Kleinian Signature Change

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Abstract

We study the propagation of plane-waves in a fixed flat background spacetime whose metric changes from a Lorentzian signature $(+ - - -)$ to a Kleinian signature $(+ + - -)$. When the change of signature is discontinuous there is an apparent difference in wave behaviour depending upon whether the wave describes matter with integer or half-integer spin. A more detailed study of the Dirac equation shows that, while it is possible to decrease this disparity, the two matter classes still do not behave identically. A more consistent picture finally emerges when the same problem is attempted with a metric which changes signature continuously. This leads to the prediction of some surprising physical phenomena.

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1 Introduction

The idea that the signature of the metric of spacetime may have changed from Euclidean to Lorentzian in the early universe, developed from the well known 'no boundary' proposal of Hartle and Hawking [1]. The recent revival of interest in signature change phenomena is due to Dray et al. [2, 3] who studied the massless scalar field, and Ellis et al. [4] who looked for cosmological solutions of the classical Einstein field equations in this context. Hayward [5] clarified much of this work, in particular, he re-emphasised a point made by Gibbons and Hartle [6] that, in order to avoid distributional contributions to the field equations, the second fundamental form of the surface at which the signature changes must be zero. There is some argument as to whether discontinuous or continuous changes in signature provide a better description of the effect, indeed both descriptions suffer from some problems [4]. In this paper, both continuous and discontinuous signature change are considered.

A metric on a 4-manifold M will be said to be of *Kleinian signature* if it has signature $(+ + - -)$. If one accepts that it is possible for the signature of the metric of spacetime to change from Euclidean to Lorentzian, then one must also accept that it is possible for the signature of the metric of spacetime to change from Lorentzian to Kleinian. Indeed, this idea of more general signature transitions was postulated by Sakharov [7].

If particles are free to move between Lorentzian and Kleinian regions then, due to the structure of the Kleinian light cone, dramatic physical effects would be observable. For example, it may be possible to use a Kleinian region for time travel (with respect to time in the Lorentzian region). It may also be possible to extract energy from a Kleinian region using a method similar to that for extracting energy from rotating black holes. If this latter effect is possible, then the amount of energy that could be extracted is unlimited.

In this paper, we examine solutions of the wave equations for various matter types on a fixed flat background spacetime which changes from a Lorentzian signature $(+ - - -)$ to a Kleinian signature $(+ + - -)$. In section 2 we solve the wave equation for the scalar, Dirac, and electromagnetic fields, using the discontinuous description of signature change. The results appear

to be dependent on whether the matter has integer or half-integer spin. In section 3 we examine the Dirac equation in further detail to try to resolve this apparent discrepancy. In section 4 we solve the wave equation for the scalar and Dirac fields using the continuous description of signature change. These final results suggest a resolution of the discrepancy found in the discontinuous case, but at the same time predict some surprising physical phenomena.

2 Discontinuous Signature Change

Here we solve the wave equations for different matter types propagating on a background spacetime with metric given by

$$ds^2 = g_{ab} dx^a dx^b = \begin{cases} dt^2 - dx^2 - dy^2 - dz^2 & z < 0 \\ dt^2 - dx^2 - dy^2 + dz^2 & z > 0. \end{cases} \quad (2.1)$$

Thus the background spacetime is flat with discontinuous signature change from Lorentzian to Kleinian across the surface $z = 0$ (see figure 1). The exact nature of the surface $z = 0$ is not specified as it is unimportant. However, since the change of signature is discontinuous, it need not be the case that the metric is degenerate at $z = 0$.

For $z < 0$ we take the standard plane wave solutions, and for $z > 0$ we seek solutions which decay exponentially in z as $z \rightarrow \infty$. Let ϕ be a linear combination of such solutions, then at $z = 0$ we impose the following natural boundary condition:

- (1) ϕ is continuous.

If the wave equation for ϕ is second order then we impose the additional boundary condition:

- (2) $\frac{\partial \phi}{\partial z}$ is continuous.

The scalar, Dirac, and electromagnetic fields are treated in this manner.

