the charge of the vacuum vanishes). As for the axial charges only one of them is non-anomalous,

$$Q_{\tilde{\theta}}^{(-)} = \int dx \bar{\psi} \gamma_0 \gamma_5 \psi - \int dx \bar{\chi} \gamma_0 \gamma_5 \chi.$$

(3.23)

Then it is easy to see that the action of $Q_{\tilde{\theta}}^{(-)}$ produces a different state,

$$Q_{\tilde{\theta}}^{(-)} \Psi_{\tilde{\theta}} = \Psi_{\tilde{\theta}^\prime} - \Psi_{\tilde{\theta}}.$$

(3.24)

This means, of course, that $\Psi_{\tilde{\theta}}$ and $\Psi_{\tilde{\theta}^\prime}$ are degenerate in energy.

The standard vacuum angle $\theta$ is introduced with respect to the large gauge transformation $S$. Then we see that the genuine vacuum angle $\tilde{\theta}$ in the model considered is twice smaller than the standard angle $\theta$ which enters the standard
The superselection rule (3.8) (with dynamical variables $\chi$ being included). The angle $\hat{\theta}$ varies within the interval $(0, 2\pi)$ – this means that we must allow $\theta$ to vary in the interval $(0, 4\pi)$. We may conjecture that a similar situation takes place in the pure Yang-Mills theory and in the Yang-Mills theory coupled to adjoint fermions where the proper range for the vacuum angle $\theta$ is not $(0, 2\pi)$, as often assumed, but rather $(0, 2\pi N_c)$ (for the unitary gauge groups; for a discussion of this question see [4]).

The fermion condensate $\langle \bar{\psi}_R \psi_L \rangle_\beta$ appears now due to nonzero matrix elements

$$\langle \text{vac} \mid \bar{\psi}_R \psi_L \mid \text{vac} \rangle, \quad \langle \text{vac} \mid \bar{\psi}_R \psi_L \mid \text{vac} \rangle, \quad \text{etc.}$$

Similarly, the fermion condensate $\langle \bar{\chi}_L \chi_L \rangle_\beta$ appears due to nonzero matrix elements

$$\langle \text{vac} \mid \bar{\chi}_L \chi_L \mid \text{vac} \rangle, \quad \langle \text{vac} \mid \bar{\chi}_L \chi_L \mid \text{vac} \rangle, \quad \text{etc.}$$

Indeed, it is clear from Fig. 3 that in passing from $n = 0$ to $n = 1$ only the levels of the field $\psi$ cross the zero-energy mark. In passing from $n = 1$ to $n = 2$ only the levels of the field $\chi$ cross the zero. And then the pattern repeats itself.

As a result, we get

$$\langle \bar{\psi}_R \psi_L \rangle_\beta = \langle \bar{\chi}_L \chi_L \rangle_\beta = e^{-i\beta} \frac{1}{2L} \exp\{-\frac{\pi}{2\mu L}\},$$

$$\langle \bar{\psi}_L \psi_R \rangle_\beta = \langle \bar{\chi}_R \chi_R \rangle_\beta = e^{i\beta} \frac{1}{2L} \exp\{-\frac{\pi}{2\mu L}\}. \quad (3.25)$$

This analysis can be easily generalized to the case of arbitrary number of flavors with the boundary conditions (1.9). The minima of the effective potential occur at

$$a = \frac{\pi}{NgL}(2n + 1), \quad n \text{ integer}. \quad (3.26)$$

The hamiltonian of the system enjoys a new global symmetry

$$\tilde{S}_N : \begin{cases} A_1(x) \to A_1(x) + \frac{2\pi}{NgL} \\ \psi_\mu(x) \to e^{2\pi i\mu/(NL)}\psi_{\mu+1 mod.}(x) \end{cases} \quad (3.27)$$

One can impose the superselection rule which picks out the states transforming as irreducible representation of $\tilde{S}_N$. This representation is characterized by a vacuum angle $\hat{\theta}_N$ varying in the range $(0, 2\pi)$. The “old” vacuum angle $\theta = N\hat{\theta}$ varies in the range $(0, 2\pi N)$. The fermion condensates are

$$\langle \bar{\psi}_R \psi_L \rangle_{\hat{\theta}_N} = \langle \bar{\chi}_L \chi_L \rangle_{\hat{\theta}_N} = e^{-i\hat{\theta}_N} \frac{1}{NL} \exp\{-\frac{\pi}{N\mu L}\} \quad (3.28)$$

with $\mu^2 = Ng^2 / \pi$. 

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4 Fermion correlator

In the previous section, we restricted ourselves to the case $gL \ll 1$ where the Born-Oppenheimer approximation works and the Hamiltonian analysis is easy. It is interesting, however, to extend our results to arbitrary $L$, especially for large $L \gg g^{-1}$ to make contact with the results of Sect. 2 referring to the infinite volume. When $gL$ is large, non-trivial nature of the boundary conditions (1.9) should be irrelevant. In particular, the fermion condensate should vanish in the limit $L \to \infty$ as it does in the theory with the standard boundary conditions.

And, indeed, the condensate vanishes in this limit. In this section, we calculate the condensate for arbitrary $L$ employing an indirect but technically the simplest way. In the next section this result will be confirmed by a direct computation of the path integral in the fracton background.

