Localized Endomorphisms of the Chiral Ising Model

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Abstract

In the frame of the treatment of the chiral Ising model by Mack and Schomerus, examples of localized endomorphisms $\varphi_1^{loc}$ and $\varphi_1^{loc}$ are presented. It is shown that they lead to the same superselection sectors as the global ones in the sense that $\pi_1 \circ \varphi_1^{loc} \equiv \pi_1$ and $\pi_0 \circ \varphi_1^{loc} \equiv \pi_1/2$ holds. For proving the latter unitary equivalence, Arakis formalism of the selfdual CAR algebra is used. Further it is shown that the localized endomorphisms obey the Ising fusion rules.

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1 Introduction

In local quantum field theory one considers a Hilbert space $\mathcal{H}$ of physical states which decomposes into orthogonal subspaces $\mathcal{H}_f$ (superselection sectors) so that observables do not make transitions between the sectors. The subspaces $\mathcal{H}_f$ carry inequivalent, irreducible representations of the observable algebra $\mathcal{A}$, possibly with some multiplicities [18]. Among the superselection sectors, there is a distinguished sector $\mathcal{H}_0$ which contains the vacuum vector $|\Omega_0\rangle$ and carries the vacuum representation $\pi_0$.

The starting point in the algebraic approach to quantum field theory is the observable algebra $\mathcal{A}$ which is usually defined as the $C^*$ inductive limit of the net of local von Neumann algebras $\{\mathcal{A}(O), O \in \mathcal{K}\}$, where $\mathcal{K}$ denotes the set of open double cones in $D$ dimensional Minkowski space. The net is assumed to satisfy the Haag-Kastler-axioms. In general, the observable algebra $\mathcal{A}$ admits a lot of inequivalent representations. Therefore one has to find an appropriate selection criterion which rules out the physically non-relevant representations. Doplicher, Haag and Roberts [7, 8, 24] developed the theory of locally generated sectors; they suggested that one has to consider only those representations $\pi_J$ which become equivalent to the vacuum representation in the restriction to the causal complement $O'$ of any sufficiently large double cone $O \in \mathcal{K}$. That means that for a representation $\pi_J$ satisfying the DHR criterion, there exists for each sufficiently large double cone $O$ a unitary $V : \mathcal{H}_0 \rightarrow \mathcal{H}_J$ such that

$$\pi_J(A) = V \pi_0(A) V^*, \quad A \in \mathcal{A}(O').$$

The DHR criterion leads to the characterization of superselection sectors by localized endomorphisms: Usually $\mathcal{A}$ and $\pi_0(\mathcal{A})$ are identified, and one defines

$$\rho_J(A) = V^* \pi_J(A) V, \quad A \in \mathcal{A}.$$  

Then $\rho_J$ is an endomorphism of the observable algebra and it is localized in $O$ in the sense that $\rho_J(A) = A$ for all $A \in \mathcal{A}(O')$. Moreover, $\pi_0 \circ \rho_J$ is a representation of $\mathcal{A}$ in $\mathcal{H}_0$ which is equivalent to $\pi_J$. The use of DHR endomorphisms allows to extract all physical information out of the vacuum sector and to work without charged, unobservable fields. It is another advantage that endomorphisms can be composed; it is possible to formulate fusion rules in terms of equivalence classes of localized endomorphisms.

Unfortunately, it seems to be very difficult to construct these endomorphisms explicitly in models. Although the conformal field theory has turned out to be an interesting and fruitful area of application of the DHR program, the $U(1)$ current algebra on the circle is the only example where localized endomorphisms are explicitly known which generate charged sectors [17]. Endomorphisms have been constructed for Level 1 WZW models [16] and, before that, for the chiral Ising model [1, 2], however, they are in no sense localized. Mack and Schomerus had already described the construction of localized endomorphisms for the chiral Ising model in [1], but it has not yet been proven if they lead to the same sectors as the global ones and, in particular, if they lead to irreducible representations. This is done in the present paper.

In two-dimensional conformal field theory one considers as basic observable the stress energy tensor which generates the space time symmetry. Its light cone components $T_{\pm}(z_{\pm})$ live separately on the compactified light cone variables $z_{\pm} \in S^1$, one deals with chiral fields. Treating each component for its own, the stress energy tensor has well-known commutation relations, fixed up to a constant $c$ [19, 20]; the stress energy tensor generates the Virasoro algebra $\text{Vir}$. In the case $c = \frac{1}{2}$ (Ising model) the Virasoro algebra admits three inequivalent positive energy representations $\pi_J$, $J = 0, \frac{1}{2}, 1$, which are lowest weight representations; $\pi_0$ is identified to be the vacuum representation. In the chiral Ising model, the stress energy tensor can be built of a free fermion field, the Majorana field [1, 20]. Smearing out the Majorana field with test functions having support in a proper subinterval $I \subset S^1$ and considering bilinear expressions of it, these objects generate the local observable algebra $\mathcal{A}(I)$. These
local algebras $\mathcal{A}(I)$ generate a global observable algebra $\mathcal{A}$. Unfortunately, the Virasoro generators are not in $\mathcal{A}$, but they are formal (unbounded) limits of elements in $\mathcal{A}$. Mack and Schomerus [1, 2] constructed a larger algebra $\overline{\mathcal{A}}$ which contains $\mathcal{A}$ and $\text{Vir}$, such that the positive energy representations of $\text{Vir}$ extend to representations of $\overline{\mathcal{A}}$, and also of $\mathcal{A}$. Moreover, they presented endomorphisms $\partial_J$, such that $\mathcal{A}$ and $\overline{\mathcal{A}}$, but not Vir is closed under their action and that $\pi_0 \circ \partial_J \cong \pi_J$, $J = \frac{1}{c}$ is fulfilled. But, as already mentioned, these endomorphisms are not localized, i.e. there is no interval $I' \neq \emptyset$ such that their action is trivial on $\mathcal{A}(I')$. In this paper we present examples $\partial^\text{loc}_{1/2}$, $\partial^\text{loc}_{1/2}$ of localized endomorphisms which are unitarily equivalent to those global ones in the composition with the vacuum representation. For our investigations, we consider the endomorphisms and representations as those of $\mathcal{A}$, i.e. we will not deal with unbounded limits.

Since the set $\mathcal{I}$ of proper subintervals on the circle is not directed, the global algebra $\mathcal{A}$ cannot be defined as the $C^*$ inductive limit of the system $\{\mathcal{A}(I), I \in \mathcal{I}\}$. Instead the global algebra is defined as the free amalgamated product of the local algebras $\mathcal{A}(I)$ [12, 15]. This leads to an exceptional property of $\mathcal{A}$: Its centre is non-trivial, generated by a unitary element $Y$ [6]. The global algebra of the punctured circle $\mathcal{A}(I_\zeta)$ where $I_\zeta = S^1 \setminus \{\zeta\}$, $\zeta \in S^1$ an arbitrary point, has a trivial centre and $\mathcal{A}$ is generated by $\mathcal{A}(I_\zeta)$ and $Y$. The local algebras $\mathcal{A}(I)$ are even subalgebras of (selfdual) CAR algebras over spaces $L^2(I)$. Also the global observable algebra is the even subalgebra of a global field algebra, the universal Majorana algebra $\mathcal{Maj}$. It has the structure of the direct sum of two selfdual CAR algebras over $L^2(S^1)$. Alternatively, it may be regarded as the algebra which is generated by an anticommuting universal Majorana field living on the double cover $S^1$ of the unit circle [1, 2]. For recovering the local algebras $\mathcal{A}(I)$ as even subalgebras of $\mathcal{Maj}$ by explicit construction, we have to fix an arbitrary reference point $\zeta$ (“point at infinity”) on the circle.

The non-trivial centre of the global algebra $\mathcal{A}$ implies that its irreducible representations can no longer be faithful. This leads to some deviations from the customary DHR program. In particular, the vacuum representation $\pi_0$ is not faithful and therefore the definition of a localized endomorphism becomes somewhat delicate: We have Haag duality on the circle, $\pi_0(\mathcal{A}(I)) = \pi_0(\mathcal{A}(I'))'$ in $\mathcal{B}(\mathcal{H}_0)$, but $\mathcal{A}(I) \neq \mathcal{A}(I')'$ in $\mathcal{A}$ ($I'$ denotes the open complement of the interval $I \subset S^1$). Therefore the requirement that an endomorphism $\partial$ acts as the identity on the algebra $\mathcal{A}(I')$ does not guarantee that $\partial$ restricts to an endomorphism of algebras $\mathcal{A}(I_\zeta)$ for larger intervals $I_\zeta$, i.e. that $\partial(\mathcal{A}(I_\zeta)) \subset \mathcal{A}(I_\zeta)$ holds for $I_\zeta \supset I$. Since this property is crucial for the whole construction, it has to be requested in addition. In section 3 we will give a definition of localized endomorphisms so that they are DHR on the algebra $\mathcal{A}(I_\zeta)$ (we assume that the reference point $\zeta$ is contained in the open complement $I'$ of the localization interval $I$).

Similar to the global endomorphisms given by Mack and Schomerus, our localized endomorphisms of $\mathcal{A}$ are restrictions of endomorphisms of $\mathcal{Maj}$, they are built of Bogoliubov endomorphisms. This fact enables us to make use of some results about representations of CAR and Bogoliubov endomorphisms. In this framework we can prove $\pi_0 \circ \partial^\text{loc}_{1/2} \cong \pi_0 \circ \partial_{1/2}$. (In comparison, this is a simple calculation for our localized $\partial^\text{loc}_{1/2}$ and the global $\partial_1$, because they are inner in $\mathcal{Maj}$.) But it is not clear that the unitary equivalence in the vacuum representation extends to an inner equivalence in $\mathcal{A}$. Therefore it is not clear if our localized endomorphisms obey the Ising fusion rules, which was proven only for the global ones in [1, 2]. Nevertheless, we can show that they really do so and, moreover, that unitary equivalence between localized endomorphisms in the vacuum representation always extends to an inner equivalence in the algebra $\mathcal{A}$. This enables us to formulate fusion rules for equivalence classes of localized endomorphisms in the usual manner. At the very end of our investigations, we construct statistics operators for localized endomorphisms and derive a left inverse. A statistics operator was already found by Mack and Schomerus [1], but its construction is based on the non-localized endomorphism $\partial_{1/2}$. We use a localized endomorphism and therefore we obtain a really local statistics operator.
2 Algebras, Representations and Endomorphisms of the Chiral Ising Model

2.1 Local and Global Algebras

We begin our investigations with a brief description of the field algebra, the local and the global observable algebras of the chiral Ising model. Our starting point is a Majorana field $\psi$ on the unit circle $S^1$ which has anticommutation relations

$$\{\psi(z)^*, \psi(w)\} = 2\pi iz\delta(z - w)$$

(1)

and hermiticity condition

$$\psi(z)^* = z\psi(z).$$

(2)

We consider smeared fields

$$\psi(f) = \int_{S^1} \frac{dz}{2\pi iz} f(z) \psi(z), \quad f(z) \in L^2(S^1).$$

(3)

These objects obey the canonical anticommutation relations (CAR) of the canonical generators of Arakis [3, 4] selfdual CAR-algebra $\mathcal{A}(\mathcal{K}, \Gamma)$ over the Hilbert space $\mathcal{K} = L^2(S^1)$ with the antiunitary involution $\Gamma$ of complex conjugation. We have

$$\{\psi(f)^*, \psi(g)\} = \langle f, g \rangle 1$$

(4)

with

$$\psi(f)^* = \psi(\Gamma f), \quad \langle f, g \rangle = \int_{S^1} \frac{dz}{2\pi iz} f(z)g(z).$$

(5)

As local algebras $\mathcal{A}(I)$ with some open interval $I \subset S^1$, $I \neq S^1$ we define those unital algebras which are generated by bilinear expressions

$$B_I(f, g) = \psi(f)\psi(g), \quad \text{supp}(f) \subset I, \text{supp}(g) \subset I$$

in the Majorana fields. These generators are complex linear in both arguments and obey relations

$$2B_I(f, f) = \langle \Gamma f, f \rangle 1,$$

(6)

$$2B_I(f, g)B_I(g, h) = \langle \Gamma g, g \rangle B_I(f, h),$$

(7)

$$B_I(f, g)^* = B_I(\Gamma g, \Gamma f),$$

(8)

where $f, g, h \in L^2(S^1)$ are functions with support in $I$. Next we consider the algebras $\mathcal{A}(I)$ as defined only by these abstract relations. Since the set $\mathcal{I}$ of open, non-empty intervals $I \neq S^1$ on the circle is not directed there is no inductive limit for the algebras $\mathcal{A}(I), I \in \mathcal{I}$. But with the additional relation

$$B_I(f, g) = B_J(f, g), \quad I \subset J$$

(9)

one can construct a global algebra $\mathcal{A}$ which is generated by all $B_I(f, g), f, g \in L^2(S^1)$ and $I \in \mathcal{I}$ [6, 15, 22]. Perhaps one could expect that the result is the even subalgebra of the selfdual CAR algebra over the whole circle $S^1$. We will show that this is actually not the case; instead there occurs a central element $Y \in \mathcal{A}$ which will finally lead to the fact that $\mathcal{A}$ is the direct sum of two of those even CAR algebras. Let now $I_1$ and $I_2$ be two disjoint intervals and let $J_+$ and $J_-$ be intervals containing both of them and $J_+ \cup J_- = S^1$. Choose real functions $f_j \in L^2(S^1)$ with $\|f_j\|^2 = 2$ with $\text{supp}(f_j) \subset I_j$, $j = 1, 2$. Then define

$$Y = B_{I_+}(f, g)B_{I_-}(g, f).$$

(10)
One finds that $Y$ is unitary, selfadjoint and independent of the special choice of $f, g, I_1, I_2, J_+, J_-$. Moreover, $Y$ is in the centre of $\mathcal{A}$. For every $\zeta \in S^1$ and $I_\zeta = S^1 \setminus \{\zeta\}$, the global algebra $\mathcal{A}$ is generated by $\mathcal{A}(I_\zeta)$ and $Y$ [6, 15, 22].

We now want to reconstruct the global, or, "universal" algebra $\mathcal{A}$ by a global field algebra, the universal Majorana algebra.

**Definition 2.1** The universal Majorana algebra $\text{Maj}$ is defined as the direct sum of the selfdual CAR algebra over $(L^2(S^1), \Gamma)$ with itself, i.e.

$$\text{Maj} = \mathcal{C}(L^2(S^1), \Gamma) \oplus \mathcal{C}(L^2(S^1), \Gamma).$$

(11)

The centre of $\text{Maj}$ is generated by the element

$$Y = (-1) \oplus 1$$

(12)

and we have the two subalgebras

$$\text{Maj}_{NS} = \frac{1}{2}(1 - Y)\text{Maj}, \quad \text{Maj}_R = \frac{1}{2}(1 + Y)\text{Maj}.$$ (13)

The universal Majorana algebra is a well defined $C^*$ algebra since $\mathcal{C}(L^2(S^1), \Gamma)$ is. For clearing the connection between our definition and the definition of $\text{Maj}$ given by Mack and Schomerus [1] we consider the following two orthonormal bases of $L^2(S^1)$

$$\{ e_r, r \in \mathbb{Z} + \frac{1}{2} \} \quad \text{and} \quad \{ e_n, n \in \mathbb{Z} \},$$

where $e_z(z) = z^a$ for $z \in S^1, a \in \frac{1}{2}\mathbb{Z}$. We define the elements of $\text{Maj}$ (Fourier modes)

$$b_r = \psi(e_r) \oplus 0, \quad r \in \mathbb{Z} + \frac{1}{2},$$

(14)

$$b_n = 0 \oplus \psi(e_n), \quad n \in \mathbb{Z}.$$ (15)

Then we have

- $\text{Maj}_{NS}$ is generated by the modes $b_r, r \in \mathbb{Z} + \frac{1}{2}$,
- $\text{Maj}_R$ is generated by the modes $b_n, n \in \mathbb{Z}$,
- $\text{Maj}$ is generated by the modes $b_a, a \in \frac{1}{2}\mathbb{Z}$,

and the Fourier modes satisfy relations

$$\{ b_z, b_{\bar{z}} \} = \frac{1}{2}(1 + (-1)^{2z}Y)b_{\bar{z}}, \quad b_{\bar{z}} = b_z,$$

(16)

$$Yb_z = (-1)^{2z}b_{\bar{z}}, \quad [Y, b_z] = 0, \quad Y = Y^*, \quad Y^2 = 1.$$ (17)

It is convenient to understand the elements of $\text{Maj}$ as smeared fields as well. We define the Hilbert space

$$\hat{\mathcal{K}} = L^2(S^1) \oplus L^2(S^1)$$

(18)

which may be identified with $L^2(\tilde{S}_1)$, where $\tilde{S}_1$ denotes the double over of $S^1$. Hence each element $\hat{f} \in \hat{\mathcal{K}}$ has the unique decomposition

$$\hat{f} = f_{NS} \oplus f_R, \quad f_{NS}, f_R \in L^2(S^1).$$ (19)
On $\hat{K}$ we have the antiunitary involution
\[ \hat{\Gamma} = \Gamma \oplus \Gamma. \]  

We define the field $\hat{\psi}(\hat{f}) \in \text{Maj}$ by
\[ \hat{\psi}(\hat{f}) = \psi(f_{NS}) \oplus \psi(f_{R}) \]  
so that we have the conjugation
\[ \hat{\psi}(\hat{f})^* = \hat{\psi}(\hat{f}^*), \]  
an anticommuation relations
\[ \{ \hat{\psi}(\hat{f})^*, \hat{\psi}(\hat{g}) \} = \frac{1}{2}(1 - Y)\langle f_{NS}, g_{NS} \rangle + \frac{1}{2}(1 + Y)\langle f_{R}, g_{R} \rangle, \]  
boundary condition
\[ Y \hat{\psi}(\hat{f}) = \hat{\psi}(y\hat{f}), \quad y = (1 - 1) \oplus 1 \in \mathcal{B}(\hat{K}), \]  
and
\[ [Y, \hat{\psi}(\hat{f})] = 0. \]  

We now want to redefine the local generators $B_I(f, g) \in \mathcal{A}(I)$ as even elements of $\text{Maj}$. For that we have to fix an arbitrary point $\zeta \in S^1$. We distinguish the two cases:

**Case 1:** For all intervals $I \in \mathcal{I}$ with $\zeta \notin I$ we set
\[ B_I(f, g) = \hat{\psi}(\hat{f}) \hat{\psi}(\hat{g}), \quad \hat{f} = f \oplus f \in \hat{K}, \quad \hat{g} = g \oplus g \in \hat{K}. \]

**Case 2:** For every interval $I \in \mathcal{I}$ with $\zeta \in I$ the point $\zeta$ splits $I$ in two disjoint intervals $I_1$ and $I_2$ so that $I = I_1 \cup \{ \zeta \} \cup I_2$. Let $\chi_j$ be the characteristic functions of $I_j$ and set $\hat{f}_j = \chi_j \hat{f}, \; \hat{g}_j = \chi_j \hat{g}, \; j = 1, 2.$ Then we set
\[ B_I(f, g) = \hat{\psi}(\hat{f}_1) \hat{\psi}(\hat{g}_1) + \hat{\psi}(\hat{f}_2) \hat{\psi}(\hat{g}_2) + Y \hat{\psi}(\hat{f}_1) \hat{\psi}(\hat{g}_2) + Y \hat{\psi}(\hat{f}_2) \hat{\psi}(\hat{g}_1), \]
\[ \hat{f}_j = f_j \oplus f_j \in \hat{K}, \quad \hat{g}_j = g_j \oplus g_j \in \hat{K}, \quad j = 1, 2. \]

It is an easy but less beautiful work to control that these $B_I(f, g)$ satisfy the relations (6) - (9), and that relation (10) is fulfilled with the $Y$ of Definition 2.1, also independent of the functions and intervals [22]. As a result we identify the global observable algebra $\mathcal{A}$ as the even part of $\text{Maj}$.  