2.1 Scalar Field

The wave equation for a massive scalar field ϕ is the Klein-Gordon equation

$$(\square + m^2)\phi = 0 \quad (2.2)$$

where $\square = g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b}$. Thus for $z < 0$, $\square$ is the standard d'Alembertian operator. ϕ has position $\mathbf{x} = (x, y, z)$ and momentum $\mathbf{p} = (p_x, p_y, p_z)$. Throughout this paper we will use the following notation:

$$\mathbf{y} = (x, y), \quad (2.3)$$

$$\mathbf{k} = (p_x, p_y), \quad (2.4)$$

$$p = p_z. \quad (2.5)$$

Solving (2.2) for $z < 0$ we find

$$\phi = e^{i(\mathbf{k} \cdot \mathbf{y} + p z - \omega t)} \quad (2.6)$$

where $\omega^2 = \mathbf{k}^2 + p^2 + m^2$. This is the standard plane-wave solution.

For $z > 0$ we require solutions of (2.2) that decay as $z \rightarrow \infty$. We find

$$\phi = e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t) - p z} \quad (2.7)$$

where, once again, $\omega^2 = \mathbf{k}^2 + p^2 + m^2$.

Set

$$\phi = e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} a e^{i p z} + b e^{-i p z} & z < 0 \\ c e^{-p z} & z > 0 \end{cases} \quad (2.8)$$

where $p > 0$.

Applying the boundary conditions discussed above we find that $b = -ia$ and $c = (1 - i)a$, and since

$$\left| \frac{\text{Incident Amplitude}}{\text{Reflected Amplitude}} \right| = \left| \frac{a}{b} \right| = 1 \quad (2.9)$$

we see that the scalar field is completely reflected at $z = 0$.

This result is typical of wave equation solutions with integer spin. Upon encountering a region of discontinuous Kleinian signature change, the incoming wave is totally reflected with a phase shift (see figure 2).

2.2 Dirac Field

The metric of spacetime g_{ab} is given by (2.1). For $z < 0$ we define the usual gamma matrices γ_L^a by $\{\gamma_L^a, \gamma_L^b\} = 2g^{ab}$, and similarly, for $z > 0$ we define the Kleinian gamma matrices γ_K^a by $\{\gamma_K^a, \gamma_K^b\} = 2g^{ab}$.

We wish to solve the following (Dirac) equations

$$(\gamma_L^a \partial_a + im)\Psi = 0 \quad z < 0, \quad (2.10)$$

$$(\gamma_K^a \partial_a + im)\Psi = 0 \quad z > 0. \quad (2.11)$$

Choosing $\gamma_K^0 = \gamma_L^0$, $\gamma_K^1 = \gamma_L^1$, and $\gamma_K^2 = \gamma_L^2$, there are two cases given by $\gamma_K^3 = \pm i\gamma_L^3$.

Case 1: $\gamma_K^3 = -i\gamma_L^3$

Let the Dirac field Ψ have the form

$$\Psi = e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \chi(z), \quad (2.12)$$

then for $z < 0$ we find using (2.10) that

$$\frac{\partial \chi}{\partial z} = -i\gamma_L^3(\gamma_L^0 \omega - \gamma_L \cdot \mathbf{k} - m)\chi \quad (2.13)$$

where $\gamma_L \cdot \mathbf{k} = \gamma_L^1 p_x + \gamma_L^2 p_y$.

Let $\chi = V_+ e^{ipz}$ then, on defining the operator

$$\mathcal{D} = -\gamma_L^3(\gamma_L^0 \omega - \gamma_L \cdot \mathbf{k} - m), \quad (2.14)$$

equation (2.13) gives the eigenvector equation

$$\mathcal{D}V_+ = pV_+. \quad (2.15)$$

Similarly, if $\chi = V_- e^{-ipz}$ we get

$$\mathcal{D}V_- = -pV_-. \quad (2.16)$$

For $z > 0$ we find using (2.11) that

$$\frac{\partial \chi}{\partial z} = \gamma_L^3(\gamma_L^0 \omega - \gamma_L \cdot \mathbf{k} - m)\chi. \quad (2.17)$$

Let $\chi = V e^{-pz}$ then (2.17) gives the eigenvector equation

$$\mathcal{D}V = pV. \quad (2.18)$$

Comparing (2.15), (2.16), and (2.18) we conclude that V_+ and V_- are linearly independent, and that we may take $V = V_+$. Thus if χ has the form

$$\chi = \begin{cases} aV_+ e^{ipz} + bV_- e^{-ipz} & z < 0 \\ cV e^{-pz} & z > 0, \end{cases} \quad (2.19)$$

then on applying the boundary conditions at $z = 0$ we are forced to set $a = c$ and $b = 0$. Thus the incoming wave is totally absorbed by the Kleinian region (see figure 3).

