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Selected Topics in String Theory and Black Hole Physics¹

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1 Introduction

The aim of these lectures is to present some selected topics in the area of string theory, in particular those where string theory should provide answers where standard Quantum Field Theory and General Relativity seem to be in conflict. Since the audience included mainly beginning graduate students and non-experts, we tried to introduce some of the recent ideas and methods without entering into the technical details. Enough references are provided for the reader to find a more complete picture of the subjects covered.

The first part of the lectures reviews some aspects of black hole physics, and in particular the many puzzles raised by Hawking when one studies quantized matter in the presence of such space-time backgrounds. Then we survey some of the current ideas (specially those inspired in string theory) which provide new scenarios where some of these problems can be addressed. These include two-dimensional dilaton black holes [14] and string black holes [16]. Finally in the last part of the lectures we briefly introduce some discrete symmetries (duality and mirror transformation) appearing in string theory and which have no analogue in Quantum Field Theory. It is likely that these symmetries will play an important rôle in determining the correct generalization of the principle of general covariance to string theory and its full symmetry group.

2 Black Holes and the Global Structure of Space-Time

The best way of visualizing the causal structure of a space time is in terms of Penrose diagrams (see, for example, [1]). The general idea is to perform a conformal transformation which brings "infinity" to a finite distance, but in such a way that the causal structure (i.e. the light-cones) is not affected. There are subtleties involved in giving a precise definition of a Penrose diagram [1], however it is not difficult to capture the main features with a few examples. The simplest example is Minkowski space. The line element in polar coordinates is given by

$$ds^2 = dt^2 - dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.1)$$

Let us introduce light-cone coordinates, $x^\pm = t \pm r$ so we can rewrite ds^2 in the form

$$ds^2 = dx^+ dx^- - ds_\perp^2, \quad (2.2)$$

where now $ds_\perp^2 = r^2 d\Omega^2$ is the line element in the transverse space, a two-dimensional sphere of radius r . In the following we are going to forget about the transverse dimensions and center our attention in the two-dimensional line element $dx^+ dx^-$. We define the new coordinates u and v as

$$x^\pm = \tan\left(\frac{u+v}{2}\right). \quad (2.3)$$

In terms of these new coordinates we can rewrite the line element as

$$ds^2 = \frac{1}{4} \cos^{-2}\left(\frac{u-v}{2}\right) \cos^{-2}\left(\frac{u+v}{2}\right) (du^2 - dv^2). \quad (2.4)$$

We can represent schematically Minkowski space as in fig. 1. The future and past timelike

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Archives

Figure 1: Penrose diagram for Minkowski space in spherical coordinates.

infinities (i.e., $t \rightarrow \pm\infty$ with $r = \text{constant}$) correspond respectively to the points $i^\pm$. Spacelike infinity ($r \rightarrow \infty$ at fixed t) is mapped into the point i^0 . The future and past null infinities are in our diagram the surfaces $I^\pm$. In the coordinate transformation $(t, r) \rightarrow (u, v)$ the angles are unchanged and then the light cones correspond to lines at 45° in the diagram. It is then very easy to extract the causal structure of Minkowski space. For example we see that the point i^+ is causally connected with any other point.

The major interest of Penrose diagrams is in the study of space-times less trivial than Minkowski, specially in the presence of event horizons. This is the case, for example, of the Schwarzschild black hole. The line element is

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} - r^2 d\Omega^2. \quad (2.5)$$

The $r^2 d\Omega^2$ piece will be unimportant in what follows. We can rewrite 2.5 as

$$ds^2 = \left(1 - \frac{2GM}{r}\right) \left[dt^2 - \left(\frac{dr}{1 - \frac{2GM}{r}}\right)^2 \right] - r^2 d\Omega^2, \quad (2.6)$$

and, if we define the new coordinate

$$r^* = r + 2GM \log \left(\frac{r}{2GM} - 1 \right), \quad (2.7)$$

we have

$$ds^2 = \left(1 - \frac{2GM}{r}\right) (dt^2 - dr^{*2}) - r^2 d\Omega. \quad (2.8)$$

Let us notice that in these new coordinates the horizon ($r = 2GM$) is still a singular point for the metric. We can define the new (light-cone) Kruskal coordinates [1]

$$u = 4GM e^{-(t-r^*)/4GM}, \quad v = 4GM e^{(t+r^*)/4GM}, \quad (2.9)$$

in terms of which the line element is expressed as

$$ds^2 = -\frac{2GM}{r} e^{-r/2GM} du dv - r^2 d\Omega^2. \quad (2.10)$$

In these new coordinates (u, v) the horizon $r = 2GM$ is no longer a singular point. The only singularity of the metric now correspond to the point $r = 0$.