### 2.2 Representations and Endomorphisms

Each of the algebras $\text{Maj}_{NS}$ and $\text{Maj}_{R}$ possesses a faithful cyclic representation $(\mathcal{H}_{NS}, \pi_{NS}, |\Omega_{NS}\rangle)$ and $(\mathcal{H}_{R}, \pi_{R}, |\Omega_{R}\rangle)$ which is characterized by
\[
\pi_{NS}(b_r)|\Omega_{NS}\rangle = 0, \quad r > 0, \quad r \in \mathbb{Z} + \frac{1}{2}, \tag{28}
\]
\[
\pi_{R}(b_n)|\Omega_{R}\rangle = 0, \quad n > 0, \quad n \in \mathbb{Z}, \tag{29}
\]
respectively. The NS-representation is uniquely characterized (all matrix-elements can be computed and the vector $|\Omega_{NS}\rangle$ is defined to be cyclic). In the R-representation, the action of the selfadjoint $b_0$ on the cyclic $|\Omega_{R}\rangle$ is not completely fixed. To determine the R-representation uniquely too, we require in addition that the vectors $|\Omega_{R}\rangle$ and $\pi_{R}(b_0)|\Omega_{R}\rangle$ are orthogonal in $\mathcal{H}_{R}$,
\[
\langle \Omega_{R} | \pi_{R}(b_0)|\Omega_{R}\rangle = 0. \tag{30}
\]

One can consider these representations as those of $\text{Maj}$ in the space $\mathcal{H}_{NS} \oplus \mathcal{H}_{R}$ by the requirement
\[
\pi_{NS}(Y) = -1, \quad \pi_{R}(Y) = 1 \tag{31}
\]
which leads automatically to

$$\pi_{NS}(b_n) = 0, \quad n \in \mathbb{Z} \quad \text{und} \quad \pi_R(b_r) = 0, \quad r \in \mathbb{Z} + \frac{1}{2}$$

(32)
i.e. $\pi_{NS}$ lives only on $\text{Maj}_{NS}$ and $\pi_R$ on $\text{Maj}_R$. Of course, both representations are then no longer faithful. The NS-representation is irreducible, the R-representation is not; it decomposes into two irreducible subrepresentations\(^1\) $(\mathcal{H}_R^+, \pi_R^+)$ and $(\mathcal{H}_R^-, \pi_R^-)$ which are generated by the action of $\pi_R(\text{Maj})$ on vectors $|\Omega_R^+\rangle$ and $|\Omega_R^-\rangle$, respectively, where

$$|\Omega_R^\pm\rangle = \frac{1}{\sqrt{2}} |\Omega_R\rangle \pm \pi_R(b_0) |\Omega_R\rangle.$$  

(33)
These states are eigenstates of $\pi_R(b_0)$ with eigenvalues $\pm 2^{-\frac{1}{2}}$. We are now interested in what happens, when the representations of $\text{Maj}$, $\pi_{NS}$ and $\pi_R$, are restricted to the observable algebra which is the even subalgebra of $\text{Maj}$, $\mathcal{A} = \text{Maj}^{+even}$. It is known that the NS-representation splits into two irreducibles,

$$\pi_{NS}|_{\mathcal{A}} = \pi_0 \oplus \pi_1, \quad \mathcal{H}_{NS} = \mathcal{H}_0 \oplus \mathcal{H}_1,$$

(34)
and the R-representation decomposes into two equivalent ones,

$$\pi_R|_{\mathcal{A}} = \pi_{1/2} \oplus \pi_{1/2}', \quad \mathcal{H}_R = \mathcal{H}_{1/2} \oplus \mathcal{H}_{1/2}'.$$

(35)
The subspaces $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_{1/2}$ und $\mathcal{H}_{1/2}'$ are spanned by vectors

$$\pi_{NS}(b_{-n_2N} \cdots b_{-n_1}) |\Omega_{NS}\rangle \in \mathcal{H}_0, \quad n_i \in \mathbb{N}_0 + \frac{1}{2}, \quad r_{2N} > r_{2N-1} > \cdots > r_1,$$

(36)
$$\pi_{NS}(b_{-n_2N+4} \cdots b_{-n_1}) |\Omega_{NS}\rangle \in \mathcal{H}_1, \quad n_i \in \mathbb{N}_0 + \frac{1}{2}, \quad r_{2N+1} > r_{2N} > \cdots > r_1,$$

(37)
$$\pi_R(b_{-n_2N} \cdots b_{-n_1}) |\Omega_R\rangle \in \mathcal{H}_{1/2}, \quad n_i \in \mathbb{N}_0, \quad n_{2N} > n_{2N-1} > \cdots > n_1,$$

(38)
$$\pi_R(b_{-n_2N+4} \cdots b_{-n_1}) |\Omega_R\rangle \in \mathcal{H}_{1/2}', \quad n_i \in \mathbb{N}_0, \quad n_{2N+1} > n_{2N} > \cdots > n_1,$$

(39)
with $N \in \mathbb{N}_0$. We remark that the subspaces $\mathcal{H}_{1/2}$ and $\mathcal{H}_{1/2}'$ do not coincide with $\mathcal{H}_R^+$ and $\mathcal{H}_R^-$. How is that possible? The reason is that the subrepresentations $\pi_R^+$ and $\pi_R^-$, when restricted to the observable algebra $\mathcal{A}$, become equivalent [3]. Therefore the decomposition into invariant subspaces is not unique.

Mack and Schomerus [1, 2] defined the following endomorphisms of $\text{Maj}$ which restrict to endomorphisms of the global observable algebra $\mathcal{A}$.

**Definition 2.2** The endomorphisms $\varrho_J$, $J = 0, \frac{1}{2}, 1$ of $\text{Maj}$ are defined by their action on the generators as follows,

$$\varrho_0 = \text{id},$$

(40)
$$\varrho_{1/2}(b_a) = \begin{cases} \frac{i}{\sqrt{2}}(\frac{\varrho_{+1/2}}{\varrho_{-1/2}}) & a \geq \frac{1}{2} \\ \frac{i}{\sqrt{2}}(\frac{\varrho_{-1/2}}{\varrho_{+1/2}}) & a \leq \frac{1}{2} \end{cases}, \quad \varrho_{1/2}(Y) = -Y,$$

(41)
$$\varrho_{1}(b_a) = \begin{cases} -b_a & a \neq 0, \pm \frac{1}{2} \\ b_a & a = 0, \pm \frac{1}{2} \end{cases}, \quad \varrho_{1}(Y) = Y.$$ 

(42)
It is shown that these endomorphisms fulfill

$$\pi_{NS} \circ \varrho_{1/2} \equiv \pi_R,$$

(43)
$$\pi_{NS} \circ \varrho_{1} \equiv \pi_{NS},$$

(44)
$$\pi_{0} \circ \varrho_{J} \equiv \pi_{J}, \quad J = 0, \frac{1}{2}, 1,$$

(45)

\(^1\)These subrepresentations are inequivalent. This is a consequence of Araki’s Lemma 10.3 in [3].
where relation (44) is the most trivial one because \( g_1 \) is inner in \( \text{Maj} \), implemented by the unitary selfadjoint
\[
R = \sqrt{2} b_0 + b_\frac{1}{2} + b_{-\frac{1}{2}} \in \text{Maj}.
\] (46)

We can define these endomorphisms by the formula
\[
g_J(\hat{\psi}(\hat{f})) = \hat{\psi}(\hat{V}_J \hat{f})
\] (47)
where \( \hat{V}_J \) are the following isometries on \( \hat{\mathcal{K}} = L^2(S^1) \oplus L^2(S^1) \),
\[
\hat{V}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{V}_{1/2} = \begin{pmatrix} 0 & V_{1/2} \\ V_{1/2}^* & 0 \end{pmatrix}, \quad \hat{V}_1 = \begin{pmatrix} V_1 & 0 \\ 0 & V_1^* \end{pmatrix},
\] (48)
and the isometries (Bogoliubov operators, see below) \( V_{1/2}, V_{1/2}', V_1, V_1' \in \mathcal{B}(L^2(S^1)) \) are defined as
\[
V_{1/2} = \frac{i}{\sqrt{2}} \left( |e_{\frac{1}{2}}\rangle \langle e_0| - |e_{-\frac{1}{2}}\rangle \langle e_0| \right) + i \sum_{n=1}^{\infty} \left( |e_{n+\frac{1}{2}}\rangle \langle e_n| - |e_{n-\frac{1}{2}}\rangle \langle e_n| \right),
\] (49)
\[
V_{1/2}' = i \sum_{n=1}^{\infty} \left( |e_n\rangle \langle e_{n-\frac{1}{2}}| - |e_n\rangle \langle e_{n+\frac{1}{2}}| \right),
\] (50)
\[
V_1 = |e_{\frac{1}{2}}\rangle \langle e_{-\frac{1}{2}}| + |e_{-\frac{1}{2}}\rangle \langle e_{\frac{1}{2}}| - \sum_{n=1}^{\infty} \left( |e_{n+\frac{1}{2}}\rangle \langle e_{n-\frac{1}{2}}| + |e_{n-\frac{1}{2}}\rangle \langle e_{n+\frac{1}{2}}| \right),
\] (51)
\[
V_1' = |e_0\rangle \langle e_0| - \sum_{n=1}^{\infty} \left( |e_n\rangle \langle e_n| + |e_{-n}\rangle \langle e_{-n}| \right).
\] (52)

It is worthy of note that the two non-vanishing entries, each in \( \hat{V}_{1/2} \) and \( \hat{V}_1 \) are actually different.

3 Localized Endomorphisms

3.1 Definition of Localized Endomorphisms

The endomorphisms of Definition 2.2 are global ones, i.e. there are no intervals \( I' \in \mathcal{I} \) such that \( g_J(A) = A \) for all \( A \in \mathcal{A}(I') \) and \( J = \frac{1}{2} \) or \( J = 1 \). In the following we are looking for localized endomorphisms but first we have to specify what ”localized” means on the circle. It is our aim to formulate a definition of localized endomorphisms such that they are DHR on the algebra of the punctured circle. For every \( I \in \mathcal{I} \) we denote by \( I' \) the open complement of \( I \), i.e. \( S^1 \setminus I \) without boundary points. As localization regions we consider only those intervals \( I \in \mathcal{I} \) with \( I' \neq \emptyset \).

**Definition 3.1** An endomorphism \( \varrho \) of the global observable algebra \( \mathcal{A} \) is called localized in an interval \( I \in \mathcal{I} \), \( I' \neq \emptyset \) if the following conditions hold:

1. \( \varrho(A) = A \), \( A \in \mathcal{A}(I') \),
2. \( \varrho(\mathcal{A}(I_0)) \subset \mathcal{A}(I_0) \), \( I_0 \in \mathcal{I}, I \subset I_0 \),
3. \( \varrho(Y) = Y \) or \( \varrho(Y) = -Y \)

In a theory with a faithful vacuum representation the second condition follows from the first by Haag duality, which holds on the circle [13]. But, however, since our vacuum representation is not faithful we have to require this property in addition. The third property will be used for proving the important
Lemma 3.2 Let $\varrho$ and $\varrho'$ be localized endomorphisms in $I \in \mathcal{I}$ ($I' \neq \emptyset$) and let $R_0 \in \mathcal{B}(\mathcal{H}_0)$ an intertwiner between $\pi_0 \circ \varrho$ and $\pi_0 \circ \varrho'$, i.e.

$$R_0 \pi_0 \circ \varrho(A) = \pi_0 \circ \varrho'(A) R_0, \quad A \in \mathcal{A}. \quad (53)$$

Then there exists a unique $R \in \mathcal{A}(I)$ with $\pi_0(R) = R_0$, which intertwiners between $\varrho$ and $\varrho'$, i.e.

$$R \varrho(A) = \varrho'(A) R, \quad A \in \mathcal{A}. \quad (54)$$

Proof. For $A = Y$ equation $(53)$ implies $\varrho(Y) = \varrho'(Y)$ if $R_0 \neq 0$. Now assume $A \in \mathcal{A}(I')$, then we have

$$R_0 \pi_0(A) = \pi_0(A) R_0$$

and by Haag duality we get

$$R_0 \in \pi_0(\mathcal{A}(I'))' = \pi_0(\mathcal{A}(I))$$

Since $\pi_0$ is faithful on $\mathcal{A}(I)$ there exists a unique $R \in \mathcal{A}(I)$ with $\pi_0(R) = R_0$. Now choose an arbitrary point $\xi \in S^1$ and consider $I_\xi = S^1 \setminus \{\xi\}$. Equation $(53)$ yields for all $A \in \mathcal{A}(I_\xi)$

$$R_0 \pi_0(\varrho(A)) = \pi_0(R \varrho(A)) = \pi_0(\varrho'(A) R) = \pi_0(\varrho'(A)) R_0$$

and since $\pi_0$ is faithful on $\mathcal{A}(I_\xi)$, too,

$$R \varrho(A) = \varrho'(A) R, \quad A \in \mathcal{A}(I_\xi).$$

The global algebra $\mathcal{A}$ is generated by $\mathcal{A}(I_\xi)$ and $Y$, but since $\varrho(Y) = \varrho'(Y) = \pm Y$ if $R_0 \neq 0$ this establishes $(54)$, and if $R_0 = 0$ then $R = 0$, q.e.d.

Our localized endomorphisms are restrictions of endomorphisms of the universal Majorana algebra $\operatorname{Maj}$, which are also defined by isometries on $\hat{\mathcal{K}}$. These isometries are built of Bogoliubov operators $V \in O^0(L^2(S^1), \Gamma)$ where

$$O^0(L^2(S^1), \Gamma) = \{V \in \mathcal{B}(L^2(S^1)) \mid V^* V = 1, [V, \Gamma] = 0\}. \quad (55)$$

Consider now intervals $I \in \mathcal{I}$ with $\zeta \in I'$. Our fixed reference point $\zeta$ divides the open complement $I'$ into two disjoint intervals $I_+, I_- \in \mathcal{I}$. A Bogoliubov operator $V \in O^0(L^2(S^1), \Gamma)$ is called pseudolocalized in $I$ if for all $f \in L^2(S^1)$

$$(V f)(z) = \sigma_+ f(z), \quad z \in I_+, \quad \sigma_+, \sigma_- \in \{1, -1\} \quad (56)$$

holds. A pseudolocalized Bogoliubov operator is called even if $\sigma_+ = \sigma_-$, and odd if $\sigma_+ = -\sigma_-$. 

Lemma 3.3 (A) Let $\hat{V} \in \mathcal{B}(\hat{\mathcal{K}})$ be an isometry defined by

$$\hat{V} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}, \quad V \in O^0(L^2(S^1), \Gamma) \quad (57)$$

where $V$ is even pseudolocalized in $I$, and let $\varrho$ be the endomorphism of $\operatorname{Maj}$ defined by

$$\varrho(\hat{V} f) = \hat{V} \varrho(f), \quad \varrho(Y) = Y. \quad (58)$$

Then, in the restriction on $\mathcal{A}$, $\varrho$ is localized in $I$.