Case 2: $\gamma_K^3 = +i\gamma_L^3$

This case is the same as the last except that (2.17) becomes

$$\frac{\partial \chi}{\partial z} = -\gamma_L^3(\gamma_L^0 \omega - \gamma_L \cdot \mathbf{k} - m)\chi, \quad (2.20)$$

and hence (2.18) becomes

$$\mathcal{D}V = -pV. \quad (2.21)$$

It follows that we may take $V = V_-$. Hence if χ is given by (2.19) then on applying the boundary conditions at $z = 0$, we are forced to set $b = c$ and $a = 0$. Thus the Kleinian region emits a wave which propagates through the Lorentzian region (see figure 3). There is no incoming wave solution.

We note that the Klein-Gordon equation solution looks like a superposition of the two Dirac equation solutions. This is not surprising, as the Klein-Gordon equation does not ‘notice’ whether $\gamma_K^3 = -\gamma_L^3$ or $\gamma_K^3 = +\gamma_L^3$.

2.3 Electromagnetic Field

The electromagnetic field may be treated using either a set of first order field equations for the electric and magnetic fields, or a second order field equation for the electromagnetic potential.

First Order

By introducing the dual tensor $\tilde{F}^{ab} = \frac{1}{2}\epsilon^{abcd}F_{cd}$, Maxwell’s equations may be written without reference to the electromagnetic potential as

$$\partial_a F^{ab} = 0, \quad (2.22)$$

$$\partial_a \tilde{F}^{ab} = 0. \quad (2.23)$$

We wish to find plane wave solutions, hence we will set

$$F^{ab} = f^{ab} e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \chi(z) \quad (2.24)$$

where $\partial_c f^{ab} = 0$. Thus from (2.22) and (2.23) we obtain

$$f^{3b} \frac{\partial \chi}{\partial z} = i(f^{0b} \omega - f^{\mu b} k_\mu) \chi, \quad (2.25)$$

$$\epsilon^{3b}_{cd} f^{cd} \frac{\partial \chi}{\partial z} = i f^{cd} (\epsilon^{0b}_{cd} \omega - \epsilon^{\mu b}_{cd} k_\mu) \chi, \quad (2.26)$$

and since f^{ab} is skew we note that at least one of the four terms in each of these equations is zero.

On the signature changing spacetime (2.1), let us write the solution as

$$F^{ab} = e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} a f_+^{ab} e^{ipz} + b f_-^{ab} e^{-ipz} & z < 0 \\ c f^{ab} e^{-pz} & z > 0 \end{cases} \quad (2.27)$$

where f_+^{ab} , f_-^{ab} and f^{ab} are the solutions of equations (2.25) and (2.26) which correspond to setting $\chi(z)$ equal to e^{ipz} , e^{-ipz} and e^{-pz} respectively.

Noting that the equation formed by putting $b = 3$ in equation (2.26) is identical in both the Lorentzian and Kleinian regions, we see that one obvious solution will have

$$f_+^{\alpha\beta} = f_-^{\alpha\beta} = f^{\alpha\beta} \quad \text{for } \alpha, \beta \in \{0, 1, 2\}. \quad (2.28)$$

Applying the boundary conditions to these components of F^{ab} , it follows that

$$a + b = c. \quad (2.29)$$

Furthermore, putting $b = 0$ in (2.25) allows us to calculate f_+^{03} , f_-^{03} , and f^{03} in terms of f^{01} and f^{02} . Applying the boundary conditions to F^{03} then gives

$$a - b = -ic. \quad (2.30)$$

Solving (2.29) and (2.30) we find $b = ia$ and $c = (1 + i)a$. Thus the incident electromagnetic wave is completely reflected at $z = 0$ (see figure 2). Note that the remaining six field equations are satisfied by this solution if we set $\omega^2 = \mathbf{k}^2 + p^2$.

Second Order

Choosing the Lorentz gauge $\partial_b A^b = 0$, Maxwell’s equations are $\square A^b = 0$. Thus results are as for the scalar field, we find

$$A^b = a^b e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} e^{ipz} - ie^{-ipz} & z < 0 \\ (1 - i)e^{-pz} & z > 0 \end{cases} \quad (2.31)$$

with $a^0 \omega = \mathbf{a} \cdot \mathbf{k} + (1 + i)a^3 p$.