Consider the Euclidean correlator (2.6). We show that in the twisted multilavor Schwinger model defined on a finite circle, this correlator tends to a constant in the limit $\tau_0 \to \infty$. The cluster decomposition implies then the existence of the fermion condensates.

To begin with, let us describe how the correlator (2.6) is calculated in the standard Schwinger model. For our purposes it is inconvenient to use the bosonization technique [22] which cannot be directly generalized to the twisted multilavor case. We, instead, use the functional integral approach developed in [23, 24] and applied for calculating the fermion correlator in (20).

Note first of all that the correlator (2.6) has the zero chiral $U_A(1)$ charge and acquires only the contributions from the topologically trivial sector. Our calculation is done in two steps. First we calculate the fermion path integral in a given gauge field background and then integrate over the gauge fields. To regularize the path integral in the infrared let us define the theory on torus with the spatial dimension $L$ and the Euclidean time dimension $T$. The latter will be assumed very large and will be sent to $\infty$ as soon as possible.

The path integral for the Schwinger model on torus has been carefully analyzed in Ref. [24]. Any topologically trivial Euclidean field $A_\mu(x)$ can be represented as

$$A_\mu(\tau, x) = A_\mu^{(0)} - \epsilon_{\mu\nu} \partial_\nu \phi(\tau, x) + \partial_\mu \lambda(\tau, x)$$

where

$$A_\mu^{(0)} = \left( \frac{2\pi}{gT} h_0, \frac{2\pi}{gL} h_1 \right), \quad 0 \leq h_{0,1} \leq 1$$

is the constant part of the potential, $\partial_\mu \lambda(\tau, x)$ is the gauge part, and the part $-\epsilon_{\mu\nu} \partial_\nu \phi(\tau, x)$ carries nontrivial dynamical information. Periodic boundary conditions on the functions $\phi(\tau, x)$ and $\lambda(\tau, x)$ are imposed in both the spatial and the Euclidean time directions. One should also impose the constraint

$$\int dx d\tau \phi(\tau, x) = 0.$$
This constraint excludes the constant part of $\phi$ which is the zero mode of the Laplace operator on compact manifolds. In two dimensions the fermion determinant can be explicitly calculated in any gauge field background. As a result, the partition function in the topologically trivial sector can be written as (including the fermion determinant)

$$Z_{v=0} = \int \prod_{\mu} dA^{(0)}_{\mu} F(A^{(0)}_{\mu}) \times$$

$$\prod_{x, \tau} d\phi(\tau, x) \exp\{-\frac{1}{2} \int_0^L dx \int_0^T d\tau \phi(\Delta^2 - \mu^2 \Delta)\phi\}. \quad (4.31)$$

A remarkable fact is that the constant harmonics $A^{(0)}_{\mu}$ completely decouple here and, in particular, the explicit form of the function $F(A^{(0)}_{\mu})$ (which can be determined, though) is irrelevant. What is relevant is that, for very large $T$, the integral is concentrated in the region

$$A^{(0)}_0 = 0, \quad A^{(0)}_1 = \frac{\pi}{gL} \quad (4.32)$$

(in this value one recognizes one of the minima of the effective potential which happens to lie within the region of integration ($0 \leq A^{(0)}_1 \leq \frac{2\pi}{gL}$).)

The path integral for the fermion correlator ($2.6$) has the form

$$C(\tau_0) = Z_0^{-1} \int \prod_{\mu} dA^{(0)}_{\mu} F(A^{(0)}_{\mu}) \prod_{x, \tau} d\phi(\tau, x) \text{Tr} \left\{ S_{\phi}(0, \tau_0) \frac{1 - \gamma^5}{2} S_{\phi}(\tau_0, 0) \frac{1 + \gamma^5}{2} \right\} \times$$

$$\exp\{-\frac{1}{2} \int_0^L dx \int_0^T d\tau \phi(\Delta^2 - \mu^2 \Delta)\phi\} \quad (4.33)$$

where the spatial coordinates of the initial and final points (set equal to zero) are not indicated explicitly. Moreover, $S_{\phi}(0, \tau_0)$ is the fermion Green’s function in the given gauge field background $-\epsilon_{\mu\nu} \partial_{\nu} \phi(\tau, x) + \delta_{\mu 1} \frac{\pi}{gL}$. Another remarkable simplification which occurs in two dimensions is that Green’s function $S_{\phi}(0, \tau_0)$ can be found exactly,

$$S_{\phi}(0, \tau_0) = \exp\{-g\gamma^5 \phi(0)\} S_0(0, \tau_0) \exp\{-g\gamma^5 \phi(\tau_0)\}, \quad (4.34)$$

where $S_0(0, \tau_0)$ is the fermion Green’s function estimated in the background ($4.32$).