(B) Let $\hat{V} \in \mathcal{B}(\hat{\mathcal{K}})$ be an isometry defined by

$$\hat{V} = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}, \quad V \in O^0(L^2(S^1), \Gamma) \quad (59)$$

where $V$ is odd pseudolocalized in $I$, and let $\varrho$ be the endomorphism of $\operatorname{Maj}$ defined by

$$\varrho(\hat{V} f) = \hat{V} \varrho(f), \quad \varrho(Y) = -Y. \quad (60)$$

Then, in the restriction on $\mathcal{A}$, $\varrho$ is localized in $I$. 


Proof. At first we show that condition 1 of Definition 3.1 is satisfied. We have to show that 
\( g \hat{\varphi}(B_{f}(f, g)) = B_{f}(f, g) \). We choose arbitrary \( f, g \in L^{2}(S^{1}) \) with support in \( P \). Let \( \chi_{+}, \chi_{-} \) be the characteristic functions of the intervals \( I_{+}, I_{-} \) and \( f_{\pm} = \chi_{\pm} f, g_{\pm} = \chi_{\pm} g, f_{\pm} = f_{\pm} \oplus f_{\pm} \in K \), \( g_{\pm} = g_{\pm} \oplus g_{\pm} \in K \). Then we have

\[
B_{f}(f, g) = \hat{\varphi}(\hat{f}_{+}) \hat{\varphi}(\hat{g}_{+}) + \hat{\varphi}(\hat{f}_{-}) \hat{\varphi}(\hat{g}_{-}) + Y \hat{\varphi}(\hat{f}_{+}) \hat{\varphi}(\hat{g}_{-}) + Y \hat{\varphi}(\hat{f}_{-}) \hat{\varphi}(\hat{g}_{+}).
\]

We have to distinguish the two cases (A) and (B).

(A). \( g \hat{\varphi}(Y) = Y, \sigma_{+} \sigma_{-} = 1; \)

\[
g \hat{\varphi}(B_{f}(f, g)) = \\
= \hat{\varphi}(\hat{V} \hat{f}_{+}) \hat{\varphi}(\hat{V} \hat{g}_{+}) + \hat{\varphi}(\hat{V} \hat{f}_{-}) \hat{\varphi}(\hat{V} \hat{g}_{-}) + Y \hat{\varphi}(\hat{V} \hat{f}_{+}) \hat{\varphi}(\hat{V} \hat{g}_{-}) + Y \hat{\varphi}(\hat{V} \hat{f}_{-}) \hat{\varphi}(\hat{V} \hat{g}_{+}) \\
= \sigma_{+}^{2} \hat{\varphi}(\hat{f}_{+}) \hat{\varphi}(\hat{g}_{+}) + \sigma_{-}^{2} \hat{\varphi}(\hat{f}_{-}) \hat{\varphi}(\hat{g}_{-}) + \sigma_{+} \sigma_{-} Y \hat{\varphi}(\hat{f}_{+}) \hat{\varphi}(\hat{g}_{-}) + \sigma_{-} \sigma_{+} Y \hat{\varphi}(\hat{f}_{-}) \hat{\varphi}(\hat{g}_{+}) \\
= B_{f}(f, g).
\]

(B). \( g \hat{\varphi}(Y) = -Y, \sigma_{+} \sigma_{-} = -1; \)

\[
g \hat{\varphi}(B_{f}(f, g)) = \\
= \hat{\varphi}(\hat{V} \hat{f}_{+}) \hat{\varphi}(\hat{V} \hat{g}_{+}) + \hat{\varphi}(\hat{V} \hat{f}_{-}) \hat{\varphi}(\hat{V} \hat{g}_{-}) - Y \hat{\varphi}(\hat{V} \hat{f}_{+}) \hat{\varphi}(\hat{V} \hat{g}_{-}) - Y \hat{\varphi}(\hat{V} \hat{f}_{-}) \hat{\varphi}(\hat{V} \hat{g}_{+}) \\
= \sigma_{+}^{2} \hat{\varphi}(\hat{f}_{+}) \hat{\varphi}(\hat{g}_{+}) + \sigma_{-}^{2} \hat{\varphi}(\hat{f}_{-}) \hat{\varphi}(\hat{g}_{-}) - \sigma_{+} \sigma_{-} Y \hat{\varphi}(\hat{f}_{+}) \hat{\varphi}(\hat{g}_{-}) + \sigma_{-} \sigma_{+} Y \hat{\varphi}(\hat{f}_{-}) \hat{\varphi}(\hat{g}_{+}) \\
= B_{f}(f, g).
\]

It remains to show that condition 2 of Definition 3.1 is satisfied. We consider an interval \( I_{0} \in \mathcal{I}, I \subset I_{0}. \) We assume that \( \zeta \in I_{0}. \) The point \( \zeta \) divides \( I_{0} \) in two intervals \( I_{1}, I_{2} \in \mathcal{I} \) and we call that one as \( I_{1} \) which contains \( I, \) i.e. \( I \subset I_{1}. \) Again we denote by \( \chi_{1}, \chi_{2} \) the characteristic function of \( I_{1}, I_{2} \) and \( f_{j} = \chi_{j} f, f_{j} = f_{j} \oplus f_{j} \in K, j = 1, 2 \), the same for \( g. \) Then we have

\[
B_{I_{0}}(f, g) = \hat{\varphi}(\hat{f}_{1}) \hat{\varphi}(\hat{g}_{1}) + \hat{\varphi}(\hat{f}_{2}) \hat{\varphi}(\hat{g}_{2}) + Y \hat{\varphi}(\hat{f}_{1}) \hat{\varphi}(\hat{g}_{2}) + Y \hat{\varphi}(\hat{f}_{2}) \hat{\varphi}(\hat{g}_{1}).
\]

(A) \( g \hat{\varphi}(Y) = Y, \)

\[
g \hat{\varphi}(B_{I_{0}}(f, g)) = \hat{\varphi}(\hat{V} \hat{f}_{1}) \hat{\varphi}(\hat{V} \hat{g}_{1}) + \hat{\varphi}(\hat{V} \hat{f}_{2}) \hat{\varphi}(\hat{V} \hat{g}_{2}) + Y \hat{\varphi}(\hat{V} \hat{f}_{1}) \hat{\varphi}(\hat{V} \hat{g}_{2}) + Y \hat{\varphi}(\hat{V} \hat{f}_{2}) \hat{\varphi}(\hat{V} \hat{g}_{1}).
\]

But \( \hat{V} f_{j} = V f_{j} \oplus V f_{j}, \hat{V} g_{j} = V g_{j} \oplus V g_{j} \) and since \( I \subset I_{1}, I \cap I_{2} = \emptyset \) we find \( \text{supp}(V f_{j}), \text{supp}(V g_{j}) \subset I_{j}, j = 1, 2. \) Therefore

\[
g \hat{\varphi}(B_{I_{0}}(f, g)) = B_{I_{0}}(V f, V g) \in \mathcal{A}(I_{0}).
\]

(B) \( g \hat{\varphi}(Y) = -Y, \)

\[
g \hat{\varphi}(B_{I_{0}}(f, g)) = \hat{\varphi}(\hat{V} \hat{f}_{1}) \hat{\varphi}(\hat{V} \hat{g}_{1}) + \hat{\varphi}(\hat{V} \hat{f}_{2}) \hat{\varphi}(\hat{V} \hat{g}_{2}) \\
= -Y \hat{\varphi}(\hat{V} \hat{f}_{1}) \hat{\varphi}(\hat{V} \hat{g}_{2}) - Y \hat{\varphi}(\hat{V} \hat{f}_{2}) \hat{\varphi}(\hat{V} \hat{g}_{1}) \\
= \hat{\varphi}(\hat{V} \hat{f}_{1}) \hat{\varphi}(\hat{V} \hat{g}_{1}) + \hat{\varphi}(\hat{V} \hat{f}_{2}) \hat{\varphi}(\hat{V} \hat{g}_{2}) \\
+ Y \hat{\varphi}(\hat{V} \hat{f}_{1}) \hat{\varphi}(\hat{V} \hat{g}_{2}) + Y \hat{\varphi}(\hat{V} \hat{f}_{2}) \hat{\varphi}(\hat{V} \hat{g}_{1}) \\
= B_{I_{0}}(V(f_{1} - f_{2}), V(g_{1} - g_{2})) \in \mathcal{A}(I_{0}).
\]

If \( \zeta \notin I_{0} \) this is the special case with \( f_{2} = g_{2} = 0, \) q.e.d.
3.2 Examples of Localized Endomorphisms

We now present examples of endomorphisms which are localized in the sense of the above definition. It will be our next aim to identify the equivalence class of representations to which they lead.

**Definition 3.4** We define the following Bogoliubov operator $W \in O^b(L^2(S^1), \Gamma)$,

$$W = |h\rangle\langle h| - 1,$$

where $h \in L^2(S^1)$ is a real function with $\|h\|^2 = 2$ and $\text{supp}(h) \subset I$ for an interval $I \in \mathcal{I}$, $\zeta \in P$. Let $\hat{W} \in \mathcal{B}(\hat{\mathcal{K}})$ be the isometry defined by

$$\hat{W} = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix},$$

and let $\varphi_1^{loc}$ be the automorphism of $\text{Maj}$ defined by

$$\varphi_1^{loc}(\hat{\psi}(\hat{f})) = \hat{\psi}(\hat{W} \hat{f}), \quad \varphi_1^{loc}(Y) = Y.$$  \hfill (63)

Because $W^2 = 1$ and therefore $(\varphi_1^{loc})^2 = id$ it is $\varphi_1^{loc}$ really an automorphism. Since $W$ is even pseudolocalized in $I$ and the conditions of Lemma 3.3 are satisfied we have the following

**Corollary 3.5** The automorphism of Definition 3.4 is in the restriction on the observable algebra $\mathcal{A}$ localized in $I$.

Consider now the unitary, selfadjoint $\hat{\psi}(\hat{h}) \in \text{Maj}$ where $\hat{h} = h \oplus h \in \hat{\mathcal{K}}$ and take into account that

$$\frac{1}{2}(1 - Y)\hat{\psi}(\hat{h}) = \hat{\psi}(h \oplus 0), \quad \frac{1}{2}(1 + Y)\hat{\psi}(\hat{h}) = \hat{\psi}(0 \oplus h).$$

It follows that for arbitrary $\hat{f} = f_{NS} \oplus f_R \in \hat{\mathcal{K}}$ one finds

$$\hat{\psi}(\hat{h})\hat{\psi}(\hat{f})\hat{\psi}(\hat{h}) = \{\hat{\psi}(\hat{h}), \hat{\psi}(\hat{f})\}\hat{\psi}(\hat{h}) - \hat{\psi}(\hat{f})\hat{\psi}(\hat{h})\hat{\psi}(\hat{h})$$

$$= \left(\frac{1}{2}(1 - Y)\langle h, f_{NS} \rangle + \frac{1}{2}(1 + Y)\langle h, f_{R} \rangle\right)\hat{\psi}(\hat{h}) - \hat{\psi}(\hat{f})$$

$$= \hat{\psi}(\langle h, f_{NS} \rangle h) - \hat{\psi}(\hat{f})$$

$$= \hat{\psi}(\langle W f_{NS} \rangle h) - \hat{\psi}(\hat{f}).$$

We conclude that the automorphism $\varphi_1^{loc}$ is inner in $\text{Maj}$. Since the global automorphism $\varphi_1$ was inner, implemented by the unitary, selfadjoint $R = \sqrt{2}b_0 + b_\frac{1}{2} + b_{-\frac{1}{2}}$ we can define the unitary

$$R_1 = R\hat{\psi}(\hat{h}) \in \mathcal{A}$$

so that

$$\varphi_1^{loc}(A) = R_1^*\varphi_1(A)R_1, \quad A \in \mathcal{A}.$$  \hfill (65)

The unitary $\pi_0(R_1)$ then realizes the equivalence of the representations $\pi_0 \circ \varphi_1^{loc}$ and $\pi_0 \circ \varphi_1$. We have proven

**Lemma 3.6** The automorphism $\varphi_1^{loc}$ of Definition 3.4 is inner in $\text{Maj}$. In the restriction to the observable algebra $\mathcal{A}$ it leads to

$$\pi_0 \circ \varphi_1^{loc} \cong \pi_1.$$  \hfill (66)
In the following we are searching for a localized endomorphism \( g_{1/2}^{\text{loc}} \) which leads to a representation being equivalent to \( \pi_{1/2} \). It turns out that the discussion becomes much more complicated. First we fix our point \( \zeta \) to be \( \zeta = -1 \), without loss of generality. Further, we choose the localization region \( I \) to be \( I_2 \),

\[
I_2 = \left\{ z = e^{i\phi} \in S^1 \left| -\frac{\pi}{2} < \phi < \frac{\pi}{2} \right. \right\}
\]

so that the open complement \( I_1 \) is divided by \( \zeta \) into \( I_- \) and \( I_+ \),

\[
I_- = \left\{ z = e^{i\phi} \in S^1 \left| -\pi < \phi < -\frac{\pi}{2} \right. \right\}, \quad I_+ = \left\{ z = e^{i\phi} \in S^1 \left| \frac{\pi}{2} < \phi < \pi \right. \right\}.
\]

The Hilbert space \( L^2(S^1) \) decomposes into a direct sum,

\[
L^2(S^1) = L^2(I_-) \oplus L^2(I_2) \oplus L^2(I_+).
\]

By \( P_{I_+}, P_{I_-} \) we denote the projections on the subspaces \( L^2(I_+), L^2(I_-) \), respectively. Define functions on \( S^1 \) by

\[
e^{(2)}_{a}(z) = \begin{cases} 
\sqrt{2}z^a & z \in I_2 \\
0 & z \in S^1 \setminus I_2
\end{cases}, \quad a \in \frac{1}{2} \mathbb{Z}.
\]

With

\[
\{ e^{(2)}_r, r \in \mathbb{Z} + \frac{1}{2} \}, \quad \{ e^{(2)}_n, n \in \mathbb{Z} \}
\]

we then obtain two orthonormal bases of the subspace \( L^2(I_2) \subset L^2(S^1) \).

**Definition 3.7** We define\(^2\) the following Bogoliubov operators \( V, V' \in O^0(L^2(S^1), \Gamma) \),

\[
V = P_{I_-} - P_{I_+} + i \sqrt{2} \left( k^{(2)}_{-\frac{\pi}{2}} \langle e^{(2)}_0 | e^{(2)}_0 \rangle - k^{(2)}_{\frac{\pi}{2}} \langle e^{(2)}_0 | e^{(2)}_0 \rangle + i \sum_{n=1}^\infty \left( k^{(2)}_{n+\frac{\pi}{2}} \langle e^{(2)}_n | e^{(2)}_n \rangle - k^{(2)}_{-n-\frac{\pi}{2}} \langle e^{(2)}_{-n} | e^{(2)}_{-n} \rangle \right) \right), \quad (71)
\]

\[
V' = P_{I_-} - P_{I_+} + i \sum_{n=1}^\infty \left( k^{(2)}_{n+\frac{\pi}{2}} \langle e^{(2)}_n | e^{(2)}_n \rangle - k^{(2)}_{-n-\frac{\pi}{2}} \langle e^{(2)}_{-n} | e^{(2)}_{-n} \rangle \right), \quad (72)
\]

Let \( \hat{V}, \hat{V}' \in \mathcal{B}(\mathcal{K}) \) be the isometries defined by

\[
\hat{V} = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}, \quad \hat{V}' = \begin{pmatrix} 0 & V' \\ V' & 0 \end{pmatrix},
\]

and let \( g_{1/2}^{\text{loc}} \) and \( \sigma_{1/2}^{\text{loc}} \) be the endomorphisms of \( \text{Maj} \) defined by

\[
\begin{align*}
\hat{g}_{1/2}^{\text{loc}}(\hat{\psi}(\hat{f})) &= \hat{\psi}(\hat{V}\hat{f}), \quad \hat{g}_{1/2}^{\text{loc}}(Y) = -Y, \\
\hat{\sigma}_{1/2}^{\text{loc}}(\hat{\psi}(\hat{f})) &= \hat{\psi}(\hat{V}'\hat{f}), \quad \hat{\sigma}_{1/2}^{\text{loc}}(Y) = -Y.
\end{align*}
\]

Since \( V \) and \( V' \) are obviously odd pseudolocalized in \( I_2 \) and the conditions of Lemma 3.3 are fulfilled, we have the following

**Corollary 3.8** The endomorphisms \( g_{1/2}^{\text{loc}} \) and \( \sigma_{1/2}^{\text{loc}} \) of Definition 3.7 are in the restriction to the observable algebra \( \mathcal{A} \) localized in \( I_2 \).

The main result of this paper is the following

**Theorem 3.9** In the restriction to the observable algebra the endomorphisms \( g_{1/2}^{\text{loc}} \) and \( \sigma_{1/2}^{\text{loc}} \) lead to

\[
\begin{align*}
\pi_0 \circ g_{1/2}^{\text{loc}} &\cong \pi_{1/2}, \\
\pi_0 \circ \sigma_{1/2}^{\text{loc}} &\cong \pi_{1/2}.
\end{align*}
\]

The proof will be elaborated in the following sections.

