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**REAL PLANCK DISTRIBUTION FOR A
COMPLEX Q-BOSON GAS**

REAL PLANCK DISTRIBUTION FOR A COMPLEX q -BOSON GAS^($\diamond$)

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Abstract – We discuss the energy density distributions for a gas of q -bosons with complex deformation parameter, by exploiting a new q -complex oscillator algebra in which both the number operator and the energy eigenvalues are real. The corresponding Planck distribution generalizes the results obtained for a real q -boson gas, including Wien's and Stephan's laws, and the role of an effective Planck constant depending on the parameter modulus.

One of the most interesting issues in the theory of quantum deformations of Lie algebras and groups [1] is to study the physical implications of the q -structures. For instance, the statistical properties of a gas of q -boson oscillators have been investigated by some authors [2-4]. In particular, recently Gupta et al [5] have studied the q -deformation of the Planck energy density distribution of a boson gas when the parameter q is complex, i.e. $q = q_1 + iq_2$, on the basis of references [6-8].

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Their starting point was the usual expression of the energy eigenvalues $E(q, n)$ for a q -harmonic oscillator, with zero point energy subtracted, i.e. [9]

$$E(q, n) = \frac{1}{2} \hbar \omega ([n] + [n+1] - 1). \quad (1)$$

The square bracket in (1) introduces the parameter q (or s) of the $SU_q(2)$ algebra according to the definition [5]

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{e^{sx} - e^{-sx}}{e^s - e^{-s}} = \frac{\sinh sx}{\sinh s} \quad (2)$$

where $q = e^s$, $s = a + ib$ and a, b are real numbers. From the above definition we see that when q is complex also the energy eigenvalues become complex. However, in our opinion the results obtained in ref.[5] by trivially generalizing eq.(1) to the complex case have no meaning, because the energy eigenvalue's form (1) is no longer valid for complex q .

Indeed, De Falco et al. [10] have recently investigated the q -boson algebra with complex deformation parameter, by using the new commutation relations:

$$AB - qBA = F(q, N, \dots) \quad (3)$$

and

$$B^+ A^+ - q^* A^+ B^+ = F^*(q^*, N, \dots) \quad (4)$$

where, A, B are non-Hermitian operators, $q \in \mathbb{C}$ ($\mathbb{C}$ complex field), and N is the usual number operator ($N|n\rangle = n|n\rangle$).

For the case where q is complex we have $B \neq A^+$. By using the bosonization method (originally introduced by Jannussis and co-workers [11], for the special case $F(q, N, \dots) = q^{-N}$ [10], we obtain the following representation

$$A = \left(\frac{q^{(N+1)} - q^{-(N+1)}}{q - q^{-1}} \cdot \frac{1}{N+1} \right)^{1/2} \cdot a = \left(\frac{[N+1]}{N+1} \right)^{1/2} a; \quad (5)$$

$$B = a^+ \left(\frac{[N+1]}{N+1} \right)^{1/2}, \quad (6)$$

where we used as usual the symbol

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (7)$$

The new generalized number operator N is defined in ref.[10] and has the form

$$N = A^+ A = B B^+ = ([N][N]^*)^{1/2}, \quad (8)$$

with action on the kets $|n\rangle$ given by

$$\begin{aligned} N|n\rangle &= \left(\frac{q^n - q^{-n}}{q - q^{-1}} \cdot \frac{q^{*n} - q^{*-n}}{q^* - q^{*-1}} \right)^{1/2} |n\rangle = \\ &= \left(\frac{|q|^{2n} + |q|^{-2n} - 2 \cos(2n\theta)}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} |n\rangle = \end{aligned} \quad (9)$$

where

$$q = |q|e^{i\theta} = q_1 + iq_2, \quad \theta = \arctan \frac{q_2}{q_1}. \quad (10)$$

