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**Q-BOSON OSCILLATOR ALGEBRA WITH
COMPLEX DEFORMATION PARAMETER**

Q-BOSON OSCILLATOR ALGEBRA WITH COMPLEX DEFORMATION PARAMETER^(*)

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Abstract – We study a q -deformed Heisenberg-Weyl algebra with complex deformation parameter. The q -boson realization of such an algebra is obtained by means of a complex generalization of the bosonization method. The spectrum of the corresponding complex deformed oscillator is derived for two special cases of the q -commutation relations. The bosonic realization of the related $SU(2)$ algebra is also briefly considered.

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In the last years the subject of deformed (quantum) groups and algebras has attracted great attention from both the mathematical and the physical point of view⁽¹⁾. In particular, deformations of the Heisenberg-Weyl algebra $h(4)$, and their related q-boson oscillator realizations, have been widely investigated⁽²⁻¹³⁾. Besides the early formulations of the q-oscillator⁽²⁻⁵⁾, various generalizations of q-deformed boson algebras have been recently considered⁽⁶⁻¹³⁾.

In this letter, we want to discuss a q-mutation oscillator algebra with a complex deformation parameter q . Complex parameters q have been considered up to now in literature only in connection with q-differential calculus in quantum plane⁽¹⁴⁻¹⁸⁾. As we shall see, the construction of the Fock representation in the complex parameter case can be readily achieved by the use of a complex version of the bosonization method (originally introduced by Jannussis and coworkers⁽¹³⁾ and recently rediscovered as a generalization of the so called Tamm-Dancoff deformation⁽⁶⁾).

Let us therefore consider the commutation relation

$$AB - qBA = F(q, N, \dots) \quad (1)$$

where A, B are non-Hermitian operators, $q \in \mathbb{C}$ ($\mathbb{C}$ complex field), N is the usual number operator ($N|n\rangle = n|n\rangle$) and F is a complex-valued function of the variables. In this case, one has to consider the adjoint relation

$$B^+ A^+ - q^* A^+ B^+ = F^*(q^*, N, \dots) \quad (2)$$

(the star denoting the usual complex conjugation).

Obviously the two commutation relations (1), (2) can only be satisfied if A, B are not conjugated to each other, i.e. $B \neq A^+$, $A \neq B^+$. We want now to find an explicit realization of the algebra (1), (2) for the two special cases

i)

$$F(q, N, \dots) = 1 \quad (3)$$

and

ii)

$$F(q, N, \dots) = q^{-N}. \quad (4)$$

Let us apply the bosonization method⁽³⁾, i.e. seek the operators A, B in the form

$$\begin{aligned} A &= f(N+1)a, \\ B &= a^+ f(N+1), \end{aligned} \quad (5)$$

where the structure function $f(N+1)$ is complex and a, a^+ are ladder operators of the usual Heisenberg-Weyl algebra ($N = a^+ a$). The representation (5) clearly implies the relations

$$(B^+)^+ = A; \quad (A^+)^+ = B \quad (6)$$

which can be regarded as the operator counterparts (in the Heisenberg representation) of the biorthonormal properties of states in the theory of non-Hermitian Hamiltonians⁽¹⁹⁾.

Replacing eqs. (5) in (1), (2), we get

$$(N+1)f^2(N+1) - qNf^2(N) = F(q, N, \dots); \quad (7)$$

$$(N+1)[f^*(N+1)]^2 - q^*N[f^*(N)]^2 = F^*(q^*, N, \dots). \quad (8)$$

Applying eqs. (7), (8) to the Fock state $|n\rangle$ we have also

$$(n+1)f^2(n+1) - qnf^2(n) = F(q, n, \dots); \quad (9)$$

$$(n+1)[f^*(n+1)]^2 - q^*n[f^*(n)]^2 = F^*(q^*, n, \dots). \quad (10)$$

Because eqs. (9), (10), are complex conjugated to each other, we can confine ourselves to solve, say, eq. (9). Let us therefore put

$$L_n = nf^2(n)q^n, \quad L_0 = 0. \quad (11)$$

Then, eq. (9) becomes

$$L_{n+1} - L_n = q^{n(n+1)} F(q, n, \dots) \quad (12)$$

whose solution is given by

$$L_{n+1} = \sum_{l=0}^n q^{-l(l+1)} F(q, l, \dots). \quad (13)$$

Let us consider the case i), i.e. $F(q, N, \dots) = 1$. Then eq. (13) yields the explicit solution

$$L_{n+1} = \sqrt{\frac{q^{n+1}-1}{q-1}} \frac{1}{n+1} = \sqrt{\frac{[n+1]}{n+1}}, \quad (14)$$

where

$$[n] = \frac{q^n - 1}{q - 1}. \quad (15)$$

On account of eqs. (11), (14) representation (5) takes the form

$$A = \sqrt{\frac{[N+1]}{N+1}} a; \quad B = a^+ \sqrt{\frac{[N+1]}{N+1}}. \quad (16)$$