\(^2\)The definition of \( V' \) was already given by Mack and Schomerus [1].
4 The Selfdual CAR Algebra and Quasifree States

4.1 Repetition of Some Well-known Results

For a better handling of our technics we give a brief repetition of Araki's selfdual CAR algebra $\mathcal{C}(\mathcal{K}, \Gamma)$ and quasifree states [3, 4]. We consider a Hilbert space $\mathcal{K}$ with an antiunitary involution $\Gamma$ (complex conjugation), $\Gamma^2 = 1$, which fulfills

$$\langle \Gamma f, \Gamma g \rangle = \langle g, f \rangle, \quad f, g \in \mathcal{K}. $$

The selfdual CAR algebra $\mathcal{C}(\mathcal{K}, \Gamma)$ is defined to be the $C^*$ norm closure of the algebra which is generated by the image of the linear mapping $\psi$, which maps elements $f \in \mathcal{K}$ to canonical generators $\psi(f)$, so that

$$\psi(f)^* = \psi(\Gamma f), \quad \{\psi(f)^*, \psi(g)\} = \langle f, g \rangle \mathbf{1}$$

holds. The elements of the set

$$O^0(\mathcal{K}, \Gamma) = \{V \in \mathcal{B}(\mathcal{K}) | [V, \Gamma] = 0, \ V^* V = 1\}$$

of $\Gamma$ commuting isometries on $\mathcal{K}$ are called Bogoliubov operators. Every Bogoliubov operator $V \in O^0(\mathcal{K}, \Gamma)$ defines an endomorphism $\varrho_V$ of $\mathcal{C}(\mathcal{K}, \Gamma)$, defined by its action on the canonical generators,

$$\varrho_V(\psi(f)) = \psi(V f).$$

Moreover, if $V \in O^0(\mathcal{K}, \Gamma)$ is surjective (i.e. unitary), then $\varrho_V$ is an automorphism.

**Definition 4.1** A state $\varphi$ of $\mathcal{C}(\mathcal{K}, \Gamma)$ is called quasifree, if for all $n \in \mathbb{N}$

\begin{align}
\varphi(\psi(f_1) \cdots \psi(f_{2n+1})) &= 0, \quad (88) \\
\varphi(\psi(f_1) \cdots \psi(f_{2n})) &= (-1)^{\frac{n(n+1)}{2}} \sum_{\sigma} \text{sign} \sigma \prod_{j=1}^{n} \varphi(\psi(f_{\sigma(i)})\psi(f_{\sigma(i+1)})) \quad (79)
\end{align}

holds. The sum runs over all permutations $\sigma \in S_{2n}$ with the property

$$\sigma(1) < \sigma(2) < \cdots < \sigma(n), \quad \sigma(j) < \sigma(j + n), \quad j = 1, \ldots, n. \quad (80)$$

Quasifree states are therefore completely characterized by their two point function. It is known that there is a one to one correspondence between the set of quasifree states and the set

$$Q(\mathcal{K}, \Gamma) = \{S \in \mathcal{B}(\mathcal{K}) | S = S^*, \ 0 \leq S \leq 1, \ S + \Gamma S \Gamma = 1\}, \quad (81)$$

given by the formula

$$\varphi(\psi(f)\psi(g)) = \langle f, S g \rangle. \quad (82)$$

The quasifree state characterized by (82) is denoted by $\varphi_S$. A quasifree state, composed with a Bogoliubov endomorphism is again a quasifree state, namely we have $\varphi_S \circ \varrho_V = \varphi_{V^* S V}$. The projections in $Q(\mathcal{K}, \Gamma)$ are called basis projections. If $P$ is a basis projection then the state $\varphi_P$ is pure and is called a Fock state. The corresponding GNS representation $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ is irreducible, it is called the Fock representation; the vector $|\Omega_P\rangle \in \mathcal{H}_P$ is called the Fock vacuum. Araki gives us [3] the following

**Lemma 4.2** Let $P$ be a basis projection and $\varphi$ be a state of $\mathcal{C}(\mathcal{K}, \Gamma)$ which satisfies

$$\varphi(\psi(f)\psi(f)^*) = 0, \quad f \in PK. \quad (83)$$

Then $\varphi$ is the Fock state $\varphi = \varphi_P$. 

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We now come to an important quasi-equivalence criterion for quasifree states. It was developed for the case of gauge invariant quasifree states by Powers and Størmer [9] and generalized for arbitrary quasifree states by Araki [3]. Unitary equivalence (denoted by \( \cong \)) or quasi-equivalence (denoted by \( \approx \)) of states means always that the corresponding GNS representations are unitarily equivalent or quasi-equivalent, respectively.

**Theorem 4.3** Two quasifree states \( \phi_s \) and \( \phi_{s'} \) of \( \mathcal{C}(K, \Gamma) \) are quasi-equivalent, if and only if
\[
S_s^\frac{1}{2} - S_{s'}^\frac{1}{2} \in \mathcal{J}_2(K),
\]
where \( \mathcal{J}_2(K) \) denotes the ideal of Hilbert Schmidt operators in \( \mathcal{B}(K) \).

We now can conclude that two Fock states \( \phi_{P_1} \) and \( \phi_{P_2} \) are unitary equivalent, if and only if \( P_1 - P_2 \) is Hilbert Schmidt class, or, if \( \phi_P \) is a Fock state and \( \varrho_{V_1} \) and \( \varrho_{V_2} \) are Bogoliubov endomorphisms, that \( \varrho_{V_1} \circ \varrho_{V_2} \approx \varrho_{V_1} \circ \varrho_{V_2} \) if and only if \( (V_1^* P V_2)^{\frac{1}{2}} - (V_2^* P V_1)^{\frac{1}{2}} \) is Hilbert Schmidt class. But in the most cases we study representations of the form \( \pi_P \circ \varrho_V \), where \( \varrho_V \) is a Bogoliubov endomorphism and \( \pi_P \) a Fock representation of \( \mathcal{C}(K, \Gamma) \). In general, \( \pi_P \circ \varrho_V \) is not cyclic, i.e. not a GNS representation of the state \( \varrho_{V^* P V} = \varrho_P \circ \varrho_V \), but a multiple of those ones. Rideau [10] gives a criterion which controls these multiplicities in the case of CAR, in [21] we find a formulation for the case of the self-dual CAR algebra.

**Theorem 4.4** Let \( \phi_P \) be a Fock state, \( (\mathcal{H}_P, \pi_P, \Omega_P) \) the corresponding Fock representation and \( \varrho_V \) a Bogoliubov endomorphism of \( \mathcal{C}(K, \Gamma) \). Define
\[
N_U = \dim(\ker U^* \cap PK).
\]
Then we have the decomposition
\[
\pi_P \circ \varrho_V = \bigoplus_{j=1}^{N_U} \pi_{U^* P V}^{[j]},
\]
where \( \pi_{U^* P V}^{[j]} \) are GNS representations of the state \( \varrho_{U^* P V} = \varrho_{U^* P V} \).

### 4.2 A Useful Theorem

The decomposition (86) in Theorem 4.4 is in general not a decomposition into irreducibles; the GNS representations \( \pi_{U^* P V}^{[j]} \) are in general not irreducible because \( \varrho_{U^* P V} \) is in general not pure. Into how many irreducibles decomposes the representation \( \pi_P \circ \varrho_V \)? The following theorem gives an answer for the case that the cokernel of the Bogoliubov operator \( U \) is two-dimensional.

**Theorem 4.5** Let \( \phi_P \) be a Fock state and \( (\mathcal{H}_P, \pi_P, \Omega_P) \) the corresponding Fock representation of \( \mathcal{C}(K, \Gamma) \). Let \( U \in O^2(K, \Gamma) \) be a Bogoliubov operator with \( \dim \ker U^* = 2 \). Let \( \{ \epsilon_+, \epsilon_- \} \) be an orthonormal basis of \( \ker U^* \) with the property that \( \epsilon_+ = \Gamma \epsilon_- \). If both numbers
\[
\alpha_\pm = \langle \epsilon_\pm, P \epsilon_\pm \rangle \neq 0,
\]
then the state \( \varrho_{U^* P V} \) is a mixture of two Fock states \( \varphi_\pm \),
\[
\varphi_P \circ \varrho_V = \alpha_+ \varphi_+ + \alpha_- \varphi_-.
\]
Moreover, the representation \( \pi_P \circ \varrho_V \) is cyclic and decomposes therefore into a direct sum of two irreducible representations.
Proof. On the Fock space $\mathcal{H}_P$ we define the projections
\begin{align}
\Pi_+ &= \pi_P(\psi(\epsilon_+)^*\psi(\epsilon_+)) = \pi_P(\psi(\epsilon_+)\psi(\epsilon_+)), \\
\Pi_- &= 1 - \Pi_+ = \pi_P(\psi(\epsilon_+)\psi(\epsilon_-)).
\end{align}
(89)
(90)
Since the orthonormal vectors $\epsilon_+, \epsilon_-$ span the kernel of $U_*$ (i.e. the cokernel of $U$), the projections $\Pi_+ , \Pi_- \in \pi_P \circ \rho_U(\mathcal{C}(\mathcal{K}, \Gamma))$,
\begin{equation}
\Pi_+ \in \pi_P \circ \rho_U(\mathcal{C}(\mathcal{K}, \Gamma))^\perp.
\end{equation}
Since $\alpha_+ \neq 0$ we have the well defined, normed vectors in $\mathcal{H}_P$,
\begin{equation}
|\Omega_\pm\rangle = \alpha_\pm^{-2}\Pi_\pm|\Omega_P\rangle.
\end{equation}
(91)
Since $P$ is a basis projection we have $0 < \alpha_\pm \leq 1$, moreover
\begin{equation}
\alpha_+ + \alpha_- = \langle \epsilon_+, Pe_+ \rangle + \langle \epsilon_-, Pe_- \rangle = \langle \epsilon_+, Pe_+ \rangle + \langle \Gamma e_+, P\Gamma e_+ \rangle = \langle (P + \Gamma P)\epsilon_+, \epsilon_+ \rangle = 1
\end{equation}
We define the states on $\mathcal{C}(\mathcal{K}, \Gamma)$ by
\begin{equation}
\varphi_\pm(x) = \langle \Omega_\pm|\pi_P \circ \rho_U(x)|\Omega_\pm\rangle = \alpha_\pm^{-1}\langle \Omega_P|\pi_P \circ \rho_U(x)\Pi_\pm|\Omega_P\rangle, \quad x \in \mathcal{C}(\mathcal{K}, \Gamma),
\end{equation}
(92)
such that we find
\begin{equation}
\varphi_P \circ \rho_U = \alpha_+ \varphi_+ + \alpha_- \varphi_-
\end{equation}
by $\Pi_+ + \Pi_- = 1$. We are able to compute the two point functions of $\varphi_+$ and $\varphi_-$ by reading the permutation formula (79),(80) for the quasifree state $\varphi_P$,
\begin{equation}
\varphi_\pm(\psi(f)\psi(g)) = \alpha_\pm^{-1}\langle \Omega_P|\pi_P \circ \rho_U(\psi(f)\psi(g))\pi_P(\psi(\epsilon_+)\psi(\epsilon_+))|\Omega_P\rangle
= \alpha_\pm^{-1}\varphi_P(\psi(Uf)\psi(Ug))\varphi_P(\psi(\epsilon_+))
= \alpha_\pm^{-1}\varphi_P(\psi(Uf)\psi(Ug))\varphi_P(\psi(\epsilon_+))
+ \alpha_\pm^{-1}\varphi_P(\psi(Uf)\psi(\epsilon_+))\varphi_P(\psi(Ug)\psi(\epsilon_+))
- \alpha_\pm^{-1}\varphi_P(\psi(Uf)\psi(\epsilon_+))\varphi_P(\psi(Ug)\psi(\epsilon_+))
= \langle \Gamma f, U^*Pu \rangle \varphi_P(\psi(Uf)\psi(Ug))
- \alpha_\pm^{-1}\langle \Gamma f, U^*Pe \rangle \varphi_P(\psi(Uf)\psi(Ug))
- \alpha_\pm^{-1}\langle \Gamma g, U^*Pe \rangle \varphi_P(\psi(Uf)\psi(Ug))
- \alpha_\pm^{-1}\langle \Gamma f, U^*Pe \rangle \varphi_P(\psi(Uf)\psi(Ug)).
\end{equation}
Since $U^*\epsilon_\pm = 0$ and $[U, \Gamma] = 0$ we find
\begin{equation}
\langle \Gamma g, U^*Pe \rangle = \langle U^*\Gamma Pe, g \rangle = \langle U^*(\Gamma - \Gamma P)e_\pm, g \rangle = -(U^*Pe_\mp, g) = -(\epsilon_\mp, Pu g).
\end{equation}
Hence we can write
\begin{equation}
\varphi_\pm(\psi(f)\psi(g)) = \langle \Gamma f, P \pm g \rangle,
\end{equation}
(93)
where
\begin{equation}
P_\pm = U^*Pu + \alpha_\pm^{-2}U^*P(\epsilon_\mp - E_\pm)PU, \quad E_\pm = |\epsilon_\pm\rangle\langle \epsilon_\pm|.
\end{equation}
(94)
Using $U^*E_\pm = 0 = E_\pm U$ and $\Gamma E_\pm = E_\mp \Gamma$ one finds easily the relation
\begin{equation}
P_\pm + \Gamma P_\pm \Gamma = 1.
\end{equation}
In the next step we show that $P_\pm^2 = P_\pm$ i.e. that $P_+$ and $P_-$ are basis projections. For simplicity we check at first only the case $P_+^2 = P_+$. We begin with some helpful formulae. Since $E_+ + E_-$ is the projection on the kernel of $U_*$ we have
\begin{equation}
UU^* = 1 - E_- - E_+.
\end{equation}
(95)
By \( \alpha_\pm = \langle \epsilon_\pm, P\epsilon_\pm \rangle \) we get
\[
E_\pm P E_\pm = \alpha_\pm E_\pm,
\]
and since
\[
\langle \epsilon_+, P\epsilon_- \rangle = \langle \epsilon_+, (1 - \Gamma \Delta)\epsilon_- \rangle = -\langle \epsilon_+, \Gamma P\epsilon_+ \rangle = -\langle P\epsilon_+, \Gamma \epsilon_+ \rangle = -\langle \epsilon_+, P\epsilon_- \rangle
\]
we find \( \langle \epsilon_\pm, P\epsilon_\mp \rangle = 0 \) and hence
\[
E_+ P E_- = E_- P E_+ = 0.
\]  
Define
\[
P_{+, 1} = U^* P U, \quad P_{+, 2} = -\alpha_+^{-1} U^* P E_+ P U, \quad P_{+, 3} = \alpha_+^{-1} U^* P E_- P U
\]
such that
\[
P_+ = P_{+, 1} + P_{+, 2} + P_{+, 3}.
\]
We obtain the following list of products by using (95), (96), and (97).
\[
P_{+, j} = U^* P U - U^* P E_+ P U - U^* P E_- P U,
\]
\[
P_{+, 1} P_{+, 2} = (1 - \alpha_+^{-1}) U^* P E_+ P U,
\]
\[
P_{+, 1} P_{+, 3} = \alpha_+^{-1} (1 - \alpha_-) U^* P E_- P U,
\]
\[
P_{+, 2} P_{+, 1} = (1 - \alpha_+^{-1}) U^* P E_+ P U,
\]
\[
P_{+, 2}^2 = (\alpha_+^{-1} - 1) U^* P E_+ P U,
\]
\[
P_{+, 2} P_{+, 3} = 0,
\]
\[
P_{+, 3} P_{+, 1} = \alpha_+^{-1} (1 - \alpha_-) U^* P E_- P U,
\]
\[
P_{+, 3}^2 = \alpha_+^{-3} \alpha_- (1 - \alpha_-) U^* P E_- P U.
\]
By using only \( \alpha_+ + \alpha_- = 1 \) we compute
\[
P_+^2 = \sum_{k,l=1}^{3} P_{+, k} P_{+, l}
\]
\[
= U^* P U + (-1 + 1 - \alpha_+^{-1} - 1 + \alpha_+^{-1} + \alpha_-^{-1} - 1) U^* P E_+ P U + (-1 + 1 + \alpha_+^{-1} - 1) U^* P E_- P U
\]
\[
= U^* P U - \alpha_+^{-1} U^* P E_+ P U + \alpha_-^{-1} U^* P E_- P U
\]
\[
= P_+.
\]
By interchanging all + and − indices this reads \( P_+^2 = P_- \). We have proven that \( P_+ \) and \( P_- \) are both basis projections, hence the states \( \varphi_+ \) and \( \varphi_- \) satisfy
\[
\varphi_\pm (\psi(f)\psi(f)^*) = 0, \quad f \in P_\pm K,
\]
and therefore they are Fock states \( \varphi_\pm = \varphi_{P_\pm} \) by Lemma 4.2.

It remains to prove that \( \pi_P \circ \varrho_V \) is cyclic, i.e. a GNS representation of the state \( \varphi_P \circ \varrho_V \). This is done with the help of Theorem 4.4: We show that the dimension of \( \ker U^* \cap PK \) is zero. Suppose there is a \( v \in PK \) with \( U^* v = 0 \). Since the kernel of \( U^* \) is spanned by \( \epsilon_+ \) and \( \epsilon_- \) we have to write
\[
v = \lambda_+ \epsilon_+ + \lambda_- \epsilon_-,
\]
\( \lambda_\pm \in \mathbb{C} \).

Since \( v \in PK \) it follows
\[
\langle \Gamma v, P \Gamma v \rangle = 0.
\]
But \( \Gamma v = \lambda_+ \epsilon_+ + \lambda_- \epsilon_- \), and by \( \langle \epsilon_+, P \epsilon_- \rangle = 0 \) this reads
\[
|\lambda_+|^2 \langle \epsilon_+, P \epsilon_- \rangle + |\lambda_-|^2 \langle \epsilon_+, P \epsilon_+ \rangle = |\lambda_+|^2 \alpha_- + |\lambda_-|^2 \alpha_+ = 0.
\]
But since \( \alpha_\pm \) both are non-zero and positive this establishes \( \lambda_+ = \lambda_- = 0 \) and therefore \( v = 0 \), q.e.d.
5 Analysis of the Localized Endomorphisms and their Corresponding Representations

5.1 The Proof of Theorem 3.9

For applying the theorems presented in the foregoing section to the representations of the universal Majorana algebra which is the direct sum of two selfdual CAR algebras, we have to develop a formalism which tells us, how to treat them as representations of \( \mathcal{C}(L^2(S^1), \Gamma) \).