In the usual Fock representation of the Hamiltonian operator $H = \frac{p^2}{2m} + \frac{m}{2} \omega^2 x^2 = \frac{\hbar\omega}{2} (AA^+ + A^+A)$, we get the following deformed spectrum

$$\begin{aligned} E_n &= \frac{\hbar\omega}{2} (|n+1| + |n|) \\ &= \frac{\hbar\omega}{2} \left[\left(\frac{|q|^{2(n+1)} + |q|^{-2(n+1)} - 2 \cos[2\theta(n+1)]}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} + \right. \\ &\quad \left. + \left(\frac{|q|^{2n} + |q|^{-2n} - 2 \cos(2\theta n)}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} \right] \end{aligned} \quad (11)$$

i.e. the energy eigenvalues are real and not complex as in ref.[5]. Nevertheless, the spectrum (11) maintains a remarkable richness of possible different cases, which are interesting from the theoretical point of view and deserve experimental verification. For instance, in the case $|q| = 1$, eq.(11) takes the form

$$E_n = \frac{\hbar\omega}{2} \frac{\sin(n+1)\theta + \sin(n\theta)}{\sin \theta} = \frac{\hbar\omega}{2} \frac{\sin(n+1/2)\theta}{\sin \theta/2}. \quad (12)$$

Let us stress again that both the new number operator (9) and the energy spectrum (11) are real and not complex as it happens in ref.[5]. The substitution of the real parameter q with complex q in the usual deformed harmonic oscillator is not allowed, in our opinion.

We want now to apply our results to getting the Planck distribution for a complex q -boson gas, thus generalizing the Martin-Delgado's findings [3].

The energy eigenvalues (11) for a q -complex harmonic oscillator, with zero point energy subtraction, are

$$\begin{aligned} \mathcal{E}(q, n) &= \frac{\hbar\omega}{2} \left[\left(\frac{|q|^{2(n+1)} + |q|^{-2(n+1)} - 2 \cos 2\theta(n+1)}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} + \right. \\ &\quad \left. + \left(\frac{|q|^{2n} + |q|^{-2n} - 2 \cos(2\theta n)}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} - 1 \right] \end{aligned} \quad (13)$$

The partition function $Z_q(\omega, T)$ for a q -boson gas in the canonical ensemble is

$$Z_q(\omega, T) = \sum_{n=0}^{\infty} e^{-\beta \mathcal{E}(n, q)}, \quad \beta = \frac{1}{K_B T}. \quad (14)$$

Then, the average energy density for this gas is defined in the usual way

$$\begin{aligned} U_{|q|}(\omega, T) &= \frac{8\pi}{c^3} \left(\frac{\omega}{2\pi} \right)^2 \left[-\frac{\partial}{\partial \beta} \ln Z_q(\omega, T) \right] = \\ &= \frac{8\pi}{c^3} \left(\frac{\omega}{2\pi} \right)^2 \frac{\sum_{n=0}^{\infty} \mathcal{E}(n, q) e^{-\beta \mathcal{E}(n, q)}}{\sum_{n=0}^{\infty} e^{-\beta \mathcal{E}(n, q)}}. \end{aligned} \quad (15)$$

For the case $q_2 = 0$ and $q_1 = 1$ we recover the classical Planck law

$$U_1(\omega, T) = \frac{8\pi}{c^3} \left(\frac{\omega}{2\pi} \right)^2 K_B T \frac{\beta \hbar \omega}{e^{\beta \hbar \omega} - 1}. \quad (16)$$

Furthermore, for $|q| \neq 1$, in dimensionless notation (with $x = \frac{\hbar \omega}{K_B T}$) we get

$$\begin{aligned} Z_q(x) &= \sum_{n=0}^{\infty} \exp \left\{ -\frac{1}{2} x \left[\left(\frac{|q|^{2(n+1)} + |q|^{-2(n+1)} - 2 \cos 2\theta(n+1)}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} + \right. \right. \\ &\quad \left. \left. + \left(\frac{|q|^{2n} + |q|^{-2n} - 2 \cos 2\theta n}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} - 1 \right] \right\}; \end{aligned} \quad (17)$$

$$u_q(x) = x^3 \left[-\frac{d}{dx} \ln Z_q(x) \right] = x^3 \frac{\sum_{n=0}^{\infty} \sigma(q, n) e^{-\frac{1}{2} x \sigma(q, n)}}{\sum_{n=0}^{\infty} e^{-\frac{1}{2} x \sigma(q, n)}}; \quad (18)$$

where

$$\begin{aligned} \sigma(q, n) &= \left(\frac{|q|^{2(n+1)} + |q|^{-2(n+1)} - 2 \cos 2\theta(n+1)}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} + \\ &\quad + \left(\frac{|q|^{2n} + |q|^{-2n} - 2 \cos 2\theta n}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2} - 1 \end{aligned} \quad (19)$$

Notice that for the above quantities the symmetry $|q| \rightarrow |q|^{-1}$ is valid as it happens for q real.