Substituting Eq. ($4.34$) into Eq. ($4.33$), we see that the path integrals for the partition function ($4.31$) and for the fermion correlator ($4.33$) are Gaussian and can be done explicitly (this is, of course, a consequence of the fact that the Schwinger model is exactly soluble). We get

$$C(\tau_0) = C_0(\tau_0) e^{4g^2[\bar{\phi}^{(0)} - \bar{\phi}(\tau_0)]} \quad (4.35)$$
where $C_0(\tau_0)$ is the fermion correlator in the background (4.32) and $G(x)$ is Green’s function of the operator $\mathcal{O} = \Delta^2 - \mu^2 \Delta$

\[(\Delta^2 - \mu^2 \Delta) G(x) = \delta^{(2)}(x)\]  
\hspace{3cm} (4.36)

Notice that in the limit $T \to \infty$ torus converts into cylinder, i.e. a non-compact manifold, and there is no need to bother with the elimination of the zero mode of the Laplace operator as is the case for torus $[24]$ and sphere $[23]$.

The free Euclidean correlator of the fermion densities can be readily found on $S_1 \otimes R_1$,

\[C_0(\tau_0) = \frac{1}{2} Tr \left\{ \int \frac{d\omega_1}{2\pi} e^{i\omega_1 \tau_0} \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{\omega_1 \gamma_0^E + k_n \gamma_1^E}{\omega_1^2 + k_n^2} \times \right. \]
\[\left. \int \frac{d\omega_2}{2\pi} e^{-i\omega_2 \tau_0} \frac{1}{L} \sum_{m=-\infty}^{\infty} \frac{\omega_2 \gamma_0^E + k_m \gamma_1^E}{\omega_2^2 + k_m^2} \right\} \]  
\hspace{3cm} (4.37)

where $\gamma_0^E = \sigma_2, \gamma_1^E = \sigma_1$ are the Euclidean $\gamma$ matrices and $k_n = \pi(2n+1)/L$. (The shift $\pi/L$ occurs due to the nonvanishing background $A_1^{(0)}$.) Alternatively, we could impose antiperiodic boundary conditions in the spatial direction. In this case the integrals (4.31) and (4.33) would pick out the value $A_1^{(0)} = 0$ with the same final result.

Doing the integrals through residues (the integration contour is closed in the upper half-plane for $\omega_1$ and in the lower half-plane for $\omega_2$) we arrive at

\[C_0(\tau_0) = \frac{1}{L^2} \left( \sum_{n=0}^{\infty} e^{-\frac{\pi \tau_0 (2n+1)}{L}} \right)^2 = \frac{1}{4L^2 \sinh^2 \frac{\pi \tau_0}{L}}. \]  
\hspace{3cm} (4.38)

The next step is calculating Green’s function $G(\tau_0)$. We have

\[G(\tau_0) = \frac{1}{L} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega \tau_0} \sum_{n=-\infty}^{\infty} \frac{1}{\lambda_{nw} (\lambda_{nw} + \mu^2)} \]
\hspace{3cm} (4.39)

where

\[\lambda_{nw} = \omega^2 + \left( \frac{2\pi}{L} \right)^2 n^2.\]

The sum in Eq. (4.39) can be calculated using the identity

\[\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a, \]  
\hspace{3cm} (4.40)

see [25], Eq. (1.217). Substituting Green’s function (4.39) obtained in this way and $C_0(\tau_0)$ from Eq. (4.38) into Eq. (4.35) we finally get

\[G(\tau_0) = \frac{1}{4L^2 \sinh^2 \frac{\pi \tau_0}{L}} \times \]
\hspace{3cm} \[\frac{1}{2} Tr \left\{ \int \frac{d\omega_1}{2\pi} e^{i\omega_1 \tau_0} \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{\omega_1 \gamma_0^E + k_n \gamma_1^E}{\omega_1^2 + k_n^2} \times \right. \]
\[\left. \int \frac{d\omega_2}{2\pi} e^{-i\omega_2 \tau_0} \frac{1}{L} \sum_{m=-\infty}^{\infty} \frac{\omega_2 \gamma_0^E + k_m \gamma_1^E}{\omega_2^2 + k_m^2} \right\} \]  
\hspace{3cm} (4.37)

where $\gamma_0^E = \sigma_2, \gamma_1^E = \sigma_1$ are the Euclidean $\gamma$ matrices and $k_n = \pi(2n+1)/L$. (The shift $\pi/L$ occurs due to the nonvanishing background $A_1^{(0)}$.) Alternatively, we could impose antiperiodic boundary conditions in the spatial direction. In this case the integrals (4.31) and (4.33) would pick out the value $A_1^{(0)} = 0$ with the same final result.

The next step is calculating Green’s function $G(\tau_0)$. We have

\[G(\tau_0) = \frac{1}{L} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega \tau_0} \sum_{n=-\infty}^{\infty} \frac{1}{\lambda_{nw} (\lambda_{nw} + \mu^2)} \]
\hspace{3cm} (4.39)

where

\[\lambda_{nw} = \omega^2 + \left( \frac{2\pi}{L} \right)^2 n^2.\]