**Definition 5.1** The projection epimorphisms \( \chi_{NS} \) and \( \chi_R \) are defined to be the homomorphic mappings from \( \text{Maj} \) to \( \mathcal{C}(L^2(S^1), \Gamma) \) with action on generators \( \hat{\psi}(\tilde{f}) \), \( \tilde{f} = f_{NS} \oplus f_R \in \mathcal{K} \),

\[
\begin{align*}
\chi_{NS}(\hat{\psi}(\tilde{f})) &= \psi(f_{NS}), \\
\chi_R(\hat{\psi}(\tilde{f})) &= \psi(f_R), \\
\chi_{NS}(Y) &= -1, \\
\chi_R(Y) &= 1.
\end{align*}
\]

We compute easily

\[
\chi_{NS}(b_a) = \begin{cases} 
\psi(\epsilon_a) & a \in \mathbb{Z} + \frac{1}{2} \\
0 & a \in \mathbb{Z} \end{cases}, \\
\chi_R(b_a) = \begin{cases} 
0 & a \in \mathbb{Z} + \frac{1}{2} \\
\psi(\epsilon_a) & a \in \mathbb{Z} \end{cases}.
\]

Now let \( \omega_{NS} \) and \( \omega_R \) be the states of \( \text{Maj} \) which satisfy for \( X \in \text{Maj} \)

\[
\begin{align*}
\omega_{NS}(X) &= \langle \Omega_{NS} | \pi_{NS}(X) | \Omega_{NS} \rangle, \\
\omega_R(X) &= \langle \Omega_R | \pi_R(X) | \Omega_R \rangle.
\end{align*}
\]

On the other hand, consider the quasifree states \( \varphi_{SN_S} \) and \( \varphi_{SN_R} \) of \( \mathcal{C}(L^2(S^1), \Gamma) \), defined by the operators \( S_{NS}, S_R \in \mathcal{Q}(L^2(S^1), \Gamma) \). The Neveu-Schwarz operator

\[
S_{NS} = \sum_{r \in \mathbb{N} + \frac{1}{2}} |\epsilon_r \rangle \langle \epsilon_r|
\]

is a basis projection, the Ramond operator

\[
S_R = \frac{1}{2} |\epsilon_0 \rangle \langle \epsilon_0| + \sum_{n \in \mathbb{N}} |\epsilon_{-n} \rangle \langle \epsilon_{-n}|
\]

is not. By \( (H_{SN_S}, \pi_{SN_S}, |\Omega_{SN_S} \rangle) \) and \( (H_{SN_R}, \pi_{SN_R}, |\Omega_{SN_R} \rangle) \) we denote the GNS triples of the states \( \varphi_{SN_S} \) and \( \varphi_{SN_R} \), respectively. We have

\[
\varphi_{SN_S}(\psi(\epsilon_r)^* \psi(\epsilon_r)) = \langle \epsilon_r, S_{NS} \epsilon_r \rangle = 0, \quad r \in \mathbb{N} + \frac{1}{2}
\]

and therefore

\[
\pi_{SN_S}(\psi(\epsilon_r)^* |\Omega_{SN_S} \rangle) = 0, \quad r \in \mathbb{N} + \frac{1}{2},
\]

as well as

\[
\varphi_{SN_R}(\psi(\epsilon_n)^* \psi(\epsilon_n)) = \langle \epsilon_n, S_R \epsilon_n \rangle = 0, \quad n \in \mathbb{N}
\]

and therefore

\[
\pi_{SN_R}(\psi(\epsilon_n)^* |\Omega_{SN_R} \rangle) = 0, \quad n \in \mathbb{N}.
\]

Since \( \varphi_{SN_R} \) is quasifree we obtain \( \varphi_{SN_R}(\psi(\epsilon_0)) = 0 \) as the correspondence to the additional requirement (30). Taking into consideration that \( \pi_{NS} \) and \( \pi_R \) are defined as cyclic representations of \( \text{Maj} \), this establishes
Lemma 5.2 The states and representations of Maj obey

\[ \omega_{NS} = \varphi_{NS} \circ \chi_{NS}, \quad \tau_{NS} \cong \pi_{NS} \circ \chi_{NS}, \quad (106) \]

\[ \omega_R = \varphi_R \circ \chi_R, \quad \tau_R \cong \pi_R \circ \chi_R. \quad (107) \]

In an analogous way we are able to associate the endomorphisms of Maj with Bogoliubov endomorphisms of \( \mathcal{C}(L^2(S^1), \Gamma) \) by the formulae

\[ \chi_{NS} \circ \vartheta_{1/2} = \vartheta_{1/2} \circ \chi_R, \quad \chi_{NS} \circ \vartheta_1 = \vartheta_1 \circ \chi_R, \quad (108) \]

\[ \chi_{NS} \circ \vartheta_{1/2} = \vartheta_{1/2} \circ \chi_N, \quad \chi_{NS} \circ \vartheta_1 = \vartheta_1 \circ \chi_N, \quad (109) \]

\[ \chi_{NS} \circ \sigma_{1/2} = \vartheta_{1/2} \circ \chi_R, \quad \chi_{NS} \circ \sigma_{1/2} = \vartheta_{1/2} \circ \chi_N, \quad (110) \]

\[ \chi_{NS} \circ \sigma_{1/2} = \vartheta_{1/2} \circ \chi_R, \quad \chi_{NS} \circ \sigma_{1/2} = \vartheta_{1/2} \circ \chi_N, \quad (111) \]

\[ \chi_{NS} \circ \vartheta_{1/2} = \vartheta_{1/2} \circ \chi_N, \quad \chi_{NS} \circ \vartheta_1 = \vartheta_1 \circ \chi_N. \quad (112) \]

Now we are able to take on informations about representations of \( \mathcal{C}(L^2(S^1), \Gamma) \) to those of Maj. We give a simple illustration. One easily checks that

\[ V^*_1 S_{NS} V_1 = S_{NS} - |\epsilon_{-\frac{1}{2}}\rangle\langle \epsilon_{-\frac{1}{2}}| + |\epsilon_{\frac{1}{2}}\rangle\langle \epsilon_{\frac{1}{2}}| \]

is a basis projection, and its difference to \( S_{NS} \) is obviously Hilbert Schmidt class. Therefore the Fock representations \( \pi_{NS} \) and \( \pi_{NS} \circ \vartheta_{V_1} \) of \( \mathcal{C}(L^2(S^1), \Gamma) \) are equivalent by Theorem 4.3, i.e. there is a unitary \( U \in \mathcal{B}(\mathcal{H}_{NS}) \) such that

\[ \pi_{NS}(\vartheta_{V_1}(x)) = U^* \pi_{NS}(x) U, \quad x \in \mathcal{C}(L^2(S^1), \Gamma). \]

Therefore

\[ \pi_{NS}(\vartheta_{V_1}(\chi_{NS}(X))) = U^* \pi_{NS}(\chi_{NS}(X)) U, \quad X \in \text{Maj}. \]

By (106) and (109) we conclude that

\[ \pi_{NS} \circ \vartheta_1 \cong \pi_{NS} \]

which is due to the fact that \( \vartheta_1 \) is inner in Maj. In a similar way, for proving Theorem 3.9, we use results of the deeper analysis of representations and Bogoliubov endomorphisms of \( \mathcal{C}(L^2(S^1), \Gamma) \) below.

The concept of the proof is the following: By exploiting Araki quasiequivalence criterion (Theorem 4.3) we can show that the representations of Maj, \( \pi_{NS} \circ \vartheta_{1/2} \) and \( \pi_{R} \), are quasiequivalent. This leads us to \( \pi_{NS} \circ \vartheta_{1/2} \cong \pi_{1/2} \) for the representations of \( \mathcal{A} \), i.e. we have unitary equivalence up to multiplicities. In the next step we consider the square \( (\vartheta_{1/2}^*)^2 \) and show that \( \pi_{NS} \circ (\vartheta_{1/2}^*)^2 \) is unitarily equivalent to the direct sum of \( \pi_{NS} \) and \( \pi_{1/2} \); our Theorem 4.5 will be used to find this result. This leads us to the fact that the open multiplicity is indeed one: \( \pi_{NS} \circ \vartheta_{1/2} \) and \( \pi_{1/2} \) are unitarily equivalent.

We begin this analysis with

Lemma 5.3 The operators of \( Q(L^2(S^1), \Gamma) \) satisfy

\[ V^* S_{NS} V - S_R \in \mathcal{J}_2(L^2(S^1)), \quad (113) \]

\[ V S_{NS} V^* - S_R \in \mathcal{J}_2(L^2(S^1)), \quad (114) \]

\[ V^* S_{NS} V' - S_R \in \mathcal{J}_2(L^2(S^1)), \quad (115) \]

\[ V' S_{NS} V - S_R \in \mathcal{J}_2(L^2(S^1)). \quad (116) \]

Because the proof is ugly work it is banished to the appendix. For drawing our first conclusions of Lemma 5.3, we remember an estimate which was given by Powers and Størmer [9]: For positive operators \( A, B \in \mathcal{B}(\mathcal{K}) \) the following inequality holds:

\[ \| A^\frac{1}{2} - B^\frac{1}{2} \|_2 \leq \| A - B \|_1, \quad (117) \]

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where for \( T \in \mathcal{B}(L^2(S^1)) \) by \( \|T\|_1 \) is denoted the trace norm
\[
\|T\|_1 = \text{tr} \left( (T^*T)^{1/2} \right),
\]
and by \( \|T\|_2 \) the Hilbert Schmidt norm
\[
\|T\|_2 = \left( \text{tr}(T^*T) \right)^{1/2}.
\]
This estimate will be used to see that the operator \( V^* S_{NS} V \) and \( V'^* S_{NS} V' \) differ from their positive square roots only by Hilbert Schmidt operators,
\[
\|(V^* S_{NS} V)^{1/2} - V^* S_{NS} V\|_2 \leq \|V^* S_{NS} V - (V^* S_{NS} V)^2\|_1 = \|V^* S_{NS}(1 - V V^*) S_{NS} V\|_1 \\
\leq \|V\|_2 \|S_{NS}\|_2 \|1 - V V^*\|_1.
\]
Since \( 1 - V V^* \) is a rank one projection and therefore trace class, the right hand side is finite. Obviously, the same calculation runs for \( V' \). More easily one finds
\[
\|S_R^{1/2} - S_R\|_2 = \left\| \frac{1}{\sqrt{2}} - \frac{1}{2} \right\| = \frac{1}{\sqrt{2}} - \frac{1}{2}.
\]
It follows immediately by Lemma 5.3:
\[
(V^* S_{NS} V)^{1/2} - S_R^{1/2} \in \mathcal{J}_d(L^2(S^1)), \quad (V'^* S_{NS} V')^{1/2} - S_R^{1/2} \in \mathcal{J}_d(L^2(S^1)).
\]
Applying Theorem 4.3 this yields
\[
\pi_{S_{NS}} \circ \varrho_V \approx \pi_{S_R}, \quad (118)
\]
and therefore
\[
\pi_{S_{NS}} \circ \varrho_{1/2}^{loc} \approx \pi_R. \quad (119)
\]
In the restriction to the observable algebra this reads
\[
\pi_0 \circ \varrho_{1/2}^{loc} \oplus \pi_1 \circ \varrho_{1/2}^{loc} \approx \pi_{1/2} \oplus \pi_{1/2} \approx \pi_{1/2},
\]
the same holds for \( \sigma_{1/2}^{loc} \). We conclude

**Lemma 5.4** *The representations of the observable algebra \( A \) obey*
\[
\pi_0 \circ \varrho_{1/2}^{loc} \approx \pi_{1/2}, \quad (120)
\]
\[
\pi_1 \circ \varrho_{1/2}^{loc} \approx \pi_{1/2}, \quad (121)
\]
\[
\pi_0 \circ \varrho_{1/2}^{loc} \approx \pi_{1/2}, \quad (122)
\]
\[
\pi_1 \circ \varrho_{1/2}^{loc} \approx \pi_{1/2}. \quad (123)
\]

We next consider the squares \( \varrho_V^2 = \varrho_{VV} \) and \( \varrho_{V'}^2 = \varrho_{V'V'} \) of Bogoliubov endomorphisms.

**Lemma 5.5** *The representations of \( \mathcal{C}(L^2(S^1), \Gamma) \) obey*
\[
\pi_{S_{NS}} \circ \varrho_{VV}^2 \approx \pi_{S_{NS}}, \quad (124)
\]
\[
\pi_{S_{NS}} \circ \varrho_{V'V'}^2 \approx \pi_{S_{NS}}. \quad (125)
\]

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Proof. If we multiply the operator in relation (114) with $V^*$ from the left and with $V$ from the right we get

$$V^*S_RV - S_{NS} \in \mathcal{J}_2(L^2(S^1)).$$

Since relation (113) holds we can replace $S_R$ by $V^*S_{NS}V$, this yields

$$V^*V^*S_{NS}VV - S_{NS} \in \mathcal{J}_2(L^2(S^1)).$$

In the same way one obtains

$$V''V''S_{NS}VV^2 - S_{NS} \in \mathcal{J}_2(L^2(S^1)).$$

Now the operators $1 - (VV)(VV)^*$ and $1 - (V'V')(V'V')^*$ are rank two projections, so that we can again conclude

$$(V^*V^*S_{NS}VV)^\frac{1}{2} - S_{NS} \in \mathcal{J}_2(L^2(S^1)).$$

and

$$(V''V''S_{NS}VV^2)^\frac{1}{2} - S_{NS} \in \mathcal{J}_2(L^2(S^1)).$$

By using Theorem 4.3 we obtain for the states

$$\varphi_{NS} \circ \vartheta_{V}^2 \approx \varphi_{NS}, \quad \varphi_{NS} \circ \vartheta_{V}^{2} \approx \varphi_{NS}.$$  

For the representations this implies, anyway if they are cyclic or not (they are, see below),

$$\pi_{NS} \circ \vartheta_{V}^2 \approx \pi_{NS}, \quad \pi_{NS} \circ \vartheta_{V}^{2} \approx \pi_{NS}, \quad \text{q.e.d.}$$

Since $\pi_{NS}$ is irreducible the representations $\pi_{NS} \circ \vartheta_{V}^2$ and $\pi_{NS} \circ \vartheta_{V}^{2}$ are unitarily equivalent to a multiple of $\pi_{NS}$. It is our next aim to keep these multiplicities under control. This is done with the help of Theorem 4.5. The Isometries $V^2$ and $V'^2$ are pseudolocalized in $I_2$ and therefore the kernel of their adjoints is spanned by functions having support in the localization region $I_2$. We start our investigation with

Lemma 5.6 Let $f \in L^2(S^1)$ be a non-zero function having support in an interval $I$ with non-vanishing open complement, i.e. $I' \neq \emptyset$. Then $f \not\in (1 - S_{NS})L^2(S^1)$, or, equivalently,

$$\langle f, S_{NS}f \rangle \neq 0. \quad (126)$$

Proof. Suppose $\langle f, S_{NS}f \rangle = 0$, i.e. $f \in (1 - S_{NS})L^2(S^1)$. Then $f(z)$ can be written as

$$f(z) = \sum_{n \in \mathbb{N} + \frac{1}{2}} f_r z^n, \quad z \in S^1, \quad f_r \in \mathbb{C}.$$  

But since $f(z) = 0$ for $z \in I'$ it is also $z^\frac{1}{2} f(z) = 0$ for $z \in I'$. But

$$z^\frac{1}{2} f(z) = \sum_{n \in \mathbb{N}} f_{n-\frac{1}{2}} z^n$$

is a holomorphic function vanishing on a non-discrete set $I' \subset \mathbb{C}$. This yields $f(z) \equiv 0$ identically, q.e.d.

Especially for pseudolocalized isometries $U \in O^\theta(L^2(S^1), \Gamma)$ we conclude that $\ker U^* \cap S_{NS}L^2(S^1) = \{0\}$, i.e. $N_U = 0$. Using Theorem 4.4 we receive the following

Corollary 5.7 If $U \in O^\theta(L^2(S^1), \Gamma)$ is a pseudolocalized Bogoliubov operator and $\vartheta_{U}$ the corresponding Bogoliubov endomorphism of $\mathcal{C}(L^2(S^1), \Gamma)$, then the representation $\pi_{NS} \circ \vartheta_{U}$ is cyclic, i.e. a GNS representation of the state $\varphi_{NS} \circ \vartheta_{U}$.  