Once again the incoming wave experiences total reflection (see figure 2).

2.4 A Wall of Kleinian Signature

The similarity between the scalar field solution and wave behaviour in a potential well encourages us to consider whether a wave which encounters a small region of Kleinian signature could tunnel through that region.

Consider the flat signature changing spacetime given by

$$ds^2 = \begin{cases} dt^2 - dx^2 - dy^2 - dz^2 & z < 0 \\ dt^2 - dx^2 - dy^2 + dz^2 & 0 < z < l \\ dt^2 - dx^2 - dy^2 - dz^2 & z > l \end{cases} \quad (2.32)$$

This spacetime consists of a wall of Kleinian signature of width l sandwiched between two Lorentzian regions. As examples of the behaviour of wave solutions on this spacetime, we will solve the Klein-Gordon and Dirac equations.

Scalar Field

Solving the Klein-Gordon equation (2.2) on this spacetime and applying the boundary conditions at $z = 0$ and $z = l$, we find

$$\phi = \frac{e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)}}{\cosh(pl)} \begin{cases} \cosh(pl)e^{ipz} - i \sinh(pl)e^{-ipz} & z < 0 \\ \frac{1}{2}(1+i)e^{p(z-l)} + \frac{1}{2}(1-i)e^{-p(z-l)} & 0 < z < l \\ e^{ip(z-l)} & z > l \end{cases} \quad (2.33)$$

Let A_i, A_r and A_t be the amplitudes of the incident, reflected and transmitted waves respectively, then

$$|A_i|^2 - |A_r|^2 - |A_t|^2 = 1 - \tanh^2(pl) - \text{sech}^2(pl) = 0. \quad (2.34)$$

Thus the wave successfully tunnels through the Kleinian region and energy is conserved (see figure 4).

Dirac Field

In the Kleinian region we take $\gamma_K^3 = -i\gamma_L^3$ then solving equations (2.10) and (2.11) and applying the continuous boundary conditions we find

$$\chi = V_* \begin{cases} e^{ipz} & z < 0 \\ e^{-pz} & 0 < z < l \\ e^{-pl}e^{ip(z-l)} & z > l \end{cases} \quad (2.35)$$

Thus the wave successfully tunnels through the Kleinian region, however, reading off the wave amplitudes we find

$$|A_i|^2 - |A_r|^2 - |A_t|^2 = 1 - e^{-2pl}. \quad (2.36)$$

It follows that energy is not conserved unless $l = 0$, in which case there would be no Kleinian region (see figure 4).

If instead we had used $\gamma_K^3 = +i\gamma_L^3$ then the same result would have been obtained, except that the waves would be propagating from right to left rather than from left to right.

3 The Dirac equation revisited

The results of the previous section predict some strange phenomena. For example, on encountering a region of Kleinian signature a π^0 -meson, described by the Klein-Gordon equation (2.2), would be totally reflected, whereas an electron, described by the Dirac equation (2.10), would be totally absorbed. Since both the first and second order formulations of the electromagnetic field give total reflection, an obvious conclusion is that regions of Kleinian signature distinguish between particles with integer and half-integer spin.

The purpose of this section is to determine whether it is possible, by altering our treatment of the Dirac equation, to remove this disparity between particles with integer and half-integer spin.

3.1 Partially reflecting solutions

In the previous section, our treatment of the Dirac equation used Kleinian gamma matrices given by $\gamma_K^\alpha = \gamma_L^\alpha$ for $\alpha \in \{0, 1, 2\}$. However, we could have chosen $\gamma_K^\alpha = -\gamma_L^\alpha$ for any or all of the γ_K^α . As we will show here, such choices give partially reflecting solutions of the Dirac equation, and thus at least give some degree of similarity between the solutions of the Dirac equation and other field equations. Note that the Klein-Gordon equation does not 'notice' whether we have chosen $\gamma_K^\alpha = \gamma_L^\alpha$ or $\gamma_K^\alpha = -\gamma_L^\alpha$.

We will consider solutions of equations (2.10) and (2.11) in the case when $\gamma_K^\alpha = -\gamma_L^\alpha$ for all $\alpha \in \{0, 1, 2\}$ and $\gamma_K^3 = -i\gamma_L^3$.