We do not discuss the convergence of the series Z_q , inasmuch this subject has already been studied by Martin-Delgado [3] with the result that the convergence is ensured for positive q (i.e. in our case the convergence is even stronger).

In the limiting case of high energies or low temperatures, i.e. $x = \frac{\hbar \omega}{K_B T} \gg 1$, the q -partition function $Z_q(x)$ can be well approximated by the first two terms of series (17):

$$Z_q(x) \simeq 1 + e^{-\frac{1}{2} x \left(\frac{|q|^4 + |q|^{-4} - 2 \cos 4\theta}{|q|^2 + |q|^{-2} - 2 \cos 2\theta} \right)^{1/2}} \simeq 1 + e^{-\frac{1}{2} x (|q| + |q|^{-1}) \left(1 - \frac{4 \sin^2 \theta}{(|q| + |q|^{-1})^2} \right)^{1/2}}$$

or

$$Z_q(x) \simeq 1 + e^{-\frac{x}{2} (|q| + |q|^{-1}) \left(1 - 2 \frac{\sin^2 \theta}{(|q| + |q|^{-1})^2} \right)} \quad (20)$$

and for $|q| = e^\tau$ we obtain

$$Z_q(x) \simeq 1 + e^{-x \cosh \tau \left(1 - \frac{\sin^2 \theta}{2 \cosh^2 \tau} \right)} \simeq 1 + e^{-x \cosh \tau A} \equiv Z_{|q|}^w(x) \quad (21)$$

where

$$A = 1 - \frac{\sin^2 \theta}{2 \cosh^2 \tau}. \quad (22)$$

Inserting in (15) the above approximation we obtain the $|q|$ -extension of Wien law for the energy density $u_{|q|}^w(x)$

$$u_{|q|}^w(x) \simeq x^3 \cosh \tau A e^{-x \cosh \tau A}. \quad (23)$$

For q real, i.e. $\theta = 0$, we recover exactly the results of Martin-Delgado [3]. Moreover, from eq.(23) it is easy to obtain the following extension of the Wien shift law for complex q

$$x_{max} = \frac{3}{\cosh \tau \left(1 - \frac{\sin^2 \theta}{2 \cosh^2 \tau} \right)} \quad (24)$$

namely

$$\omega_{max} = \frac{3}{\hbar \cosh \tau \left(1 - \frac{\sin^2 \theta}{2 \cosh^2 \tau} \right)} K_B T \quad (25)$$

which increases for q complex in analogy with the case of real q [3].

The $|q|$ -Stefan law can be readily obtained by integrating the Wien approximation (23) of the energy density, thus getting

$$U_\tau^w(x) = \frac{6K_B^4}{\pi^2 (\hbar \cosh \tau A c)^3} T^4 \quad (26)$$

From (25) we see that the effect of the deformation in this approximation is easily understood by introducing an effective Planck constant $\hbar_\tau(\theta)$ as follows:

$$\hbar_\tau(\theta) = \hbar \cosh \tau \left(1 - \frac{\sin^2 \theta}{2 \cosh^2 \tau} \right). \quad (27)$$

The above effective Planck constant for $\theta = 0$, i.e. q real, coincides exactly with that given by Martin-Delgado [3], i.e.

$$\hbar_\tau \rightarrow \hbar \cosh \tau. \quad (28)$$

Summarizing, we have studied the Planck distribution for a complex q -boson gas by using a new q -boson oscillator algebra with complex deformation parameter [10].

The main advantage of such an algebra is that it allows us to get real (not complex [5]) number operator and energy eigenvalues for the deformed harmonic oscillator. We obtain generalized results for the q -complex Planck distribution, which for real q coincide with those already given for a gas of q -bosons [3]. This constitutes a first application of the new q -complex boson algebra introduced in [10]. Other applications will be given elsewhere.

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