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Now let us come back to our representations $\pi_{\text{NS}, s} \circ \varrho_V^{\gamma}$ and $\pi_{\text{NS}, s} \circ \varrho_V^{\gamma'}$. The isometries $V$ and $V'$ are of codimension one and therefore $V^2$ and $V'^2$ are of codimension two and, in addition, even pseudodthesized. For applying Theorem 4.5, we have to find in both cases an orthonormal basis $\{e_+, e_\pm\}$, $e_+ = \Gamma e_-$, spanning the kernel of the adjoints $(V^*)^2$ and $(V'^*)^2$, respectively. The required relations $\langle e_+ , S_{\text{NS}} e_\pm \rangle \neq 0$ are guaranteed by Lemma 5.6, because $V^2$ and $V'^2$ are pseudodthesized and therefore the functions $e_\pm$ are supported in $I_2$. First we consider the isometry $V^2$. The kernel of $V^*$ is spanned by the $\Gamma$-invariant, normed vector

$$f_0^{(2)} = \frac{1}{\sqrt{2}} (e_0^{(2)} + e_0^{(2)}) .$$

Therefore

$$\ker(V^*)^2 = \text{span} \left\{ f_0^{(2)} , V f_0^{(2)} \right\} ,$$

we remark that $f_0^{(2)}$ and $V f_0^{(2)}$ are orthonormal. By setting

$$e_\pm = \frac{1}{\sqrt{2}} (f_0^{(2)} \pm i V f_0^{(2)}) ,$$

this is an orthonormal basis of $\ker(V^*)^2$ satisfying $e_+ = \Gamma e_-$. By Theorem 4.5, $\varphi_{\text{NS}, s} \circ \varrho_V^{\gamma}$ is a mixture of two Fock states, and $\pi_{\text{NS}, s} \circ \varrho_V^{\gamma}$ decomposes into two irreducible representations. On the other hand,

$$\ker(V'^*)^2 = \text{span} \left\{ e_0^{(2)}, V' e_0^{(2)} \right\} .$$

By setting

$$e_\pm = \frac{1}{\sqrt{2}} (e_0^{(2)} \pm i V' e_0^{(2)}) ,$$

these orthonormal vectors satisfy $e_+ = \Gamma e_-$. Analogously, the representation $\pi_{\text{NS}, s} \circ \varrho_V^{\gamma'}$ decomposes into two irreducibles. Respecting Lemma 5.5, we obtain

**Lemma 5.8** The representations of $\mathcal{C}(L^2(S^1), \Gamma)$ obey

$$\pi_{\text{NS}, s} \circ \varrho_V^{\gamma} \cong \pi_{\text{NS}, s} \oplus \pi_{\text{NS}, s} ,$$

$$\pi_{\text{NS}, s} \circ \varrho_V^{\gamma'} \cong \pi_{\text{NS}, s} \oplus \pi_{\text{NS}, s} .$$

For the representation of Maj this reads

$$\pi_{\text{NS}} \circ \varrho_{\text{loc}}^{\text{loc}} \cong \pi_{\text{NS}} \oplus \pi_{\text{NS}} ,$$

$$\pi_{\text{NS}} \circ \sigma_{\text{loc}}^{\text{loc}} \cong \pi_{\text{NS}} \oplus \pi_{\text{NS}} .$$

and in the restriction to the observable algebra $A$ this yields

$$\pi_0 \circ \varrho_{\text{loc}}^{\text{loc}} \cong \pi_0 \oplus \pi_1 \oplus \pi_0 \oplus \pi_1 ,$$

$$\pi_0 \circ \sigma_{\text{loc}}^{\text{loc}} \cong \pi_0 \oplus \pi_1 \oplus \pi_0 \oplus \pi_1 .$$

We have to assign the irreducible representations on the right hand side to the representations on the left. By Lemma 5.4 we find $\pi_0 \circ \varrho_{\text{loc}}^{\text{loc}} \approx \pi_1 \circ \varrho_{\text{loc}}^{\text{loc}}$ and therefore $\pi_0 \circ \varrho_{\text{loc}}^{\text{loc}} \varrho_{\text{loc}}^{\text{loc}} \cong \pi_1 \circ \varrho_{\text{loc}}^{\text{loc}} \varrho_{\text{loc}}^{\text{loc}}$. Using the same argument, one obtains $\pi_0 \circ \sigma_{\text{loc}}^{\text{loc}} \sigma_{\text{loc}}^{\text{loc}} \approx \pi_1 \circ \sigma_{\text{loc}}^{\text{loc}} \sigma_{\text{loc}}^{\text{loc}}$. These facts admit only

**Lemma 5.9** The representations of the observable algebra $A$ obey

$$\pi_0 \circ \varrho_{\text{loc}}^{\text{loc}} \cong \pi_0 \oplus \pi_1 ,$$

$$\pi_1 \circ \varrho_{\text{loc}}^{\text{loc}} \cong \pi_0 \oplus \pi_1 ,$$

$$\pi_0 \circ \sigma_{\text{loc}}^{\text{loc}} \cong \pi_0 \oplus \pi_1 ,$$

$$\pi_1 \circ \sigma_{\text{loc}}^{\text{loc}} \cong \pi_0 \oplus \pi_1 .$$
Suppose now that the representation $\pi_0 \circ g_{1/2}^{\text{loc}}$ is reducible. This means that it is unitarily equivalent to a multiple of $\pi_{1/2}$ by Lemma 5.4. Therefore $\pi_0 \circ g_{1/2}^{\text{loc}} g_{1/2}^{\text{loc}}$ should be unitarily equivalent to a multiple of $\pi_{1/2} \circ g_{1/2}^{\text{loc}}$. But Lemma 5.9 establishes that this representation is equivalent to a direct sum of two inequivalent representations. Since this is a contradiction it follows that $\pi_0 \circ g_{1/2}^{\text{loc}}$ is irreducible, i.e. $\pi_0 \circ g_{1/2}^{\text{loc}} \cong \pi_{1/2}$. Analogously, we obtain $\pi_0 \circ g_{1/2}^{\text{loc}} \cong \pi_{1/2}$, the proof of Theorem 3.9 is complete, q.e.d.

5.2 Fusion Rules of Localized Endomorphisms

Our endomorphism $g_{1/2}^{\text{loc}}$ is localizable in any interval $I$ with $\zeta \in I'$. By conjugation with rotations on the circle, we can localize it in arbitrary intervals. The endomorphism $g_{1/2}^{\text{loc}}$ is localized in the interval $I_2$, by conjugation with conformal transformations it is localizable in arbitrary intervals. Now let $\sigma_1$ and $\sigma_2$ be any two endomorphisms of $A_0$, localized in intervals $J_1$ and $J_2$, respectively, such that we can choose an interval $J = J_1 \cup J_2$ with $J' \neq \emptyset$. By definition, $\sigma_1$ and $\sigma_2$ are both localized in $J$ as well. Suppose now that $\pi_0 \circ \sigma_1 \cong \pi_0 \circ \sigma_2$. By Lemma 3.2 we find a unitary intertwiner $U \in A(J) \subset A$ in the observable algebra, i.e. the unitary equivalence in the vacuum representation extends to an inner equivalence in the observable algebra. By iterating the procedure we conclude that for arbitrary pairs of localized endomorphisms which are unitarily equivalent in the vacuum representation, there exist unitary intertwiners in the observable algebra. This argument leads us to the fact that we can decipher the fusion rules for equivalence classes of localized endomorphisms by computing it for special representatives. This is actually not the case for non-localizable endomorphisms. Counterexamples can be constructed; it is possible, for instance, to construct an endomorphism $\mu_{1/2}$ which satisfies $\pi_0 \circ \mu_{1/2} \cong \pi_0 \circ g_{1/2}^{\text{loc}}$, but $\pi_0 \circ \mu_{1/2} \neq \pi_0 \circ g_{1/2}^{\text{loc}}$ [22]. The reason is that the observable algebra $A$ has a direct sum structure of an NS-part and an R-part, and the vacuum representation, being not faithful, vanishes on the R-part. Therefore the composition of a non-localized endomorphism with the vacuum representation does not contain enough information, e.g. the representation $\pi_0 \circ g_{1/2}^{\text{loc}}$ is completely independent of the Bogoliubov operator $V_{1/2}^{\text{loc}}$ but the representation $\pi_0 \circ g_{1/2}^{\text{loc}}$ is. On the other hand, the requirements in the definition of a localized endomorphism $g_{1/2}^{\text{loc}}$ imply that its action on the NS-part and the R-part of $A$ is the same; the properties of $g_{1/2}^{\text{loc}}$ are encoded in the representation $\pi_0 \circ g_{1/2}^{\text{loc}}$.

It is no problem to compute the fusion rules for our special examples of localized endomorphisms, we just have to summarize some of our previous results. Lemma 5.9 gives us the first fusion rule, we established

$$\pi_0 \circ g_{1/2}^{\text{loc}} g_{1/2}^{\text{loc}} \cong \pi_0 \circ \pi_1.$$  

The second fusion rule we obtain by the fact that $\pi_1 \circ g_{1/2}^{\text{loc}} \cong \pi_{1/2}$ (Lemma 5.4). Again, we can only avoid a contradiction to Lemma 5.9, relation (139), if $\pi_1 \circ g_{1/2}^{\text{loc}}$ is irreducible, i.e. unitarily equivalent to $\pi_{1/2}$. Hence we conclude

$$\pi_0 \circ g_{1/2}^{\text{loc}} g_{1/2}^{\text{loc}} \cong \pi_1 \circ g_{1/2}^{\text{loc}} \cong \pi_{1/2}.$$  

Since $g_{1/2}^{\text{loc}}$ and $g_{1/2}^{\text{loc}}$ commute, if we choose the localization region of $g_{1/2}^{\text{loc}}$ disjoint to that of $g_{1/2}^{\text{loc}}$, we also obtain

$$\pi_0 \circ g_{1/2}^{\text{loc}} g_{1/2}^{\text{loc}} \cong \pi_{1/2}.$$  

Trivially, the fact that $(g_{1/2}^{\text{loc}})^2 = \text{id}$ leads us to the third fusion rule

$$\pi_0 \circ g_{1/2}^{\text{loc}} g_{1/2}^{\text{loc}} \cong \pi_0.$$  

Denoting by $g_0$ the identity endomorphism (everywhere localized) and by $[g_J]$ the equivalence class of localized endomorphisms being unitarily equivalent to $g_{1/2}^{\text{loc}}$ in the vacuum representation, we summarize

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Theorem 5.10 (Fusion rules of localized endomorphisms)

\[
[e_1^2] = [e_0] + [e_1], \quad (142)
\]
\[
[e_1^2 e_1] = [e_1 e_1^2], \quad (143)
\]
\[
[e_1] = [e_0], \quad (144)
\]

i.e. the localized endomorphisms obey the Ising fusion rules.

5.3 Statistics Operator and Left Inverse

According to the general theory of superslection sectors [8, 11, 24], we expect that for each endomorphism \( \varrho \) which is localized in an interval \( I \) there exist a unitary \( \varepsilon_\varrho \in \mathcal{A}(I) \) which commutes with \( \varrho^2(\mathcal{A}) \),

\[
\varepsilon_\varrho \in \varrho^2(\mathcal{A})^\prime
\]

and fulfills

\[
\varepsilon_\varrho \varrho(\varepsilon_\varrho) \varepsilon_\varrho = \varrho(\varepsilon_\varrho) \varepsilon_\varrho \varrho(\varepsilon_\varrho).
\]

Therefore elements \( \tau_i = \varrho^{i-1}(\varepsilon_\varrho), i = 1, 2, \ldots \), satisfy the Artin relations and determine a representation of the braid group \( B_\infty \) [8, 11]. The statistics operator is given by the formula

\[
\varepsilon_\varrho = U^{-1} \varrho(U)
\]

where \( U \) is unitary such that the (equivalent) endomorphism \( \tilde{\varrho} \), defined by

\[
\tilde{\varrho}(A) = U \varrho(A) U^{-1}, \quad A \in \mathcal{A}
\]

is localized in an interval \( I_0 \subset I \). The statistics operator does not depend on the special choice of \( \tilde{\varrho} \) but it may depend on the fact whether \( I_0 \) lies in the left or the right complement of \( I \) with respect to our “point at infinity” \( \zeta \). The computation of \( \varepsilon_\varrho \) is straightforward for \( \varrho = \varrho_1^{\text{loc}} \). Let \( \varrho \) be induced by a real function \( h \in L^2(S^1) \) with support in an interval \( I \), \( ||h||^2 = 2 \), as described in Definition 3.4. Analogously, let \( \tilde{\varrho} \) be induced by a function \( h_0 \) with \( \text{supp}(h_0) \subset I_0, I_0 \cap I = \emptyset \) and write \( \tilde{h} = h \oplus h \in \tilde{K} \), the same for \( h_0 \). Since these endomorphisms are implemented in \( \text{Maj} \) (Lemma 3.6) we find

\[
U = \tilde{\psi}(h_0) \tilde{\psi}(\tilde{h}), \quad U^{-1} = \tilde{\psi}(\tilde{h}) \tilde{\psi}(h_0)
\]

and

\[
\varrho(U) = \tilde{\psi}(\tilde{h}_0) \tilde{\psi}(\tilde{h}) = -U
\]

so that

\[
\varepsilon_\varrho = -1.
\]

We now want to construct the statistics operator \( \varepsilon_\sigma \) for our localized endomorphism \( \sigma = \sigma_1^{\text{loc}} \). It seems to be very difficult to do that by the formula (147) but it is much easier to determine it by its properties. The statistics operator commutes with \( \sigma^2(\mathcal{A}) \). The commutant \( \sigma^2(\mathcal{A})^\prime \) is generated by elements \( 1, Y, \Pi \) where the projection \( \Pi \) is defined by

\[
\Pi = \tilde{\psi}(\tilde{e}_+) \tilde{\psi}(\tilde{e}_-), \quad \tilde{e}_\pm = e_\pm \oplus e_{\pm} \in \tilde{K}, \quad e_\pm = \frac{1}{\sqrt{2}} \left( e_0^{(2)} \pm i V^\prime e_0^{(2)} \right).
\]

(Remember that the functions \( e_\pm \) span the kernel of \( (V^\prime V^\prime)^* \) and satisfy \( e_+ = \Gamma e_- \).) But since \( \varepsilon_\sigma \in \mathcal{A}(I_0) \) this admits only the ansatz

\[
\varepsilon_\sigma = \alpha(1 + \gamma \Pi), \quad \alpha, \gamma \in \mathbb{C}.
\]

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Now $\varepsilon_\sigma$ is unitary,
\[ \varepsilon_\sigma \varepsilon_\sigma^* = |\alpha|^2 (1 + (\gamma + \bar{\gamma} + \gamma \bar{\gamma}) I) = 1. \]

Therefore $\gamma + \bar{\gamma} + \gamma \bar{\gamma} = 0$, $|\alpha|^2 = 1$, we write $\alpha = e^{i \psi}, \varphi$ real. The statistics operator satisfies equation (146); we exclude the case $\gamma = 0$ and set
\[ 0 = e^{-2i \psi} \gamma^{-1} (\varepsilon_\sigma \sigma (\varepsilon_\sigma) - \sigma (\varepsilon_\sigma) \varepsilon_\sigma (\varepsilon_\sigma)) \]
\[ = (\gamma + 1) (\Pi - \sigma (\Pi)) + \gamma^2 (\Pi \sigma (\Pi) - \sigma (\Pi) \Pi \sigma (\Pi)). \]

We define $\hat{\varepsilon}_b^{(2)} = \varepsilon_b^{(2)} \oplus \varepsilon_b^{(2)} \in \tilde{K}$ and $\hat{V}'$ as in Definition 3.7. Then we find
\[ \Pi = \frac{1}{2} \left( 1 + 2i \hat{\psi} (\hat{V}' \hat{\varepsilon}_b^{(2)}) \right), \quad \sigma (\Pi) = \frac{1}{2} \left( 1 + 2i \hat{\psi} (\hat{V}' \varepsilon_b^{(2)}) \right). \]

The fields obey
\[ \hat{\psi} (\hat{V}' \hat{\varepsilon}_b^{(2)})^2 = \hat{\psi} \left( \hat{V}' \hat{\varepsilon}_b^{(2)} \right)^2 = \hat{\psi} (\hat{V}' \varepsilon_b^{(2)})^2 = \frac{1}{2} \mathbf{1}, \]
\[ \{ \hat{\psi} (\hat{V}' \hat{\varepsilon}_b^{(2)}), \hat{\psi} (\varepsilon_b^{(2)}) \} = \{ \hat{\psi} (\hat{V}' \varepsilon_b^{(2)}), \hat{\psi} (\varepsilon_b^{(2)}) \} = \{ \hat{\psi} (\hat{V}' \varepsilon_b^{(2)}), \hat{\psi} (\hat{V}' \varepsilon_b^{(2)}) \} = 0. \]

Using these relations one finds$^3$
\[ \Pi \sigma (\Pi) \Pi = \frac{1}{2} \Pi, \quad \sigma (\Pi) \Pi \sigma (\Pi) = \frac{1}{2} \sigma (\Pi) \]
so that we obtain
\[ (\gamma + 1 + \frac{1}{2} \gamma^2) (\Pi - \sigma (\Pi)) = 0. \]

Since $\Pi - \sigma (\Pi) \neq 0$ we find
\[ \gamma^2 + 2 \gamma + 2 = 0 \implies \gamma = -1 \pm i \]
and therefore
\[ \varepsilon_\sigma = e^{i \psi} (1 - (1 \pm i) I). \quad (150) \]

According to the general theory [8, 11, 24], we expect that there exists also a left inverse $\Phi_\sigma$ to our endomorphism $\sigma$ such that $\Phi_\sigma \circ \sigma = \text{id}$. The left inverse is a unital, positive mapping from $A$ to $A$ which satisfies $\Phi_\sigma (\mathcal{A}(I)) \subset \mathcal{A}(I)$ if $I \supset I_2$. Since $\sigma$ is not an automorphism $\Phi_\sigma$ does in general not respect products but
\[ \Phi_\sigma (\sigma (A) B \sigma (C)) = A \Phi_\sigma (B) C \quad (151) \]
holds for $A, B, C \in \mathcal{A}$. In the following we want to derive an explicit description for $\Phi_\sigma$. We introduce an arbitrary orthonormal basis $\{ v_n, n \in \mathbb{Z} \}$ of $L^2(S^1)$ with $v_0 = \varepsilon_b^{(2)}$ and $\Gamma v_n = v_{-n}$. We define $\hat{\varepsilon}_n = v_n \oplus v_{-n} \in \tilde{K}$. Then every element $A \in \mathcal{A}$ can be written as
\[ A = A_1 + Y A_2 \quad (152) \]
where $A_1$ and $A_2$ are sums over monoms $X$ of the form
\[ X = \hat{\psi} (\hat{\varepsilon}_{n_1}) \hat{\psi} (\hat{\varepsilon}_{n_2}) \cdots \hat{\psi} (\hat{\varepsilon}_{n_{2k}}). \quad (153) \]