If we suppose that the line element (2.5) describes the geometry in the exterior of a spherically symmetric collapsing object then we can immediately draw the Penrose diagram (fig 2). The shaded region corresponds to the interior of the collapsing body and the horizon is the line $u = 0$. We see that once an observer has crossed the horizon he/she is doomed to end up in the singularity. On the contrary, being outside the horizon, although arbitrarily closed to it, it is still possible to avoid the singularity and scape to infinity. We say that Schwarzschild space-time is asymptotically flat since it looks like Minkowski space at infinity. This can be easily checked by comparing figs 1 and 2. In particular in both cases we have the past and future null infinities $I^\pm$ as well as the space and timelike past and future infinities $i^0, i^\pm$.

If we forget about the fact that the black hole is formed from a collapsing body, we can continue the coordinates (u, v) and draw the corresponding Penrose diagram (fig. 3) which has four regions. Region I and III are asymptotically flat exterior of the black hole and the region inside the horizon respectively that we have described in the previous paragraph. II is a new asymptotically flat region. In region IV it is worth stressing that the singularity represents the time reversal of a black hole; every point in the whole Penrose diagram is causally connected with it. This white hole is considered unphysical and only the result of extending the new coordinates beyond the physical region.

We can introduce some definitions that are very useful in the study of a black hole space-times [1]. We define the chronological past of the point p as the set $J^-(p)$ of all points which can be causally connected to the future to p . In the same way we define the chronological future of the point p as the set $J^+(p)$ formed by all points which are causally connected with p in the past. In terms of these definitions we define a black hole space time as an asymptotically flat space time M for which $J^-(I^+) \neq M$. The physical meaning of this definition is clear: there exist points in M from which it is impossible to scape to infinity. The set of such points is the black hole region $B = M - J^-(I^+)$ where the boundary $H = \partial B$ is the horizon.

As we have seen some paragraphs above, the singularity in the Schwarshild black hole is in the black hole region, i.e., it is covered by a horizon in such a way that no external observer can see it. In principle one can think of more general singularities which are *naked*, not covered by any kind of horizon. This is the case, for example, of the white hole in the analytic extension of the Schwarzschild space time. However we have said that region IV is

Figure 2: Penrose diagram describing the formation of a Schwarzschild black hole by gravitational collapse.

not physical in the sense that it is absent from the Penrose diagram describing the formation of a black hole by gravitational collapse (fig. 2). If this is the case for more general situation is something that has not been proved up to now. However it is widely believed that this would be the case and this has led to the formulation of the Cosmic Censorship Hypothesis which states that the complete gravitational collapse of a body under certain reasonable conditions always results in a black hole and never in a naked singularity [2].

3 The laws of black hole thermodynamics

The classical properties of black holes were extensively studied in the sixties and seventies. In particular it was noticed by Hawking and Bekenstein that the area of the horizon plays a rôle very similar to the thermodynamical entropy [4, 3], while the surface gravity on the horizon κ is the quantity associated with the temperature [5]. This analogy can be pushed further to construct the zeroth law of black hole thermodynamics stating that the surface gravity κ is constant in the horizon. This would be analogous to the statement that any

Figure 3: Penrose diagram for the complete Schwarzschild black hole.

thermodynamical system in equilibrium has a constant temperature.

The same analogy can be made in the case of the first law. In thermodynamics this simply states the conservation of energy, in differential form

$$dE = \dot{d}Q + \dot{d}W, \quad (3.1)$$

where $\dot{d}W$ is the work made on the system and $\dot{d}Q$ is the heat absorbed. The second law implies that $\dot{d}Q = TdS$, where S is the entropy and T the absolute temperature. In the case of a black hole we find a similar relation between the mass M , the area of the horizon A and its angular momentum J

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ, \quad (3.2)$$

where κ is the surface gravity and Ω_H is the angular velocity of the black hole.