Let Ψ have the form

$$\Psi = e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} aV_+ e^{ipz} + bV_- e^{-ipz} & z < 0 \\ cV e^{-pz} & z > 0, \end{cases} \quad (3.1)$$

then we find the relations

$$\mathcal{D}V_+ = pV_+, \quad (3.2)$$

$$\mathcal{D}V_- = -pV_-, \quad (3.3)$$

$$\mathcal{D}V = (2m\gamma_L^3 - p)V, \quad (3.4)$$

where the operator $\mathcal{D}$ is defined by equation (2.14).

To proceed further, it is necessary to choose an explicit representation of the gamma matrices. We will use the 'standard representation' in which

$$\gamma_L^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma_L^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \quad j = 1, 2, 3, \quad (3.5)$$

where $\mathbf{1}$ is the 2×2 identity matrix and the Pauli matrices σ_j are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.6)$$

For simplicity, let us set $\mathbf{k} = 0$, then solving equations (3.2), (3.3) and (3.4) for V_+ , V_- and V respectively, we find

$$V_+ = \begin{pmatrix} 1 \\ 0 \\ \frac{\omega-m}{p} \\ 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} 1 \\ 0 \\ \frac{\omega-m}{-p} \\ 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 \\ 0 \\ \frac{\omega+m}{-p} \\ 0 \end{pmatrix}. \quad (3.7)$$

Continuity of Ψ at $z = 0$ gives $b = -\left(\frac{\omega}{m}\right)a$ and $c = \left(\frac{m-\omega}{m}\right)a$, and thus Ψ has the form

$$\Psi = a e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} V_+ e^{ipz} - \left(\frac{\omega}{m}\right)V_- e^{-ipz} & z < 0 \\ \left(\frac{m-\omega}{m}\right)V e^{-pz} & z > 0, \end{cases} \quad (3.8)$$

Defining the current $j_0 = \Psi \Psi^\dagger$ we see that

$$0 < \left(\frac{\text{Incident } j_0}{\text{Reflected } j_0} \right) = 1 - \frac{p^2}{\omega^2} < 1, \quad (3.9)$$

and thus the incoming wave is partially reflected at $z = 0$. A similar calculation with $\gamma_K^3 = +i\gamma_L^3$ gives a solution in which the reflected wave has a greater current than the incoming wave. However, in both cases, there is no solution which totally reflects, since this would correspond to setting $p = 0$. These solutions are similar to those obtained in a Klein Paradox* situation (see eg. [8]).

3.2 Squaring the Dirac equation

In Minkowski space, each component of a solution to the Dirac equation satisfies the standard Klein-Gordon equation. However, even in the case of a

*The Klein of 'Kleinian signature' is not the same Klein as the Klein of 'Klein Paradox'. The former was the leading nineteenth century mathematician Felix Klein, who is best known for his work in projective geometry. The latter was the physicist Oskar Klein, who is well known for his work in relativistic quantum mechanics and his unification of general relativity and electromagnetism.

flat signature changing spacetime the analogue of this statement is not true. Here we derive the second order equation which each component of a Dirac equation solution must satisfy if the spacetime metric is given by (2.1).

Let $\theta(z)$ be the usual step function given by

$$\theta(z) = \begin{cases} 1 & z > 0 \\ 0 & z < 0, \end{cases} \quad (3.10)$$

then on defining $\tilde{\gamma}^\alpha = \gamma_L^\alpha$ for $\alpha = 0, 1, 2$, and

$$\tilde{\gamma}^3 = [\theta(-z) \pm i\theta(z)] \gamma_L^3 \quad (3.11)$$

the Dirac equations (2.10) and (2.11) may be concisely written as

$$(\tilde{\gamma}^a \partial_a + im)\Psi = 0. \quad (3.12)$$

The distributional identities

$$\begin{aligned} \delta(z)[\theta(z) - \theta(-z)] &= 0, \\ \delta(z)[\theta(z) + \theta(-z)] &= \delta(z), \end{aligned} \quad (3.13)$$

may be obtained using the definition of a product of two distributions given in [9]. This definition avoids the need to define $\theta(0)$.