The sum in Eq. (4.39) can be calculated using the identity

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\hspace{3cm} (4.40)

see [25], Eq. (1.217). Substituting Green’s function (4.39) obtained in this way and $C_0(\tau_0)$ from Eq. (4.38) into Eq. (4.35) we finally get

\[G(\tau_0) = \frac{1}{4L^2 \sinh^2 \frac{\pi \tau_0}{L}} \times \]
\hspace{3cm} \[\frac{1}{2} Tr \left\{ \int \frac{d\omega_1}{2\pi} e^{i\omega_1 \tau_0} \frac{1}{L} \sum_{n=-\infty}^{\infty} \frac{\omega_1 \gamma_0^E + k_n \gamma_1^E}{\omega_1^2 + k_n^2} \times \right. \]
\[\left. \int \frac{d\omega_2}{2\pi} e^{-i\omega_2 \tau_0} \frac{1}{L} \sum_{m=-\infty}^{\infty} \frac{\omega_2 \gamma_0^E + k_m \gamma_1^E}{\omega_2^2 + k_m^2} \right\} \]  
\hspace{3cm} (4.37)
The expression in the exponent has an infrared singularity at $\omega = 0$. It is not difficult to understand how it must be treated. Recalling that we actually start from torus with a very large $T$ and eliminate the zero mode of the Laplace operator we conclude that the integral in the exponent must be understood as the principal value. Then the factor $(1 - e^{i\omega \tau_0})$ can be substituted by $(1 - \cos \omega \tau_0)$ and the integral becomes well-defined.

The correlator (4.41) tends to a constant in the limit $\tau_0 \to \infty$. After some algebra it is not difficult to find

$$C(\tau_0 \to \infty) = \left(\frac{\mu}{4\pi}\right)^2 e^{(2\gamma-21)}$$

where $\gamma$ is Euler’s constant and

$$I = \int_0^{\infty} \frac{d\omega}{\sqrt{\omega^2 + \mu^2}} \left( \coth \frac{L\sqrt{\omega^2 + \mu^2}}{2} - 1 \right). \quad (4.42)$$

The cluster decomposition yields us then the value of the fermion condensate, up to a phase factor,

$$\langle \bar{\psi}_R \psi_L \rangle \frac{\mu e^\gamma}{4\pi} \epsilon^{-1}, \quad (4.43)$$

in agreement with [21, 24, 18]. The phase $\alpha$ is nothing else than the vacuum angle $\theta$, as follows from the exact Ward identities (which are not specific for two dimensions but hold also in the four-dimensional QCD).

One can be interested not only in the final result but also in the question of what particular configuration of the gauge field (4.29) saturates the path integral, or, in other words, what the stationary point of the functional integral (4.33) is. As was shown in [20] this is the instanton-antiinstanton configuration,

$$A_{\mu}^{\text{inst}}(\tau, x) = A_{\mu}^{\text{inst}}(\tau, x) - A_{\mu}^{\text{inst}}(\tau - \tau_0, x). \quad (4.44)$$

The explicit result for $A_{\mu}^{\text{inst}}(\tau, x)$ can be derived for any $L$ but the results are particularly simple in the limits $gL \gg 1$ and $gL \ll 1$. As was already mentioned (see [20] for a detailed discussion), the question is much more meaningful in the small $gL$ limit where quantum fluctuations are suppressed and the situation is quasiclassical.

In the gauge $A_0 = 0$, the field $A_{\mu}^{\text{inst}}$ does not depend on $x$ and is given by the formula

$$A_{\mu}^{\text{inst}}(\tau) = \begin{cases} \frac{\pi}{gL} \exp{\mu \tau}, & \tau \leq 0 \\ \frac{\pi}{gL}[2 - \exp{-\mu \tau}], & \tau \geq 0 \end{cases}. \quad (4.45)$$
It has essentially the same form as the fracton configuration in Eq. (2.8), but the photon mass $\mu$ is evaluated at $N = 1$ and the overall factor $1/N$ is absent. The configuration (4.45) carries the unit topological charge (1.10). It is, indeed, the instanton.

We described in such detail the functional integral calculation of the correlator (2.6) in the standard Schwinger model because its generalization to multiflavor case is straightforward.

Consider first the multiflavor model with the standard periodic boundary conditions. Again, the correlator is given by the formula (4.35). Again, the functional integrals (4.31) and (4.33) pick up the value $A_{1}^{(0)} = \frac{2\pi}{\mu L}$, and the result for the correlator in the constant background (4.38) is the same as previously. Green’s function $\mathcal{G}(\tau_{0}, 0)$ is modified a little bit, but the modification is trivial and amounts to a rescaling of $\mu$. As a result we get

$$C_{N}(\tau_{0}) = \frac{1}{4L^{2} \sinh^{2} \frac{\pi m}{T}} \times \exp \left\{ \frac{1}{N} \int_{-\infty}^{\infty} d\omega (1 - e^{i\omega \eta}) \left[ \frac{1}{\omega} \coth \frac{\omega L}{2} - \frac{1}{\sqrt{\omega^{2} + \mu^{2}}} \coth \frac{L\sqrt{\omega^{2} + \mu^{2}}}{2} \right] \right\}.$$

(4.46)

The origin of the factor $1/N$ is easy to understand if turning to Eq. (4.35) we rewrite $4g^{2}$ in the exponent as $4\pi\mu^{2}/N$.