We can now define the generalized number operator

$$\mathcal{N} = A^+ A = B B^+ = \sqrt{[N]} [N]^* \quad (17)$$

whose action on the kets $|n\rangle$ is given by

$$\mathcal{N}|n\rangle = \sqrt{\frac{q^n - 1}{q - 1}} \cdot \frac{(q^*)^n - 1}{q^* - 1} |n\rangle =$$

$$= \sqrt{\frac{|q|^{2n} - 2|q|^n \cos n\theta + 1}{|q|^2 - 2|q| \cos \theta + 1}} |n\rangle \quad (18)$$

where we put $q = |q|e^{i\theta}$.

Following ref. [3], we can express A, A^+ in terms of the operators x, p , as follows:

$$A = \frac{1}{\sqrt{(q+1)\hbar}} [\sqrt{m\omega}x + \frac{ip}{\sqrt{m\omega}}]; \quad (19)$$

$$A^+ = \frac{1}{\sqrt{(q^*+1)\hbar}} [\sqrt{m\omega}x - \frac{ip}{\sqrt{m\omega}}]; \quad (20)$$

or

$$x = \frac{1}{2} \sqrt{\frac{\hbar}{m\omega}} (\sqrt{q+1}A + \sqrt{q^*+1}A^+); \quad (21)$$

$$p = -\frac{1}{2} \sqrt{\hbar m\omega} (\sqrt{q+1}A - \sqrt{q^*+1}A^+). \quad (22)$$

Representation (19) - (22) implies noncanonical commutation relations between the operators x, p ⁽⁸⁾. In such a representation the Hamiltonian reads

$$H = \frac{p^2}{2m} + \frac{m}{2} \omega^2 x^2 = \frac{\hbar\omega}{4} \sqrt{(q+1)(q^*+1)} (AA^+ + A^+A), \quad (23)$$

whose eigenvalues $E_n (n = 0, 1, 2, \dots)$ are given by

$$E_n = \frac{\hbar\omega}{4} \sqrt{\frac{|q|^2 + 2|q| \cos \theta + 1}{|q|^2 - 2|q| \cos \theta + 1}} \left(\sqrt{|q|^{2(n+1)} - 2|q|^{n+1} \cos(n+1)\theta + 1} + \sqrt{|q|^{2n} - 2|q|^n \cos n\theta + 1} \right). \quad (24)$$

For q real ($\theta = 0$), eq. (24) reduces to the usual expression of the eigenvalues of the q -deformed oscillator, i.e.

$$E_n = \frac{\hbar\omega}{4} (q+1) \frac{q^{n+1} + q^n - 2}{q-1} \quad (25)$$

whereas for $|q| = 1$ the energy spectrum of the complex q -deformed oscillator becomes

$$E_n = \frac{\hbar\omega}{2} \frac{\cos(\theta/2)}{\sin(\theta/2)} \left\{ \sin\left[\frac{(n+1)\theta}{2}\right] + \sin\left(\frac{n\theta}{2}\right) \right\}. \quad (26)$$

For the usual Fock representation⁽⁴⁾

$$x = \sqrt{\frac{\hbar}{2m\omega}} (A + A^+); p = -i\sqrt{\frac{\hbar m\omega}{2}} (A - A^+) \quad (27)$$

we get the following deformed energy spectrum

$$E_n = \frac{\hbar\omega}{2} \left[\sqrt{\frac{|q|^{2(n+1)} - 2|q|^{n+1} \cos(n+1)\theta + 1}{|q|^2 - 2|q| \cos \theta + 1}} + \sqrt{\frac{|q|^{2n} - 2|q|^n \cos n\theta + 1}{|q|^2 - 2|q| \cos \theta + 1}} \right] \quad (28)$$

which, for $|q| = 1$, becomes

$$E_n = \frac{\hbar\omega}{2} \left\{ \frac{\sin\left[\frac{(n+1)\theta}{2}\right] + \sin\left(\frac{n\theta}{2}\right)}{\sin(\theta/2)} \right\}. \quad (29)$$