Operating on (3.12) with $\tilde{\gamma}^b \partial_b$, and using the results (3.13), we obtain the second order equation

$$\left[g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} + \delta(z) \frac{\partial}{\partial z} + m^2 \right] \Psi = 0. \quad (3.14)$$

Hence, in order to avoid distributional terms, we must restrict our solution space to functions Ψ which satisfy $\frac{\partial \Psi}{\partial z}|_{z=0} = 0$. This obviously excludes the wave solutions of the Dirac equation found in section 2. Interestingly, this condition is identical to that found by Hayward [5] when he considered the case of Euclidean signature change in which a scalar field was coupled to gravity.

3.3 Modifying the Dirac Equation

We have seen that the Dirac equations (2.10, 2.11) square up to give the Klein–Gordon equation plus an extra distributional term. It is well known that, in curved space, the square of the Dirac equation gains an extra term proportional to the scalar curvature, thus one might conclude that the signature changing spacetime has a distributional part in its curvature scalar. However, this is not the case, it is easily checked from first principals that the curvature scalar of the spacetime (2.1) is everywhere zero. Hence this additional term is a new signature effect.

The historical motivation behind the original derivation of the Dirac equation was to find an equation which squared to give the Klein–Gordon equation and which was first order. We have seen that in a discontinuously signature changing spacetime the Dirac equation (2.10, 2.11) does not satisfy these properties. Thus one possible conclusion is that, in the case of signature change, a new ‘Dirac equation’ is required. Here we will investigate whether it is possible to modify the Kleinian side of the Dirac equation so that it squares to give the Klein–Gordon equation, and so that it is satisfied by solutions which have a similar form to those found in section 2.

Once again let us set

$$\Psi = e^{i(k \cdot y - \omega t)} \begin{cases} aV_+ e^{ipz} + bV_- e^{-ipz} & z < 0 \\ cV e^{-pz} & z > 0, \end{cases} \quad (3.15)$$

then solving the Lorentzian Dirac equation for $z < 0$ gives

$$\mathcal{D}V_\pm = \pm pV_\pm \quad (3.16)$$

where $\mathcal{D}$ is given by equation (2.14). If we now require that Ψ is continuous then we must have

$$\mathcal{D}V = p \left(V - \frac{2b}{c} V_- \right). \quad (3.17)$$

Consider the vector space which is spanned by the vectors V_+ and V_- , and define the operator $\mathcal{P}$ by $\mathcal{P}V_- = V_-$ and $\mathcal{P}V_+ = 0$. Thus $\mathcal{P}$ projects onto the

subspace spanned by V_- . It is easily shown that $\mathcal{P}$ commutes with $\mathcal{D}$, and furthermore, from the continuity of Ψ , that $\mathcal{P}V = \frac{1}{c}V_-$. We now see that equation (3.17) can be written

$$\mathcal{D}(1 - 2\mathcal{P})V = pV. \quad (3.18)$$

The ‘Dirac equation’ which gives the equations (3.16) and (3.18) can be shown to be

$$[\tilde{\gamma}^a \partial_a + im - 2\theta(z)(\gamma_L^a \partial_a + im)\mathcal{P}] \Psi = 0. \quad (3.19)$$

Squaring this equation by applying $\tilde{\gamma}^b \partial_b$, some miraculous cancellations conspire to give

$$\left[g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} + m^2 + \delta(z) \left(\frac{\partial}{\partial z} - (1+i)\mathcal{D}\mathcal{P} \right) \right] \Psi = 0. \quad (3.20)$$

It is now easy to check by explicit substitution of Ψ that the distributional term does not cancel unless Ψ is everywhere zero. Hence even by modifying the Dirac equation in the Kleinian region we are unable to find an equation which squares to give the Klein–Gordon equation (2.2).

4 Continuous Signature Change

In this section we will examine wave solutions of the Dirac and Klein–Gordon equations in a spacetime which continuously changes signature. Thus the spacetime metric is degenerate at the surface of signature change. We will choose g^{ab} to be everywhere finite and thus g_{ab} will become infinite on the surface of signature change. This choice is more convenient since it is g^{ab} that appears in the field equations.

4.1 The scalar field

With coordinates $\{t, x, y, z\}$ we will take the metric to be

$$(g^{ab}) = \text{diag}(1, -1, -1, z). \quad (4.1)$$

We wish to solve

$$(\square + m^2)\phi = 0 \quad (4.2)$$

where $\square$ is given by

$$\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + z^{\frac{1}{2}} \frac{\partial}{\partial z} z^{\frac{1}{2}} \frac{\partial}{\partial z}. \quad (4.3)$$

We find that

$$\phi = e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} ae^{-2pz^{\frac{1}{2}}} + be^{2pz^{\frac{1}{2}}} & z < 0 \\ ce^{-2pz^{\frac{1}{2}}} & z > 0 \end{cases} \quad (4.4)$$

satisfies the wave equation if $\omega^2 = \mathbf{k}^2 + p^2 + m^2$.