This factor $1/N$ in the exponent has drastic consequences. In contrast to the correlator (4.41) $C_{N}(\tau_{0})$ falls off now exponentially

$$C_{N}(\tau_{0} \to \infty) \propto \exp \left\{ -\frac{2\pi \tau_{0}}{L} \left( 1 - \frac{1}{N} \right) \right\}$$

at large $T$ and the condensate is not generated.

Consider finally the twisted multiflavor case (for simplicity, we restrict ourselves to the case $N = 2$). The exponential factor in (4.35) is the same since Green’s function $G(\tau_{0})$ knows nothing about the fermion boundary conditions. What is modified is the preexponential factor $C_{0}(\tau_{0})$ because the constant background in which the correlator $C_{0}(\tau_{0})$ is evaluated is now different. The values of $A_{1}^{(0)}$ on which the path integrals (4.31) and (4.33) are concentrated coincide with the minima of the effective potential (3.15), i.e. $A_{1}^{(0)} = \pi(2n + 1)/(2gL)$. One can readily convince oneself that all these minima give identical contributions both to the partition function and the correlator (the latter property is specific to $N = 2$). Calculating the integrals over $\omega_{1}$ and $\omega_{2}$ in Eq. (4.37) we then get

$$\tilde{C}_{0}^{N}(\tau_{0}) = \frac{1}{L^{2}} \left[ \left( \sum_{n=0}^{\infty} e^{-\frac{\pi m}{2L} - \frac{2\pi m}{L}} \right)^{2} + \left( \sum_{n=0}^{\infty} e^{-\frac{3\pi m}{2L} - \frac{2\pi m}{L}} \right)^{2} \right] = \frac{\cosh \frac{\pi m}{L}}{2L^{2} \sinh \frac{\pi m}{L}}.$$

(4.47)
Substituting it in Eq. (4.35) with the same exponential as in Eq. (4.46) we see that
the correlator does tend now to a constant in the limit $\tau_0 \to \infty$ (we remind that
$N = 2$). The explicit value for the square root of this constant is

$$\langle \bar{\psi}_R \psi_L \rangle = \langle \bar{\psi}_L \psi_R \rangle^* = e^{-i\alpha} \sqrt{\frac{\mu e^{-\gamma}}{16\pi L}} e^{-1/2}. \quad (4.48)$$

The small $gL$ asymptotics of this expression coincides with the result (3.25) derived
in the previous section. The phase $\alpha$ coincides with the vacuum angle $\bar{\theta} = \theta/2$ as
required by the Ward identities.

At large $gL$ the condensate behaves as

$$\langle \bar{\psi}_R \psi_L \rangle = \langle \bar{\psi}_L \psi_R \rangle^* = e^{-i\bar{\theta}} \sqrt{\frac{\mu e^{-\gamma}}{16\pi L}}, \quad (4.49)$$

and disappears in the limit $L \to \infty$. In is instructive to confront the result (4.49)
with the behavior $\propto 1/\tau_0$ of the correlator (2.6) in the infinite volume limit as follows
from the result (2.5). In the twisted model with finite $L$ this falloff is frozen when
$\tau_0$ reaches $L$. In the model with the standard boundary conditions the correlator
continues to fall off exponentially in the region $\tau_0 \gg L$.

5 Fracton path integral

This is, perhaps, the central section of the paper. We derive the result (4.48) directly
by calculating the functional integral for the partition function in the fracton sector
with the topological charge $\nu = \pm 1/2$ (for simplicity, we restrict ourselves to the
two-flavor case).

In order to do this we must, however, first define what this functional integral
actually means. The standard definition

$$Z_{\nu = 1/2} = \int \prod_{\tau, \bar{\tau}, \mu} dA_\mu(\tau, x) \det(iD - m) \det(iD - m) \exp \left\{ -\frac{1}{4} \int d^2 x F_{\mu\nu}^2 \right\} \quad (5.1)$$

is not suitable here because the Dirac operator for an individual fermion $\psi$ or $\chi$
is not defined on a compact manifold with the background carrying a fractional
topological charge – self-consistent solutions for the eigenvalue Dirac equation are
absent.

Still, in the case of the twisted boundary conditions (3.14) the configurations
with $\nu = \pm 1/2, \pm 3/2, \ldots$ do contribute to the partition function. To understand this
we have to go back to basics and recall how topologically nontrivial path integrals
appear in a field theory in the first place.
5.1 Path integral in the standard Schwinger model

In a simple-minded theory with no gauge fields and no topological effects (say, in the Yukawa theory in four dimensions), the vacuum partition function is defined as

\[
Z = \lim_{T \to -\infty} \sum_n e^{-TE_n} = \lim_{T \to -\infty} \int \prod_x d\phi(x) d\psi(x) d\psi^+(x) \exp\{-\psi^+(x) \psi(x)\} \times K[\phi(x), \psi(x); \phi(x), \psi^+(x); T]
\]

(5.2)

where

\[
K[\phi(x), \psi(x); \phi(x), \psi^+(x); T] = \sum_n \Phi_n(\psi(x), \phi(x)) (\Phi_n(\psi(x), \phi(x)))^* e^{-TE_n}
\]

(5.3)

is the Euclidean evolution operator. The expression (5.2) can be written as a path integral

\[
Z = \lim_{T \to -\infty} \int \prod_{\tau,x} d\phi(\tau, x) d\psi(\tau, x) d\bar{\psi}(\tau, x) \times \\
\exp\left\{- \int_0^T d\tau \int dx \mathcal{L}[\phi(\tau, x), \bar{\psi}(\tau, x), \psi(\tau, x)]\right\}
\]

(5.4)

with the boundary conditions in the Euclidean time \(^6\)

\[
\phi(\tau + T, x) = \phi(\tau, x), \\
\psi(\tau + T, x) = -\psi(\tau, x),
\]

(5.5)

(and if the theory is defined on a spatial box some boundary conditions in spatial directions must be imposed as well).