According to the second law of black hole thermodynamics the area of the event horizon of a black hole never decreases with time,

$$\delta A \geq 0. \quad (3.3)$$

As we have indicated the area of the horizon is equivalent to the entropy in thermodynamics. In particular the second law implies that it is not possible for a black hole to divide spontaneously into two black holes, since the sum of the areas would be less than the initial one. It is important to notice that the analogy between the area of the event horizon and the thermodynamical entropy is a purely formal one. In particular if one take seriously this

identification it could be possible to define a density of states of the black hole in a similar fashion as it is done in thermodynamics

$$S = k \log \Omega(E) \longrightarrow A = k \log \Omega(M), \quad (3.4)$$

since the area is proportional to the mass squared of the black hole we find that

$$\Omega(M) \sim e^{\alpha M^2}. \quad (3.5)$$

Physical black holes are expected to form from gravitational collapse. This poses the problem of which aspects of the collapse will be relevant in order to predict the properties of the resulting black hole. In principle the gravitational collapse of a star is a very complicated process in which one should take into account an enormous number of degrees of freedom. However with respect to the final state the situation simplifies dramatically since the so-called *no hair* theorems ensure that the final state in a gravitational collapse only keeps memory of three conserved quantities: mass, charge and angular momentum. It is necessary to stress that these theorems are purely classical, they contain a number of assumptions and it is not clear how they should be modified once quantum of semiclassical effects are taken into account.

4 Hawking radiation

Up to now we have been dealing with the classical properties of a black hole. Nevertheless, it would be interesting to know what happens when we put quantum fields in the peculiar geometry of a black hole. To study this we need to face the problem of the quantization of fields in curved space-time [6]. When trying to quantize fields in a curved background we find that many of the properties of the quantum field theory which hold in Minkowski space either do not apply, or change radically. This is the case, for example, of the vacuum, which in Minkowski space has several well-known properties:

- It is unique.
- There is a well defined concept of localized excitations (particles).
- It determines the symmetries of the world and their realizations.

In the presence of a gravitational field the vacuum state loses the absolute meaning it has in Minkowski space. The reason is that strong or rapidly varying gravitational fields can produce particles. It is easy to see that this makes the concept of particle ambiguous (observer-dependent); let us consider a region of space-time in which we have a very strong gravitational field (for example near the horizon of a black hole). A static observer will see, because of the strong gravity, that particles are created in pairs. On the other hand a free-falling observer, by the equivalence principle, will not observe (locally) any gravitational field, so for him there is no creation of particles at all. This means that what the free-falling observer sees as the vacuum state of his quantum field theory is not a vacuum for the static observer. There are a number of elements that we need in the construction of a quantum field theory in a fixed background [7]:

- A Hilbert space of states $\mathcal{H}$.
- A classical space-time $(M, g_{\mu\nu})$.
- Fields operators acting on $\mathcal{H}$.
- A set of canonical commutation relations obeyed by these fields.
- Wave equations.
- Rules for constructing “particles” observables and Fock basis of $\mathcal{H}$.
- Regularization and renormalization schemes to make expectation values like $\langle T_{\mu\nu} \rangle$ finite.

However, in quantizing fields in a curved background we face some difficult problems. First of all we are performing the quantization of the theory in a given background metric $g_{\mu\nu}$. It is natural to ask about the back-reaction of the quantum fields over the space-time geometry. To be more concrete, when solving the quantum theory we usually impose the consistency condition

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = 8\pi G \langle T_{\mu\nu} \rangle. \quad (4.1)$$

In general our energy-momentum tensor will not verify (4.1), so it is necessary to proceed *à la* Hartree-Fock: we quantize the theory in the presence of a “test” background metric $g_{\mu\nu}^{(0)}$, and we evaluate the vacuum expectation value of the energy-momentum tensor $\langle T_{\mu\nu} \rangle$. Then we solve (4.1) with $\langle T_{\mu\nu} \rangle$ and so we obtain a first-corrected metric $g_{\mu\nu}^{(1)}$. Repeating the process we get a series of metrics $\{g_{\mu\nu}^{(0)}, g_{\mu\nu}^{(1)}, g_{\mu\nu}^{(2)}, \dots, g_{\mu\nu}^{(N)}, \dots\}$ which, hopefully, should eventually converges to a self-consistent solution $g_{\mu\nu}$.