Let us now consider the case ii) (eq.(4)). The bosonization procedure (see eqs. (5) - (14)) provides us with the following representation of the operators A, B

$$A = \sqrt{\frac{q^{(N+1)} - q^{-(N+1)}}{q - q^{-1}}} \frac{1}{N+1} a = \sqrt{\frac{[N+1]}{N+1}} a \quad (30a)$$

$$B = a^+ \sqrt{\frac{q^{(N+1)} - q^{-(N+1)}}{q - q^{-1}}} \frac{1}{N+1} = a^+ \sqrt{\frac{[N+1]}{N+1}} \quad (30b)$$

where now

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad (31)$$

and A, B , still satisfy relations (6). In the usual Fock representation (27) of the operator A, A^+ , we get the following deformed spectrum of Hamiltonian (23):

$$\begin{aligned} E_n &= \frac{\hbar\omega}{2} (|[n+1]| + |[n]|) = \\ &= \frac{\hbar\omega}{2} \left(\sqrt{\frac{|q|^{2(n+1)} - |q|^{-2(n+1)} - 2\cos[2\theta(n+1)]}{|q|^2 + |q|^{-2} - 2\cos 2\theta}} + \right. \\ &\quad \left. + \sqrt{\frac{|q|^{2n} + |q|^{-2n} - 2\cos(2n\theta)}{|q|^2 + |q|^{-2} - 2\cos 2\theta}} \right). \end{aligned} \quad (32)$$

For $|q| = 1$ eq. (32) yields

$$E_n = \frac{\hbar\omega}{2 \sin \theta} \{ \sin[(n+1)\theta] + \sin(n\theta) \}. \quad (33)$$

For $\theta=0$ one recovers the spectrum of the standard harmonic oscillator, whereas for $\theta = \pi/2$ we get

$$E_n = (-1)^n \frac{\hbar\omega}{2}. \quad (34)$$

Let us finally construct the deformed $SU(2)$ algebra $(3,4,20,22)$ with complex parameter. If $A_j, A_j^\dagger (j = 1, 2)$ are the operators of two independent, deformed oscillators with complex deformation parameters q_1, q_2 , we have, in representation (30), (31):

$$A_j = \sqrt{\frac{[N_j + 1]}{N_j + 1}} a_j; A_j^\dagger = a_j^\dagger \sqrt{\frac{[N_j + 1]^*}{N_j + 1}}, j = 1, 2. \quad (35)$$

The $SU(2)$ generators are obtained by the Jordan-Schwinger map

$$J_+ = A_1^\dagger A_2; J_- = A_2^\dagger A_1; 2J_z = [J_+, J_-]. \quad (36)$$

After some algebra we find for instance ($q_j = |q_j|e^{i\theta_j}$):

$$2J_z = \sqrt{\frac{|q_1|^{2N_1} + |q_1|^{-2N_1} - 2\cos(2\theta_1 N_1)}{|q_1|^2 + |q_1|^{-2} - 2\cos(2\theta_1)}} \cdot \sqrt{\frac{|q_2|^{2(N_2+1)} + |q_2|^{-2(N_2+1)} - 2\cos[2(N_2+1)\theta_2]}{|q_2|^2 + |q_2|^{-2} - 2\cos(2\theta_2)}} - \sqrt{\frac{|q_1|^{2(N_1+1)} + |q_1|^{-2(N_1+1)} - 2\cos[2(N_1+1)\theta_1]}{|q_1|^2 + |q_1|^{-2} - 2\cos(2\theta_1)}} \cdot \sqrt{\frac{|q_2|^{2N_2} + |q_2|^{-2N_2} - 2\cos(2N_2\theta_2)}{|q_2|^2 + |q_2|^{-2} - 2\cos(2\theta_2)}} \quad (37)$$

Let us write down the explicit form of the action of J_+, J_-, J_z on the Fock states for the special case $|q_1| = |q_2| = 1$:

$$2J_z |n_1, n_2\rangle = \left\{ \frac{\sin(\theta_1 n_1) \cdot \sin[\theta_2(n_2 + 1)] - \sin[\theta_1(n_1 + 1)] \cdot \sin(\theta_2 n_2)}{\sin\theta_1 \sin\theta_2} \right\} |n_1, n_2\rangle; \quad (38)$$

$$J_+ |n_1, n_2\rangle = \sqrt{\frac{\sin(\theta_1 n_1) \cdot \sin[\theta_2(n_2 + 1)]}{\sin\theta_1 \sin\theta_2}} |n_1 - 1, n_2 + 1\rangle; \quad (39)$$

$$J_- |n_1, n_2\rangle = \sqrt{\frac{\sin[\theta_1(n_1 + 1)] \cdot \sin(\theta_2 n_2)}{\sin\theta_1 \sin\theta_2}} |n_1 + 1, n_2 - 1\rangle. \quad (40)$$

In conclusion, we hope that the complex deformation of the Heisenberg-Weyl algebra may be fit for physical applications (like those discussed in ref.[23]), due to the richness of its mathematical structure and the malleableness of the energy spectrum of the related deformed oscillator.

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