In gauge theories one should take into account in the sum (5.2) only the physical states annihilated by the constraints. The trace of the constrained evolution operator can also be written as a path integral involving additional integrations over the gauge transformation parameters. Let us do it for the standard Schwinger model defined on torus,

\[
Z^{SSM} = \lim_{T \to -\infty} \int \prod_x d\lambda(x) dA_1(x) d\psi(x) d\psi^+(x) \exp\{-\psi^+(x) \psi(x)\} \times \\
\int \prod_{\tau,x} d\phi(\tau, x) d\psi(\tau, x) d\bar{\psi}(\tau, x) \times \\
\exp\left\{- \int_0^T d\tau \int dx \mathcal{L}[\phi(\tau, x), \bar{\psi}(\tau, x), \psi(\tau, x)]\right\}
\]

\(^6\)Then \(T\) can be treated as inverse temperature. The fact that the boundary conditions to be imposed on the fermion fields should be antiperiodic in the Euclidean time is widely known, but its accurate proof involves some intricacies. A very good pedagogical derivation can be found in Ref. [26].
where $\lambda(x)$ is a periodic gauge transformation function continuously deformable to zero (the integration over such functions takes complete account of the constraint (3.2), the Gauss law); the superscript $(\lambda)$ marks the gauge-transformed quantities, the superscript SSM refers to the standard Schwinger model.

It is rather obvious, however, that (5.6) is not the full partition function of the theory but only a part of it corresponding to the topologically trivial gauge field configurations. Path integrals calculated with the prescription (5.6) do not satisfy the cluster decomposition property. For example, the fermion condensate in the topologically trivial sector is zero, but the $\tau_0 \rightarrow \infty$ limit of the correlator (2.6) is not.

To write the physical partition function, one should impose the superselection rule on top of the Gauss law constraint (3.2). The correct partition function in the sector with the given vacuum angle is given by

$$Z_{SSM} = \lim_{T \rightarrow \infty} e^{i\theta} \sum_{n} \exp \left\{ -\psi(x)\lambda(x) \right\} \times$$

$$K[A_0(x), \psi(x); A_1^{(\lambda)}(x), \psi^{(\lambda)}(x); T]$$

(5.7)

where the global symmetry transformation $S$ is defined in (3.7). Let us take a closer look, say, at the term with $n = 1$. After trading off the integral over $\lambda(x)$ for the integral over $A_0(x)$ (the variable canonically conjugated to the Gauss law constraint) and presenting the evolution operator as a path integral over the Euclidean time we get

$$Z_1^{SSM} = e^{-i\theta} \lim_{T \rightarrow \infty} \int \prod_{\tau,x,\mu} dA_0(x) d\bar{\psi}(x) d\psi(x) \exp \left\{ -S_E^{SM}(A_\mu, \bar{\psi}, \psi) \right\}$$

(5.8)

where the integration goes over the fields satisfying the boundary conditions

$$A_0(\tau + T, x) = A_0(\tau, x),$$

$$A_1(\tau + T, x) = A_1(\tau, x) + \frac{2\pi}{gL},$$

$$\psi(\tau + T, x) = -e^{2\pi i x/L} \psi(\tau, x),$$

(5.9)

(plus periodicity for all fields in the spatial direction). The boundary conditions (5.9) describe the field configurations with the unit topological charge, the instantons $^7$.

Likewise, the term $n = 2$ in the sum in (5.7) corresponds to Euclidean configurations with the double topological charge, etc.

$^7$We hope that the reader will not be confused by a little bit different description of the instanton configurations in [24, 20] where the trivial boundary conditions (periodic for bosons and antiperiodic for fermions) in the Euclidean time direction were chosen along with the nontrivial boundary conditions in the spatial direction.
5.2 Twisted model

The partition function for the twisted model in the sector with a given vacuum angle \( \hat{\theta} \) entering the superselection rule (3.20) is given by the sum

\[
Z = \lim_{T \to \infty} \sum_n e^{-i n \hat{\theta}} \int \prod_x d\lambda(x) dA_1(x) d\psi(x) d\psi^+(x) \times \\
\exp\{-\psi^+(x)\psi(x)\} d\chi(x) d\chi^+(x) \exp\{-\chi^+(x)\chi(x)\} \times \\
\mathcal{K}[A_1(x), \psi(x), \chi(x); \tilde{\xi}^n A_1^{(1)}(x), \tilde{\xi}^n \psi^{(+)}(x), \tilde{\xi}^n \chi^{(+)}(x); T].
\]