Using the anticommutation relations,
\[ \{ \hat{\psi} (\hat{\varepsilon}_n), \hat{\psi} (\hat{\varepsilon}_{-n}) \} = \delta_{n,-n} \mathbf{1}, \]

with statistical dimension $d(\sigma) = \sqrt{2}$.
in particular $\hat{\psi}(\hat{\epsilon}_0)^2 = \frac{1}{2}1$, we can write every monom $X$ such that $\hat{\psi}(\hat{\epsilon}_0)$ appears mostly once. If every $A \in \mathcal{A}$ is written in that way we define $\Phi_\sigma$ as the linear mapping with
\[
\Phi_\sigma(1) = 1, \quad \Phi_\sigma(Y A_2) = -Y \Phi_\sigma(A_2), \quad \Phi_\sigma(X) = \hat{\psi}(\hat{\epsilon}_{n_1})\hat{\psi}(\hat{\epsilon}_{n_2})\cdots \hat{\psi}(\hat{\epsilon}_{n_{2k}}) \quad (154)
\]
where
\[
\hat{\epsilon}_{n_j} = (\hat{V}^\dagger)^* \hat{\epsilon}_{n_j} = \hat{V}^* \hat{\epsilon}_{n_j} \oplus \hat{V}'^* \hat{\epsilon}_{n_j} \in \hat{\mathcal{K}}.
\]
It is no problem to check that $\Phi_\sigma$ is well defined and has the required properties. The general theory says
\[
\Phi_\sigma(\varepsilon_\sigma) = \frac{\omega_\sigma}{d(\sigma)} 1 \quad (155)
\]
where $\omega_\sigma$ is a phase factor (“statistical phase”) and the positive real number $d(\sigma)$ is called statistical dimension. Since $\sigma = \sigma_{\dot{1}/\dot{2}}^{1\sigma}$ belongs to the sector $[\dot{q}_{\dot{1}/\dot{2}}]$ we expect that $d(\sigma) = \sqrt{2}$. Using our formula for $\Phi_\sigma$ we find (respecting that $\hat{V}'\hat{\epsilon}_0^{(2)}$ is orthogonal to $\varepsilon_0^{(2)}$)
\[
\Phi_\sigma(\Pi) = \Phi_\sigma \left( \frac{1}{2} \left( 1 + 2i \hat{\psi}(\hat{V}'\hat{\epsilon}_0^{(2)}) \hat{\psi}(\hat{\epsilon}_0^{(2)}) \right) \right) = \frac{1}{2} 1.
\]
We conclude
\[
\Phi_\sigma(\varepsilon_\sigma) = e^{i\omega} \left( 1 - \left( \frac{1}{2} \pm \frac{i}{2} \right) \right) 1 = e^{i((\varphi + \frac{\pi}{2})/\sqrt{2})},
\]
in agreement with $d(\sigma) = \sqrt{2}$. At the end we find
\[
\varepsilon_\sigma = \frac{\omega_\sigma}{\sqrt{2}}((1 \pm i)1 \mp 2i \Pi). \quad (156)
\]
By the spin and statistics theorem [12] we expect that the statistical phase is given by $\omega_\sigma = e^{2\pi i s}$ where $s$ is the infimum of the conformal energy operator $L_\text{in}$ in the representation $\pi_0 \circ \sigma$. Since $\sigma$ belongs to the sector $[\dot{q}_{\dot{1}/\dot{2}}]$ we have $s = \frac{1}{16}$ and therefore $\omega_\sigma = e^{\frac{\pi}{8}}$. But, however, we did not succeed in computing $\omega_\sigma$ directly. Moreover, we observe the freedom to choose the $\pm$-sign in our formula for the statistics operator $\varepsilon_\sigma$. The change of this sign corresponds to the replacement of $\varepsilon_\sigma$ by $\varepsilon_\sigma^*$. The fact that $\varepsilon_\sigma \neq \varepsilon_\sigma^*$ goes back to the non-trivial spacetime topology which is the origin of braid statistics. At the end we remark that the same calculations we have done for $\sigma = \sigma_{\dot{1}/\dot{2}}^{1\sigma}$ run for the endomorphism $\sigma_{\dot{1}/\dot{2}}^{1\sigma}$; we just have to replace $\hat{V}'$ by $\hat{V}$ and $\varepsilon_0^{(2)}$ by $f_0^{(2)}$.

\section{Appendix: The Proof of Lemma 5.3}

An essential fact we use for the proof of Lemma 5.3 is presented in the following

\begin{lemma}
The difference of the two odd pseudolocalized Bogoliubov operators, given in Definition 3.7 is Hilbert Schmidt class,
\[
V - V' \in \mathcal{J}_2(L^2(S^1)).
\]
\end{lemma}

\begin{proof}
Since $1 - P_0^{(2)}$ is Hilbert Schmidt class, where
\[
P_0^{(2)} = 1 - |\varepsilon_0^{(2)}\rangle\langle \varepsilon_0^{(2)}| = P_{t+} + P_{t-} + \sum_{n=1}^{\infty} \left( |\varepsilon_n^{(2)}\rangle\langle \varepsilon_n^{(2)}| + |\varepsilon_n^{-2}\rangle\langle \varepsilon_n^{-2}| \right),
\]
it is equivalent to prove
\[
\Sigma_0 = \| (1 - P_0^{(2)})(V - V')(1 - P_0^{(2)}) \|_2^2 < \infty.
\]

\end{proof}
We remember that the square of the Hilbert Schmidt norm is the sum over the squares of all matrix elements in any Hilbert space basis. Obviously, the Bogoliubov operators \( V \) and \( V' \) differ only on the subspace \( L^2(L_2) \subset L^2(S^1) \). We compute

\[
\Sigma_0 = \sum_{n \geq 0} \sum_{m \neq n} \left| \langle \epsilon_n^{(2)}, V \epsilon_m^{(2)} \rangle - \langle \epsilon_n^{(2)}, V' \epsilon_m^{(2)} \rangle \right|^2
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| i \langle \epsilon_n^{(2)}, \epsilon_m^{(2)} \rangle - i \langle \epsilon_n, \epsilon_m \rangle \right|^2 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| -i \langle \epsilon_n^{(2)}, \epsilon_{m+\frac{1}{2}} \rangle - i \langle \epsilon_n, \epsilon_m \rangle \right|^2
\]

\[
+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| i \langle \epsilon_n^{(2)}, \epsilon_{m+\frac{1}{2}} \rangle + i \langle \epsilon_n, \epsilon_m \rangle \right|^2 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| -i \langle \epsilon_n^{(2)}, \epsilon_{m+\frac{1}{2}} \rangle + i \langle \epsilon_n, \epsilon_m \rangle \right|^2.
\]

Since \( \langle \epsilon_n^{(2)}, \epsilon_m^{(2)} \rangle = \langle \epsilon_n, \epsilon_m \rangle \) the first and the fourth summation vanishes, so that one finds by substituting to positive summation indices

\[
\Sigma_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \langle \epsilon_n^{(2)}, \epsilon_{m+\frac{1}{2}} \rangle + \langle \epsilon_n, \epsilon_{m+\frac{1}{2}} \rangle \right|^2 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \langle \epsilon_n^{(2)}, \epsilon_{m+\frac{1}{2}} \rangle + \langle \epsilon_n, \epsilon_{m+\frac{1}{2}} \rangle \right|^2
\]

\[
= 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \langle \epsilon_n^{(2)}, \epsilon_{m+\frac{1}{2}} \rangle + \langle \epsilon_n, \epsilon_{m+\frac{1}{2}} \rangle \right|^2,
\]

we used \( \langle \epsilon_n^{(2)}, \epsilon_m^{(2)} \rangle = \langle \epsilon_n, \epsilon_m \rangle \). The remaining matrix elements are easily computed,

\[
\langle \epsilon_n^{(2)}, \epsilon_{m+\frac{1}{2}} \rangle = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2i(m+n+\frac{1}{2})\phi} \frac{d\phi}{2\pi} = (\frac{-1}{\pi})^{m+n} \frac{1}{m + n + \frac{1}{2}},
\]

\[
\langle \epsilon_n^{(2)}, \epsilon_{m+\frac{1}{2}} \rangle = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2i(m-n+\frac{1}{2})\phi} \frac{d\phi}{2\pi} = (\frac{-1}{\pi})^{m+n} \frac{1}{m + n - \frac{1}{2}}.
\]

It follows

\[
\Sigma_0 = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{((n+m)^2 - \frac{1}{4})^2} = \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{k}{((k+1)^2 - \frac{1}{4})^2} < \infty, \quad \text{q.e.d.}
\]

Now we can start proving Lemma 5.3. We introduce the following notations

\[
B = V^* S_N S V - S_R,
\]

\[
P_0 \equiv |\epsilon_0\rangle\langle \epsilon_0 | + |\epsilon_1\rangle\langle \epsilon_1 | + |\epsilon_2\rangle\langle \epsilon_2 |,
\]

\[
P_1 = \sum_{n=1}^{\infty} |\epsilon_{-2n-1}\rangle\langle \epsilon_{-2n-1} |,
\]

\[
P_2 = \sum_{n=1}^{\infty} |\epsilon_{2n+1}\rangle\langle \epsilon_{2n+1} |,
\]

\[
P_3 = \sum_{n=1}^{\infty} |\epsilon_{-2n}\rangle\langle \epsilon_{-2n} |,
\]

\[
P_4 = \sum_{n=1}^{\infty} |\epsilon_{2n}\rangle\langle \epsilon_{2n} |.
\]
such that we find
\[
\sum_{i=0}^{4} P_i = 1, \quad \Gamma P_1 = P_3 \Gamma, \quad \Gamma P_3 = P_1 \Gamma. \tag{164}
\]

At first we have to show, that \( \|B\|_2 < \infty \). Since \( P_0 \) is Hilbert Schmidt class it is equivalent to prove that
\[
\|(1 - P_0)B(1 - P_0)\|_2 = \left\| \sum_{i,j=1}^{4} P_i B P_j \right\|_2 \leq \sum_{i,j=1}^{4} \|P_i B P_j\|_2 < \infty. \tag{165}
\]
This will be done by estimating each term \( \|R_i B P_j\|_2 \) for its own. Since \( B = B^* \) we find
\[
\|P_i B P_j\|_2 = \| (P_i B P_j)^* \|_2 = \|P_j B P_i\|_2,
\]
so that we are allowed to treat only those ten of sixteen terms with \( i \leq j \). Further, by
\[
\Gamma BT = V^* \Gamma S_{NS} \Gamma V - \Gamma S_R \Gamma = V^* (1 - S_{NS}) V - (1 - S_R) = -B
\]
we find the identity
\[
\|P_1 B P_1\|_2 = \|\Gamma P_1 B P_1 \|_2 = \|\Gamma B T P_3\|_2 = \|P_3 B P_3\|_2,
\]
and in the same way
\[
\|P_3 B P_3\|_2 = \|P_3 B P_3\|_2, \quad \|P_3 B P_3\|_2 = \|P_1 B P_1\|_2, \quad \|P_3 B P_3\|_2 = \|P_2 B P_2\|_2.
\]
in each term on the right hand side one of the projections \( P_i \) or \( P_4 \) appears, but since
\[
P_2 S_R = S_R P_2 = P_4 S_R = S_R P_4 = 0
\]
we have only to prove the finiteness of the six norms
\[
\|P_1 V^* S_{NS} V P_2\|_2, \quad \|P_1 V^* S_{NS} V P_2\|_2, \quad \|P_1 V^* S_{NS} V P_2\|_2,
\]
\[
\|P_2 V^* S_{NS} V P_2\|_2, \quad \|P_2 V^* S_{NS} V P_2\|_2, \quad \|P_2 V^* S_{NS} V P_2\|_2,
\]
and, since \( V - V' \) is Hilbert Schmidt class, this is equivalent to prove the finiteness of
\[
\|P_2 V^* S_{NS} V P_2\|_2, \quad \|P_1 V^* S_{NS} V P_2\|_2, \quad \|P_1 V^* S_{NS} V P_2\|_2,
\]
\[
\|P_2 V^* S_{NS} V P_2\|_2, \quad \|P_2 V^* S_{NS} V P_2\|_2, \quad \|P_2 V^* S_{NS} V P_2\|_2,
\]
At first we consider
\[
\Sigma_1 = \|P_2 V^* S_{NS} V P_2\|_2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \langle e_{2n+1}, V^* S_{NS} V e_{2m+1} \rangle \right|^2
\]
\[
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r \in \mathbb{N} + \frac{1}{2}} \left| \langle e_{-r}, V e_{2n+1} \rangle \langle e_{r}, V e_{2m+1} \rangle \right|^2.
\]
Since \( \langle e_{m+\frac{1}{2}}^{(2)}, e_{2n+1} \rangle = 2^{-\frac{1}{2}} \delta_{n,m} \) the action of \( V \) on odd basis vectors \( e_{2n+1} \) is simple, one reads by definition
\[
(V e_{2n+1})(z) = \begin{cases} e_{2n+1}(z) & z \in I_2 \\ i e_{2n+1}(z) & z \in I_4 \\ -e_{2n+1}(z) & z \in I_4 \end{cases}, \quad n \in \mathbb{N}.
\]
This leads us to
\[
\langle \epsilon_{-r}, V^{q} \epsilon_{2m+1} \rangle = \int_{-\pi}^{\pi} \epsilon^{i(2n+1+q)} \frac{d\phi}{2\pi} + i \int_{-\pi}^{\pi} \epsilon^{i(2n+2+q)} \frac{d\phi}{2\pi} - \int_{-\pi}^{\pi} \epsilon^{i(2n+1+q)} \frac{d\phi}{2\pi} = \frac{i(-1)^n}{\pi} \sin \left( \frac{r\pi}{2} \right) \frac{1}{(2n+1+r)(2n+2+r)}.
\]

Substituting to integer summation indices we obtain
\[
\Sigma_{1} = \frac{64}{\pi} \sum_{n=1}^{\infty} \left( \sum_{m=-\infty}^{\infty} \sigma_{n,m,i}^{(1)} \right)^2
\]
where
\[
\sigma_{n,m,i}^{(1)} = \frac{1}{(4n+2l+3)(4n+2l+5)(4m+2l+3)(4m+2l+5)}.
\]
We put off the estimate of this summation for some time and pass over to the next sum.
\[
\Sigma_{2} = \| P_{1} V^{q} S_{NS} V^{p} \|_{2}^{2} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \sum_{r \in \mathbb{N}_{0} + \frac{1}{2}} \langle \epsilon_{-r}, V^{q} \epsilon_{2n-1} \rangle \langle \epsilon_{-r}, V^{q} \epsilon_{2m+1} \rangle \right|^{2}.
\]
The action of \( V^{q} \) on vectors \( \epsilon_{-2n-1} \) is
\[
(V^{q} \epsilon_{-2n-1})(z) = \begin{cases} 
\epsilon_{-2n-1}(z) & z \in I_{-} \\
-\epsilon_{-2n-2}(z) & z \in I_{2} \\
-\epsilon_{-2n-3}(z) & z \in I_{+} 
\end{cases} \quad n \in \mathbb{N}.
\]
This leads us to
\[
\langle \epsilon_{-r}, V \epsilon_{2n-1} \rangle = \frac{i(-1)^n}{\pi} \sin \left( \frac{r\pi}{2} \right) \frac{1}{(2n+1+r)(2n+2-r)},
\]
so that
\[
\Sigma_{2} = \frac{64}{\pi^{2}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \sigma_{n,m,l}^{(2)} \right)^2
\]
where
\[
\sigma_{n,m,l}^{(2)} = \frac{1}{(4n-2l+1)(4n-2l+3)(4m+2l+3)(4m+2l+5)}.
\]

Analogously,
\[
\Sigma_{3} = \| P_{1} V^{q} S_{NS} V_{P} \|_{2}^{2} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \sum_{r \in \mathbb{N}_{0} + \frac{1}{2}} \langle \epsilon_{-r}, V^{q} \epsilon_{2n} \rangle \langle \epsilon_{-r}, V \epsilon_{2m} \rangle \right|^{2}.
\]
The action of \( V \) on basis vectors \( \epsilon_{2m} \) is
\[
(V \epsilon_{2m})(z) = \begin{cases} 
\epsilon_{2m}(z) & z \in I_{-} \\
\epsilon_{2m+1}(z) & z \in I_{2} \\
-\epsilon_{2m}(z) & z \in I_{+} 
\end{cases} \quad n \in \mathbb{N}.
\]
This leads to
\[
\langle \epsilon_{-r}, V \epsilon_{2m} \rangle = -\frac{i(-1)^n}{\pi} \cos \left( \frac{r\pi}{2} \right) \frac{1}{(2n+r)(2n+r+1)},
\]
so that
\[
\Sigma_{3} = \frac{64}{\pi^{2}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \sigma_{n,m,l}^{(3)} \right)^2
\]