Continuity of ϕ at $z = 0$ gives $a + b = c$, and continuity of $z^{\frac{1}{2}} \frac{\partial \phi}{\partial z}$ at $z = 0$ gives $a - b = c$. Hence $a = c$ and $b = 0$.

This result is remarkably similar to the Dirac equation solution in the case of discontinuous signature change (see figure 3).

4.2 The Dirac field

The Dirac equation with spacetime metric (4.1) can be shown to be

$$\left(\gamma_L^0 \frac{\partial}{\partial t} + \gamma_L^1 \frac{\partial}{\partial x} + \gamma_L^2 \frac{\partial}{\partial y} + i\gamma_L^3 z^{\frac{1}{2}} \frac{\partial}{\partial z} + im \right) \Psi = 0. \quad (4.5)$$

Taking

$$\Psi = V_{\pm} e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} e^{\pm 2pz^{\frac{1}{2}}} \quad (4.6)$$

we find that

$$\mathcal{D}V_{\pm} = \pm pV_{\pm} \quad (4.7)$$

where the operator $\mathcal{D}$ is given by equation (2.14).

Thus if we set

$$\Psi = e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} aV_+ e^{2pz^{\frac{1}{2}}} + bV_- e^{-2pz^{\frac{1}{2}}} & z < 0 \\ cV_- e^{-2pz^{\frac{1}{2}}} & z > 0 \end{cases} \quad (4.8)$$

then continuity at $z = 0$ gives $aV_+ + bV_- = cV_-$. Applying $\mathcal{D}$ to this gives $aV_+ - bV_- = -cV_-$, hence $a = 0$ and $b = c$.

Thus, for the Dirac equation, we obtain qualitatively the same result that was found in the case of discontinuous signature change. There are two solutions (depending upon whether for $z < 0$ one takes $z^{\frac{1}{2}} = -i|z|^{\frac{1}{2}}$ or $z^{\frac{1}{2}} = +i|z|^{\frac{1}{2}}$) corresponding to wave absorption by the Kleinian region and wave emission from the Kleinian region (see figure 3).

4.3 A Wall of Kleinian Signature

With coordinates $\{t, x, y, z\}$ we will take the metric to be

$$(g^{ab}) = \text{diag}(1, -1, -1, z(l-z)). \quad (4.9)$$

As in the case of discontinuous signature change, this spacetime consists of a wall of Kleinian signature of width l sandwiched between two Lorentzian regions. As examples of the behaviour of wave equation solutions on this spacetime, we will solve the Klein-Gordon and Dirac equations.

Scalar Field

We wish to solve

$$(\square + m^2)\phi = 0 \quad (4.10)$$

where $\square$ is now given by

$$\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + z^{\frac{1}{2}}(l-z)^{\frac{1}{2}} \frac{\partial}{\partial z} z^{\frac{1}{2}}(l-z)^{\frac{1}{2}} \frac{\partial}{\partial z}. \quad (4.11)$$

With the requirement that both ϕ and $z^{\frac{1}{2}}(l-z)^{\frac{1}{2}} \frac{\partial \phi}{\partial z}$ are continuous at $z = 0$ and $z = l$, we find the solution

$$\phi = ae^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} \exp[-ip \cosh^{-1}(\frac{l-2z}{l})] & z < 0 \\ e^{-p\pi} \exp[p \cos^{-1}(\frac{2z-l}{l})] & 0 < z < l \\ e^{-p\pi} \exp[ip \cosh^{-1}(\frac{2z-l}{l})] & z > l. \end{cases} \quad (4.12)$$

Hence comparing wave amplitudes we see that

$$\left| \frac{\text{Incident Amplitude}}{\text{Transmitted Amplitude}} \right| = \left| \frac{a}{ae^{-p\pi}} \right| = e^{p\pi}. \quad (4.13)$$

Thus the wave successfully tunnels through the Kleinian region but energy is not conserved. The wave amplitude decreases as it passes through the Kleinian region (see figure 4).