This expression differs from Eq. (5.7) by an extra pair of the fermion variables and, what is more essential, by the substitution \( \theta \to \hat{\theta}, \quad S \to \tilde{S} \). Consider again the term with \( n = 1 \). It can be presented in the path integral form

\[
Z_{1/2}^{TSM} = e^{-i \hat{\theta}} \lim_{T \to \infty} \int \prod_{\tau,x,\mu} dA_\mu(x) d\tilde{\psi}(x) d\psi(x) d\tilde{\chi}(x) d\chi(x) \times \\
\exp\{-S_{E}^{TSM}(A_\mu, \tilde{\psi}, \psi)\tilde{\chi}, \chi\}
\]

where the integration goes over the fields satisfying the boundary conditions

\[
A_0(\tau + T, x) = A_0(\tau, x), \\
A_1(\tau + T, x) = A_1(\tau, x) + \frac{\pi}{gL},
\]

\[
\psi(\tau + T, x) = -e^{\pi i z/L} \chi(\tau, x),
\]

\[
\chi(\tau + T, x) = -e^{\pi i z/L} \psi(\tau, x),
\]

and the superscript TSM refers to the twisted Schwinger model. The shift in \( A_1 \) is twice smaller compared to the instanton case (5.9), and the topological charge of the gauge field is also twice smaller. It is a fracton field configuration (thereby the subscript 1/2 in Eq. (5.11)).

We understand now how the difficulty of defining the Dirac operator in the fracton background is resolved. We see that here the two fermion fields are coupled to each other by the boundary conditions and, though we cannot formulate the eigenvalue problem for an individual fermion, the coupled-channels eigenvalue problem

\[
\begin{cases}
\mathcal{D} \psi_n = \mu_n \psi_n, \\
\mathcal{D} \chi_n = \mu_n \chi_n
\end{cases}
\]
with eigenfunctions $\psi_n(\tau, x)$, $\chi_n(\tau, x)$ satisfying the boundary conditions (5.12) is perfectly well-defined. Hence, the path integral (5.11) is well-defined too. The only remaining problem is to calculate it.

We are interested, eventually, in the fermion condensate $\langle \bar{\psi}_R \psi_L \rangle$ which in the limit of very small mass is given by the ratio

$$\langle \bar{\psi}_R \psi_L \rangle = - \lim_{m \to 0} \frac{Z_{1/2}}{m Z_0}. \quad (5.14)$$

It is worth reminding that the condensate is the logarithmic derivative with respect to the mass of the full partition function as in Eq. (1.6), but the higher-$n$ terms in the sum (5.10) involve the suppression factor $m^{k_l}$ and do not contribute in the limit $m \to 0$.

Let us first carefully calculate the partition function in the topologically trivial sector following the technique developed in Ref. [24]. In the sector $\nu = 0$ there is no problem whatsoever with calculating the integrals over $d\bar{\psi}d\psi$ and over $d\bar{\chi}d\chi$ separately, and we get

$$Z_0^{TSM} = \mathcal{N} \int_0^1 \int_0^1 dh_0 dh_1 \times$$

$$\int \prod_{\tau, x} d\phi(\tau, x) \exp \left\{ -\frac{1}{2} \int d^2 x \phi \Delta^2 \phi \right\} \text{det}(i\mathcal{D}) \text{det}(i\mathcal{D})$$

where $h_{0,1}$ and $\phi(\tau, x)$ are the gauge invariant variables characterizing the field $A_\mu(\tau, x)$ as written in Eq. (4.29) and $\mathcal{N}$ is an unspecified normalization factor. Notice that the mass term can be set equal to zero here. The determinants of the Dirac operator on torus involve an intricate dependence on the constant harmonics $h_0, h_1$ through the Jacobi $\Theta$ function; in the limit of very large $T$ the results are greatly simplified, however. We have

$$\text{det}(i\mathcal{D}) \approx e^{\pi T/6L} \exp \left\{ -\frac{2\pi T}{L} \left( h_1 - \frac{1}{2} \right)^2 \right\} \exp \left\{ \frac{g^2}{2\pi} \int d^2 x \phi \Delta \phi \right\},$$

$$\text{det}(i\mathcal{D}) \approx e^{\pi T/6L} \left[ \exp \left\{ -\frac{2\pi T}{L} h_1^2 \right\} + \exp \left\{ -\frac{2\pi T}{L} (h_1 - 1)^2 \right\} \right] \times$$

$$\exp \left\{ \frac{g^2}{2\pi} \int d^2 x \phi \Delta \phi \right\}. \quad (5.16)$$

$(T \gg L, \ 0 \leq h_1 \leq 1)$. Substituting (5.16) in (5.15) and integrating over $h_1$, we get

$$Z_0^{TSM} = \frac{\mathcal{N}}{\sqrt{T/L}} e^{\pi T/12L} \int \prod_{\tau, x} d\phi(\tau, x) \exp \left\{ -\frac{1}{2} \int d^2 x \phi (\Delta^2 - \mu_2^2 \Delta) \phi \right\}. \quad (5.17)$$

Now, let us find the fracton contribution to the partition function. The result of integration over the fermion variables in (5.11) is the determinant of the matrix
Dirac operator (5.13) (i.e. the product of all its eigenvalues $\mu_n$). Fortunately, we need not calculate it anew but may use again the results of Ref. [24] exploiting the following trick. Consider the function
\[
\Psi(\tau, x) = \psi(\tau, x) + \chi(\tau, x). \tag{5.18}
\]
This function does not satisfy definite boundary conditions on the original torus $(T, L)$ but it does satisfy the periodic boundary conditions on the twice “thicker” torus $(T, 2L)$
\[
\Psi(\tau, x + 2L) = \Psi(\tau, x)
\]
(we will call it “large” torus). The fields $\psi(\tau, x)$ and $\chi(\tau, x)$ may be considered as the sum of all even and, correspondingly, all odd Fourier spatial harmonics of $\Psi(\tau, x)$ defined on the box $(T, 2L)$.

