Exact Solutions for Cosmological Perturbations with Collisionless Matter

Dominik J. Schwarz

Institut für Theoretische Physik, Technische Universität Wien,
Wiedner Hauptstraße 8-10/136, A-1040 Wien, Austria

Abstract

All regular and singular cosmological perturbations in a radiation dominated Einstein-de Sitter Universe with collisionless particles can be found by a generalized power series ansatz.

Although the study of cosmological perturbations has a long history [1], exact solutions in the linear regime are known for a few perfect fluid models only (see, e.g., [2]). Within a more fundamental description of matter no such solutions have been found until recently [3, 4, 5]. In its early epochs the Universe contains various collisionless, massless particles. Examples are background gravitons generated during the Planck epoch and neutrinos after their decoupling from the electrons.

Cosmological perturbations for collisionless matter have been analyzed numerically (e.g., [7]) and analytically in the short- and long-wavelength approximations [8]. Two kinds of solutions show up: Regular solutions are specified by bounded metric perturbations at the Big Bang singularity, whereas for singular solutions the metric blows up, changing the type of the initial singularity. The latter show superhorizon oscillations as discussed by Zakharov [8] and Vishniac [9]. The general solution is a combination of both types. Only the regular part of the solution depends on the distribution of the collisionless particles at the singularity.

We obtain exact solutions for scalar perturbations by a generalized power series ansatz. The matter under investigation is a mixture of collisionless particles and a perfect fluid in a radiation dominated Einstein-de Sitter background. The corresponding discussion for the vector and tensor perturbations can be found in [4, 5]. The equations of motion are formulated gauge invariantly. The matter perturbations can either be obtained from the Boltzmann equation [7, 10] or from the finite-temperature graviton self-energy [6]. Reference [5] compares the two approaches.

We use Bardeen’s gauge invariant metric potentials [11]. This is useful since the perturbation of the energy-momentum tensor

\[ \delta T^{\mu\nu} = \delta \left( \frac{2}{\sqrt{-g}} \frac{\delta \Gamma^M}{\delta g_{\mu\nu}} \right) \]

is gauge invariant from the definition of the effective action \( \Gamma^M \) (for background-covariant gauges). The scalar metric potentials are denoted by \( \Phi \) and \( \Pi \). They are related to the gauge invariant density contrast \( \epsilon_m \) (which is the density contrast \( \delta \) in the local rest-frame of matter) and the anisotropic pressure \( \pi_T \) by the Einstein equations:

\[ \frac{x^2}{3} \Phi = \epsilon_m, \]

\[ x^2 \Pi = \pi_T. \]

The variable \( x \) is the product of the conformal time and the (fixed) wavenumber of the perturbation.

\(^1\text{E-mail: dschwarz@esph.fh-wien.ac.at}\)
The high-temperature limit of the graviton self-energy (the second variation of $\Gamma^M$ in Eq. (1)) for temperatures below the Planck scale and the kinetic theory yield the same equations of motion [5] for the metric potentials. They form a closed set of equations allowing isentropic perturbations in the perfect fluid component only. With $\alpha$ being the ratio of the energy density in collisionless matter and the total energy density, and $\Phi = \Phi_{\text{rel}} + \Phi_{PF}$, the trace of the Einstein equations reads:

$$\Phi'' + \frac{4}{x} \Phi' + \frac{1}{3} \Phi + \frac{2}{x} \Pi' - \frac{2}{3} \Pi = 0. \tag{4}$$

Equations (2) and (3) lead to

$$\frac{x^2}{3} \Phi_{\text{cl}} = -2\alpha \left[ -\Phi - 2\Pi + 2 \int_0^x dx' \left( j_0(x - x') + \frac{3}{x} j_1(x - x') \right) (\Phi + \Pi)'(x') \right] + 2 \sum_{n=0}^{\infty} \gamma_n (j_0 + \frac{3}{x} j_1)^{(n)}(x) \tag{5}$$

and

$$x^2 \Pi = -12\alpha \left[ \int_0^x dx' j_2(x - x')(\Phi + \Pi)'(x') + \sum_{n=0}^{\infty} \gamma_n j_2^{(n)}(x) \right]. \tag{6}$$

The $\gamma_n$'s are related to the $n$-th momenta of the collisionless particles' initial distribution function [3]. The evolution starts at the Big Bang ($x = 0$).

To solve the equations (4) – (6) we make the ansatz

$$F(x) = C_1 F_{\text{reg}}(x) + C_2 x^\sigma F_{\text{sing}}(x), \tag{7}$$

where $C_1$ and $C_2$ are arbitrary constants and

$$F_{\text{reg,sing}}(x) = \sum_{n=0}^{\infty} c_{n}^{\text{reg,sing}} \frac{x^n}{n!}. \tag{8}$$

The singular solutions do not depend on the initial conditions $\gamma_n$. They can be added to any regular solution specified by the initial conditions. The exponent

$$\sigma_\pm = \frac{5}{2} \pm \sqrt{\frac{5 - 32\alpha}{20}} \tag{9}$$

takes complex values for $\alpha > \alpha_{\text{crit}} = 5/32$. Thus superhorizon size oscillations occur. They can not be taken seriously up to the initial singularity, because nonlinear effects of the Einstein equation will be important. But they are necessary to match a mixmaster scenario [12] or quantum generated perturbations from a preceding inflationary epoch [13]. A strictly Friedmannian singularity rules out the singular solutions.

In the Figs. 1 and 2 we consider a model with three massless neutrino species and a perfect fluid consisting of relativistic electrons and photons in equilibrium. For such a model $\alpha \sim 1/2$ [14].

Fig. 1 shows the regular solutions for $\Pi(0) = 0$ and $\Phi(0) = 1, \Phi'(0) = 0$, i.e., a perfect fluid at the origin. The density contrast grows on superhorizon scales and oscillates after horizon crossing ($x/\pi = 1$). On subhorizon scales the perfect fluid component has constant amplitude, whereas the collisionless component decays ($\sim 1/x$) due to directional dispersion.
Figure 1: Regular scalar perturbations in a two component universe ($\alpha = 1/2$). The various lines show regular solutions for $|\delta_{\text{cll}}|$ (full line), $|\delta_{PF}|$ (dotted line), and $|\pi_T|$ (dashed line). The initial conditions are specified by $\gamma_0 = 5/64$, $\gamma_2 = 42\gamma_0$, and $\gamma_4 = 49\gamma_0$. All other $\gamma_n$ vanish.

Figure 2: Singular solutions corresponding to $\sigma_+$. Same variables as in Fig. 1. Notice the logarithmic scale in $x$ — the solutions are essentially singular as $x \to 0$. The second solution $\sigma_-$ differs by a phase.
In Fig. 2 the singular solutions are plotted. The perfect fluid has the same behaviour as in Fig. 1, but the collisionless component shows superhorizon oscillations as mentioned above. These oscillations are of geometrical, not of causal, origin. Lukash et al. showed that collisionless matter can isotropize an initial anisotropy [15]. Reversing time these solutions reflect the instability of an isotropic collapse. The sound velocity inside the horizon equals to the speed of light for all solutions.

I am very grateful to the organizers for a fellowship to attend the conference. I thank A. K. Rebhan for a most enjoyable collaboration. This work was supported by the Austrian “Fonds zur Förderung der wissenschaftlichen Forschung” under contract no. P9005-PHY.

References