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where
\[
\sigma^{(3)}_{n,m,l} = \frac{(-1)^l}{(4n - 2l + 1)(4n - 2l + 3)(4m + 2l + 1)(4m + 2l + 3)}.
\]  
(169)

Further,
\[
\Sigma_4 = \| P_3 V^* S_{NS} V P_4 \|_2^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r \in \mathbb{N} + \frac{1}{2}} \langle \epsilon_{-r}, V \epsilon_{2m+1} \rangle \langle \epsilon_{-r}, V \epsilon_{2m} \rangle = \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \sigma^{(4)}_{n,m,l} \right)^2
\]

where
\[
\sigma^{(4)}_{n,m,l} = \frac{(-1)^l}{(4n + 2l + 3)(4n + 2l + 5)(4m + 2l + 1)(4m + 2l + 3)}.
\]  
(170)

In the same way we compute
\[
\Sigma_5 = \| P_3 V^* S_{NS} V P_4 \|_2^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r \in \mathbb{N} + \frac{1}{2}} \langle \epsilon_{-r}, V \epsilon_{-2m} \rangle \langle \epsilon_{-r}, V \epsilon_{2m} \rangle = \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \sigma^{(5)}_{n,m,l} \right)^2
\]

The action of \( V \) on basis vectors \( \epsilon_{-2n} \) is
\[
(V \epsilon_{-2n})(z) = \begin{cases} 
\epsilon_{-2n}(z) & z \in I_- \\
-\overline{\epsilon}_{-2n-1}(z) & z \in I_1 \\
-\epsilon_{-2n}(z) & z \in I_n 
\end{cases}
\]

This leads us to
\[
\langle \epsilon_{-r}, V \epsilon_{-2m} \rangle = \frac{i(-1)^n}{\pi} \cos \left( \frac{r\pi}{2} \right) \frac{1}{(2n - r)(2n - r + 1)},
\]
so that
\[
\Sigma_5 = \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \sigma^{(5)}_{n,m,l} \right)^2
\]

where
\[
\sigma^{(5)}_{n,m,l} = \frac{1}{(4n - 2l - 1)(4n - 2l + 1)(4m + 2l + 1)(4m + 2l + 3)}.
\]  
(171)

Finally,
\[
\Sigma_6 = \| P_4 V^* S_{NS} V P_4 \|_2^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r \in \mathbb{N} + \frac{1}{2}} \langle \epsilon_{-r}, V \epsilon_{2m} \rangle \langle \epsilon_{-r}, V \epsilon_{2m} \rangle \langle \epsilon_{-r}, V \epsilon_{-2m} \rangle \langle \epsilon_{-r}, V \epsilon_{-2m} \rangle = \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \sigma^{(6)}_{n,m,l} \right)^2
\]

where
\[
\sigma^{(6)}_{n,m,l} = \frac{1}{(4n + 2l + 1)(4n + 2l + 3)(4m + 2l + 1)(4m + 2l + 3)}.
\]  
(172)

Next, we turn to the discussion of the operator
\[
C = V S_{NS} V^* - S_R.
\]  
(173)
For showing that \( \|C\|_2 < \infty \) we prove that
\[
\left\| (1 - P_0) C (1 - P_0) \right\|_2 = \left\| \sum_{i,j=1}^k P_i C P_j \right\|_2 \leq \sum_{i,j=1}^k \| P_i C P_j \|_2 < \infty.
\]
(174)

Because \( C = C^* \) we have again only to treat those terms with \( i \leq j \). Further, by
\[
\Gamma C T = V (1 - S_{NS}) V^* - (1 - S_R) = (VV^* - 1) - C
\]
and since \( VV^* - 1 \) is a rank one projection (i.e. \( \| VV^* - 1 \|_2 = 1 \)), we find
\[
\| P_i C P_j \|_2 = \| \Gamma P_i C P_j \|_2 = \| \Gamma P_2 \|_2 \leq \| P_2 (VV^* - 1 - C) P_2 \|_2 \leq \| P_2 C \|_2 + 1.
\]
In the same way one obtains
\[
\| P_i C P_j \|_2 \leq \| P_i C P_k \|_2 + 1, \quad \| P_i C P_j \|_2 \leq \| P_i C P_k \|_2 + 1, \quad \| P_i C P_j \|_2 \leq \| P_i C P_k \|_2 + 1.
\]
Again, \( S_R \) is annihilated by \( P_2 \) or \( P_2 \) in this terms. Using once more that \( V - V' \) is Hilbert Schmidt class, we conclude that it is sufficient to prove the finiteness of the following six terms:
\[
\| P_2 V S_{NS} V^* P_2 \|_2, \quad \| P_1 V S_{NS} V^* P_2 \|_2, \quad \| P_1 V S_{NS} V^* P_4 \|_2,
\| P_2 V S_{NS} V^* P_4 \|_2, \quad \| P_2 V S_{NS} V^* P_4 \|_2, \quad \| P_2 V' S_{NS} V^* P_4 \|_2.
\]

Now we have to work again,
\[
\Sigma_7 = \| P_2 V S_{NS} V^* P_2 \|_2^2 = \sum_{n=1}^\infty \sum_{m=1}^\infty \left( \sum_{r \in N_{n+\frac{1}{2}}} \langle e_{-r}, V^* e_{2n+1} \rangle \langle e_{-r}, V^* e_{2m+1} \rangle \right)^2.
\]
The action of \( V^* \) on basis vectors \( e_{2n+1} \) is
\[
(V^* e_{2n+1})(z) = \begin{cases} e_{2n+1}(z) & z \in I_- \\ -i e_{2n}(z) & z \in I_2 \\ -e_{2n+1}(z) & z \in I_+. \end{cases}
\]
This leads us to
\[
\langle e_{-r}, V^* e_{2n+1} \rangle = -i (-1)^n \sin \left( \frac{r \pi}{2} \right) \frac{1}{(2n + r)(2n + r + 1)},
\]
so that
\[
\Sigma_7 = \frac{64}{\pi^2} \sum_{n=1}^\infty \sum_{m=1}^\infty \left( \sum_{l=0}^{\infty} \sigma_{n,m,l}^{(7)} \right)^2
\]
where
\[
\sigma_{n,m,l}^{(7)} = \frac{1}{(4n + 2l + 1)(4n + 2l + 3)(4m + 2l + 1)(4m + 2l + 3)}.
\]
(175)

Further,
\[
\Sigma_8 = \| P_1 V S_{NS} V^* P_2 \|_2^2 = \sum_{n=1}^\infty \sum_{m=1}^\infty \sum_{r \in N_{n+\frac{1}{2}}} \left( \sum_{l=0}^{\infty} \langle e_{-r}, V^* e_{2n+1} \rangle \langle e_{-r}, V^* e_{2m+1} \rangle \right)^2.
\]
The action of \( V^* \) on basis vectors \( e_{-2n-1} \) is
\[
(V^* e_{-2n-1})(z) = \begin{cases} e_{-2n-1}(z) & z \in I_- \\ ie_{-2n}(z) & z \in I_2 \\ -e_{-2n-1}(z) & z \in I_+ \end{cases} \quad n \in \mathbb{N}.
\]
This leads us to
\[ \langle \epsilon_{-r}, V^* \epsilon_{-2n-1} \rangle = -\frac{i(-1)^n}{\pi} \sin \left( \frac{r \pi}{2} \right) \frac{1}{(2n - r)(2n - r + 1)}, \]
so that
\[ \Sigma_9 = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{i=0}^{\infty} \sigma_{n,m,i}^{(8)} \right)^2, \]
where
\[ \sigma_{n,m,i}^{(8)} = \frac{1}{(4n - 2l - 1)(4n - 2l + 1)(4m + 2l - 1)(4m + 2l + 1)}. \]
Further,
\[ \Sigma_9 = \|P_1 V S \epsilon_{-2n} V^* P_4\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{r \in \mathbb{N} + \frac{1}{2}} \langle \epsilon_{-r}, V^* \epsilon_{-2n-1} \rangle \langle \epsilon_{-r}, V^* \epsilon_{2m} \rangle \right)^2. \]
The action of $V^*$ on basis vectors $\epsilon_{2n}$ is
\[ (V^* \epsilon_{2n})(z) = \begin{cases} \epsilon_{2n}(z) & z \in I_- \\ -i \epsilon_{2n-1}(z) & z \in I_0 \\ -\epsilon_{2n}(z) & z \in I_+ \end{cases} \]
This leads us to
\[ \langle \epsilon_{-r}, V^* \epsilon_{2n} \rangle = \frac{i(-1)^n}{\pi} \cos \left( \frac{r \pi}{2} \right) \frac{1}{(2n + r)(2n + r - 1)}, \]
so that
\[ \Sigma_9 = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{i=0}^{\infty} \sigma_{n,m,i}^{(8)} \right)^2, \]
where
\[ \sigma_{n,m,i}^{(8)} = \frac{(-1)^i}{(4n - 2l - 1)(4n - 2l + 1)(4m + 2l - 1)(4m + 2l + 1)}. \]
Further,
\[ \Sigma_{10} = \|P_2 V S \epsilon_{2n} V^* P_4\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{r \in \mathbb{N} + \frac{1}{2}} \langle \epsilon_{-r}, V^* \epsilon_{2n+1} \rangle \langle \epsilon_{-r}, V^* \epsilon_{2m} \rangle \right)^2 = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{i=0}^{\infty} \sigma_{n,m,i}^{(10)} \right)^2, \]
where
\[ \sigma_{n,m,i}^{(10)} = \frac{(-1)^i}{(4n + 2l + 1)(4n + 2l + 3)(4m + 2l - 1)(4m + 2l + 1)}. \]
Further,
\[ \Sigma_{11} = \|P_3 V^* S \epsilon_{-2n} V^* P_4\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{r \in \mathbb{N} + \frac{1}{2}} \langle \epsilon_{-r}, V^* \epsilon_{-2n} \rangle \langle \epsilon_{-r}, V \epsilon_{2m} \rangle \right)^2. \]
The action of $V^*$ on basis vectors $\epsilon_{-2n}$ is
\[ (V^* \epsilon_{-2n})(z) = \begin{cases} \epsilon_{-2n}(z) & z \in I_- \\ i \epsilon_{-2n+1}(z) & z \in I_0 \\ -\epsilon_{-2n}(z) & z \in I_+ \end{cases} \]
\[ n \in \mathbb{N}. \]
This leads us to
\[
\langle \epsilon_{-r}, V^* \epsilon_{-2n} \rangle = -i(-1)^r \cos \left( \frac{r \pi}{2} \right) \frac{1}{(2n - r)(2n - r - 1)},
\]
so that
\[
\Sigma_{11} = \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \sigma_{n,m,l}^{(11)} \right)^2
\]
where
\[
\sigma_{n,m,l}^{(11)} = \frac{1}{(4n - 2l - 3)(4n - 2l - 1)(4m + 2l - 1)(4m + 2l + 1)}.
\]
Finally,
\[
\Sigma_{12} = \left\| P_4 V^* S_{NS} V^* P_4 \right\|^2 = \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \sigma_{n,m,l}^{(12)} \right)^2
\]
where
\[
\sigma_{n,m,l}^{(12)} = \frac{1}{(4n + 2l - 1)(4n + 2l + 1)(4m + 2l - 1)(4m + 2l + 1)}.
\]
We have the following estimate of absolute values of the \( \sigma_{n,m,l}^{(j)} \) for \( j = 1, 4, 6, 7, 10, 12 \):
\[
\sigma_{n,m,l}^{(1)} \geq |\sigma_{n,m,l+2}^{(j)}|, \quad j = 4, 6, 7, 10, 12, \quad n, m \in \mathbb{N}, \quad l \in \mathbb{N}_8.
\]
If we omit in our summations \( l = 0 \) and \( l = 1 \) terms, this corresponds to the replacement of \( S_{NS} \) by
\[
S_{NS}' = S_{NS} - |\langle \epsilon_{-\frac{1}{2}} \rangle| |\langle \epsilon_{-\frac{3}{2}} \rangle| - |\langle \epsilon_{-\frac{3}{2}} \rangle| |\langle \epsilon_{-\frac{1}{2}} \rangle|.
\]
Since the difference \( S_{NS} - S_{NS}' \) is obviously Hilbert Schmidt class, this has no influence of the property of \( \Sigma_j \) to be finite or infinite. Hence the estimate (181) tells us that for the proof of \( \Sigma_j < \infty \), \( j = 1, 4, 6, 7, 10, 12 \), it is sufficient to prove it for \( j = 1 \). We compute
\[
\Sigma_1 = \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} \frac{1}{(4n + 2l + 3)(4n + 2l + 5)} \right)^2
\]
\[
< \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(2m + 1)^4} \left( \sum_{l=0}^{\infty} \frac{1}{(4n + 2l + 3)} \right)^2
\]
\[
= \frac{2}{\pi^4} \sum_{n=1}^{\infty} \left( \sum_{l=0}^{\infty} \frac{1}{(4n + 2l + 3)} \right)^2
\]
\[
< \frac{2}{\pi^4} \sum_{n=1}^{\infty} \left( \frac{1}{4n + 1} \right)^2
\]
\[
= \frac{1}{96} \sum_{n=1}^{\infty} \frac{1}{n^2}
\]
\[
= \frac{\pi^2}{96}.
\]
\( \Sigma_1 \) is finite. On the other hand we find for \( n, m \in \mathbb{N}, \ l \in \mathbb{N}_0 \)

\[
\begin{align*}
|\sigma_{n,m,l}^{(1)}| &> |\sigma_{n,m,l+1}^{(2)}|, \\
|\sigma_{n,m,l}^{(1)}| &> |\sigma_{n,m,l+2}^{(3)}|, \\
|\sigma_{n,m,l}^{(1)}| &> |\sigma_{n,m,l+1}^{(2)}|, \\
|\sigma_{n,m,l}^{(1)}| &> |\sigma_{n,m,l+1}^{(8)}|, \\
|\sigma_{n,m,l}^{(1)}| &> |\sigma_{n,m,l+1}^{(9)}|.
\end{align*}
\]

(182) (183) (184) (185) (186)

By the same argument, for the proof of \( \Sigma_j < \infty, \ j = 2, 3, 5, 8, 9, 11 \), it is sufficient to prove that

\[
\tilde{\Sigma}_{11} = \frac{64}{\pi^3} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \sum_{l=0}^{\infty} |\sigma_{n,m,l}^{(1)}| \right)^2 < \infty.
\]

For this purpose, we decompose the sum over the index \( l \) into three parts,

\[
\sum_{l=0}^{\infty} |\sigma_{n,m,l}^{(1)}| = \sum_{l=0}^{2n-2} |\sigma_{n,m,l}^{(1)}| + |\sigma_{n,m,2n-1}^{(1)}| + \sum_{l=1}^{\infty} |\sigma_{n,m,l}^{(1)}| = \sum_{l=0}^{2n-2} |\sigma_{n,m,l}^{(1)}| - \sigma_{n,m,2n-1}^{(1)} + \sum_{l=1}^{\infty} |\sigma_{n,m,l}^{(1)}|.
\]

We begin with estimating the first part. By reversing the order of summation we obtain

\[
\begin{align*}
\sum_{l=0}^{2n-2} |\sigma_{n,m,l}^{(1)}| &= \sum_{l=0}^{2n-2} \frac{1}{(2l+1)(2l+3)(4m+4n-2l-3)(4m+4n-2l-5)} \\leq \frac{1}{(2n+2m-3)(4n+4m-2l-5)^2} \\
&= \sum_{l=0}^{n+m-2} \frac{1}{(2l+1)^2(4n+4m-2l-5)^2} + \sum_{l=n+m-1}^{2n+2m-3} \frac{1}{(2l+1)(4n+4m-2l-5)^2} \\
&= \frac{2}{(4n+4m-5)^2} + \frac{1}{(2n+2m-3)(4n+4m-2l-5)^2} \\
&= \frac{2}{(4n+4m-5)^2} + \frac{1}{x^2(4n+4m-4-x)^2} \\
&= \frac{1}{(4n+4m-4)^2} - \frac{1}{(4n+4m-4)(2n+2m-1)(4n+4m-5)} + \frac{2}{(4n+4m-4)^3} \ln(4n+4m-5) \\
&= \frac{2}{(4n+4m-4)^3} \ln \left( \frac{2n+2m-1}{2n+2m-3} \right) + \frac{2}{(4n+4m-4)^3} \ln(4n+4m-5) \\
&< \frac{1}{(2n+2m-3)^2}.
\end{align*}
\]

The last estimate is very rough but correct. In our computation we have used the fact that in an area of strict decrease a summation can be estimated by an integral plus the first summand. Next we consider the only negative term,

\[
-\sigma_{n,m,2n-1}^{(1)} = \frac{1}{(4n+4m-3)(4n+4m-1)} < \frac{1}{(2n+2m-3)^2},
\]

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and finally the remaining summation,

\[
\sum_{l=2n}^{\infty} \sigma_{l,n,m}^{(11)} = \sum_{l=0}^{\infty} \frac{1}{(2l+1)(2l+3)(4n+4m+2l-1)(4n+4m+2l+1)} < \frac{1}{(4n+4m-1)^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)(2l+3)} = \frac{1}{(4n+4m-1)^2} < \frac{1}{(2n+2m-3)^2}.
\]

We now can conclude that

\[
\tilde\Sigma_{11} < \frac{64}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{8}{(2n+2m-3)^2} \right)^2 = \frac{4096}{\pi^4} \sum_{k=0}^{\infty} \frac{k+1}{(2k+1)^4} < \infty,
\]

the proof of Lemma 5.3 is complete, q.e.d.

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References