Consider now the Dirac operator for the field $\Psi$ on the large torus in the gauge field background which consists of two copies: the field $A_\mu(\tau, x)$ in the interval $L \leq x \leq 2L$ coincides identically with the field in the interval $0 \leq x \leq L$.

It is not difficult to see now that the eigenvalues of the Dirac operator thus defined coincide identically with the eigenvalues of the original matrix Dirac operator in Eq. (5.13) [the simplest way to establish the equivalence of the two fermion path integrals is to write them in terms of Fourier harmonics of the fields $\psi(\tau, x)$, $\chi(\tau, x)$, and $\Psi(\tau, x)$].

The field $A_\mu(\tau, x)$ defined on the large torus carries the topological charge $2 \times \frac{1}{2} = 1$. Thus, the fermion path integral in (5.11) coincides with the determinant of the Dirac operator in the instanton background. The latter has been calculated in [24]. The result is
\[
\det(i\mathcal{D} - m) = m\sqrt{LT} \int \frac{1}{2LT} \int d^2 x e^{-2g\phi(\tau, x)} \exp \left\{ \frac{g^2}{2\pi} \int d^2 x \phi \Delta \phi \right\} \tag{5.19}
\]
where the limits of integration are $0 \leq \tau \leq T$, $0 \leq x \leq 2L$ and the field $\phi$ satisfies the condition $\phi(\tau, x + L) = \phi(\tau, x)$.

The factor $m$ in Eq. (5.19) comes from one complex zero mode of the field $\Psi$ in the instanton background (and, correspondingly, one zero mode for the fields $\psi(\tau, x)$, $\chi(\tau, x)$ in a fracton background). The product of all nonzero eigenmodes is not sensitive to $m$ if it is small enough and just coincides with Eq. (3.9) of Ref. [24]. Substituting the result (5.19) in the path integral for $Z_{1/2}$ and substituting
\[
\frac{1}{2LT} \int d^2 x e^{-2g\phi(\tau, x)} \rightarrow e^{-2g\phi(0)}
\]
(we can do this due to the translational invariance of the path integral), we get
\[
Z_{1/2} = e^{-il} \mathcal{N} m \sqrt{LT} \int \prod_{\tau, x} d\phi(\tau, x) e^{-2g\phi(0)} \exp \left\{ -\frac{1}{2} \int d^2 x (\Delta^2 - \mu^2 \Delta) \phi \right\}. \tag{5.20}
\]
Here the integral runs over the original torus and the factor 2 in (5.19) is taken into account in the definition of $\mu_0^2 = 2g^2/\pi$. The normalization factor $\mathcal{N}$ is the same as in Eq. (5.17). Substituting Eqs. (5.20) and (5.17) in Eq. (5.14) we arrive at the following expression for the fracton contribution in the fermion condensate:

$$\langle \bar{\psi}_R \psi_L \rangle = -e^{i\theta} \frac{1}{L} e^{-\pi T/12L} e^{i\theta^2 G(0)}$$

(5.21)

where $G$ is the Green’s function of the operator $\Delta^2 - \mu_0^2 \Delta$ defined in Eq. (4.39) with $\tau_0$ set equal to zero. Performing the sums by means of Eq. (4.40) we reproduce the result (4.48) obtained in the previous section.

6 Conclusions

It is known for a long time that in the standard Schwinger model on a circle the topology of the functional space is nontrivial. The functional space of the gauge fields contains a non-contractible path, $S_1$. Since the configurational space is also $S_1$ the path integral is decomposed in a sum corresponding to different winding numbers in the mapping $S_1 \rightarrow S_1$. The size of the non-contractible path (i.e. what points are to be identified) is determined by the large gauge transformation.

In this work we suggested a model (the twisted Schwinger model on a circle) which exhibits a remarkable feature. The non-contractible path is still there, but the identification of points in the functional space is decided by a transformation of fields that is not pure gauge. Thus, in the two-flavor twisted model the original circle of the standard Schwinger model is further glued in such a way that two points lying on one diameter are identified. It seems quite plausible that this lesson may be of a paramount importance for the non-abelian gauge theories, in solving such long-standing problems as, say, the problem of Witten’s index in $O(N)$ SUSY Yang-Mills theories.

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Figure captions

Fig. 1. Fracton-antifracton configuration.

Fig. 2. Fermion levels in the standard Schwinger model.

Fig. 3. Fermion levels in the twisted two-flavor Schwinger model.
References