Dirac Field

The appropriate Dirac equation for the metric (4.9) is

$$\left(\gamma_L^0 \frac{\partial}{\partial t} + \gamma_L^1 \frac{\partial}{\partial x} + \gamma_L^2 \frac{\partial}{\partial y} + i\gamma_L^3 [z(l-z)]^{\frac{1}{2}} \frac{\partial}{\partial z} + im \right) \Psi = 0. \quad (4.14)$$

Solving this equation with the requirement that Ψ is continuous gives

$$\Psi = aV_- e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} \exp[-ip \cosh^{-1}(\frac{l-2z}{l})] & z < 0 \\ e^{-p\pi} \exp[p \cos^{-1}(\frac{2z-l}{l})] & 0 < z < l \\ e^{-p\pi} \exp[ip \cosh^{-1}(\frac{2z-l}{l})] & z > l. \end{cases} \quad (4.15)$$

Thus the wave successfully tunnels through the Kleinian region, however, as in the scalar field case, the wave amplitude decreases by a factor of $e^{p\pi}$ (see figure 4).

5 Conclusion

The solutions of the wave equations in a spacetime where the metric signature changes continuously are consistent and independent of matter type, although they do predict some strange phenomena. An incoming wave can propagate through a thin wall of Kleinian signature, but in the process it would lose energy. If the wall is of infinite length then the wave is completely absorbed. For example, a beam of light could pass through a region of Kleinian signature, and although its frequency would remain unchanged, it would become less intense.

The continuous signature change results also suggest a possible resolution of the disparity between the integer and half-integer spin solutions in the case of discontinuous signature change. Although, in both approaches to signature change, the boundary conditions which we applied seemed the most natural, at closer inspection they are inequivalent. It is, however, possible to modify the boundary conditions in the discontinuous case so that they mimic those in the continuous case. Let ϕ be a scalar field on a spacetime experiencing discontinuous signature change at $z = 0$, and let us take the following boundary conditions:

(1) ϕ is continuous,

$$(2) \lim_{z \rightarrow 0^-} \frac{\partial \phi}{\partial z} = \pm i \lim_{z \rightarrow 0^+} \frac{\partial \phi}{\partial z}.$$

Solving the wave equation with these boundary conditions gives two solutions (depending on whether + or - is taken in boundary condition (2)). These solutions are identical in form to the Dirac equation solutions of section 2. Thus by employing these boundary conditions, the wave solutions in the presence of discontinuous signature change are consistent and independent of matter type.

An alternate view point, for those who believe that the discontinuous approach to signature change with the original boundary conditions is correct, is that regions of Kleinian signature are totally 'harmless'. On encountering a thin wall of Kleinian signature, an incoming wave splits into a reflected

wave and a transmitted wave, and thus energy is conserved. If the wall is of infinite length the wave is totally reflected. The spin- $\frac{1}{2}$ results can be regarded as spurious since the components of the Dirac equation solution do not satisfy the singular Klein-Gordon equation (3.14). Furthermore, it is not possible to modify the Dirac equation (in the Kleinian region) so that its square is satisfied by solutions of the same form as those found in section 2.

In either case, it is interesting to note the important role played by boundary conditions. By choosing them carefully, it is possible to obtain qualitatively the same physical results with both the discontinuous and continuous descriptions of signature change.

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Figure 1. Light cones in Lorentzian and Kleinian regions of spacetime. The vectors v_L and v_K show typical timelike vectors.

Figure 2. One type of behaviour of wave equation solutions in a space-time that changes signature from Lorentzian to Kleinian. The incident and reflected waves are labelled i and r respectively. The incident wave is completely reflected with a phase shift.

Figure 3. A second type of behaviour of wave equation solutions in a space-time that changes signature from Lorentzian to Kleinian. In (a) the incident wave, labelled i , is totally absorbed by the Kleinian region, and in (b) a wave, labelled e , is emitted from the Kleinian region.

Figure 4. Two types of wave behaviour which could occur when a wave encounters a wall of Kleinian signature. The incident, reflected and transmitted waves are labelled i , r , and w respectively. In (a) energy is conserved, the incident wave splits into a reflected wave and a transmitted wave. In (b) energy is lost, the incident wave tunnels through the Kleinian region without reflection, and the transmitted wave has a smaller amplitude than the incident wave.