A second problem is that, in order to solve wave equations, we need our background space-time to be globally hyperbolic [1] (this means that there exist a global Cauchy hypersurface Σ). In the case in which we have singularities or Cauchy horizons we would need certain (unclear) boundary conditions.

In ordinary quantum field theory the divergent vacuum energies of the fields can be subtracted without any further problem since the subtraction amounts to a redefinition of the origin of energies. In the presence of gravity vacuum energies contributes to the value of the cosmological constant. As soon as we have symmetry breaking, the quantum contributions are unacceptably large. In view of this, the smallness of the cosmological constant is probably the most mysterious fact which receives no explanation from any theory of Quantum Gravity, including String Theory.

After this preamble about general aspects of the quantization of fields in curved space-time, we proceed to the study of the Hawking radiation [8]. Let us consider a scalar field ϕ living in Schwarzschild space-time. The action functional for such a field can be easily written

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi, \quad (4.2)$$

which, upon variation, leads to the equation of motion $\square\phi = 0$. Looking at the Penrose diagram in fig. 2, we see that we need to specify the boundary conditions for the field in $\mathcal{I}^-$ (the past null infinity) and $\mathcal{I}^+$ (the future null infinity), as well as over the horizon H .

In $\mathcal{I}^-$ we can decompose ϕ_{in} in terms of the normal mode solutions of $\square$ with positive an negative frequency, $\{f_i, f_i^*\}$

$$\phi_{in} = \sum_i (a_i f_i + a_i^* f_i^*) , \quad (4.3)$$

with $\{a_i^+, a_i\}$ the corresponding creation-annihilation operators. In the same way we can perform the same kind of decomposition for ϕ_{out}

$$\phi_{out} = \sum_i (b_i g_i + b_i^* g_i^*) + \sum_i (c_i q_i + c_i^* q_i^*) , \quad (4.4)$$

where now the g 's and the q 's are the normal modes at $\mathcal{I}^+$ and H respectively. Hawking found that the creation-annihilation operators in $\mathcal{I}^+$ are related with those in $\mathcal{I}^-$ by a Bogoliubov transformation

$$b_i = \sum_j (\alpha_{ij} a_j + \beta_{ij} a_j^*) . \quad (4.5)$$

If we define the *in* vacuum $|0_{in}\rangle$ as the state annihilated by all the a_i (which represent the state with no incoming particles) the number of particles in the i -th outgoing state is given by

$$\langle 0_{in} | b_i^\dagger b_i | 0_{in} \rangle = \sum_j |\beta_{ij}|^2 . \quad (4.6)$$

By studying the flux of outgoing particles in the $|0_{in}\rangle$ state Hawking found that the emission rate corresponded to that of a black body at temperature

$$T = \frac{1}{8\pi GM} = \frac{\kappa}{2\pi} . \quad (4.7)$$

We see that the *temperature* of the black hole is essentially equal to the surface gravity κ as suggested by the thermodynamical analogy.

This result means that black holes ain't really black but they emit particles as if they where black bodies with a temperature inversely proportional to their masses. This means that if they do not accrete matter from its surroundings the continuous emission will result in a continuous loss of mass. Since the temperature is inversely proportional to the mass and the total amount of energy radiated per unit time and per unit area is proportional to the fourth power of the temperature we find that, assuming the outlined semiclassical approach all the way, the black hole will evaporate completely in a time proportional to M_0^3 with M_0 the initial mass of the black hole. A very unintuitive fact from a thermodynamical point of view is that the black hole heats up as it radiates.

Black hole evaporation gives rise to serious conceptual problems (for a review see [9]). Hawking found that the radiation emitted by a black hole is thermal; if this would exactly correct, it would mean that the radiation is in a mixed quantum state. In classical GR no information from inside the horizon can escape to the asymptotic region. It would seem natural in quantizing in the presence of a black hole to always take a trace over the states inside the horizon. This produces a mixed state. If we imagine a matter system in a pure state which collapse to form a black hole, and if we consider the complete evaporation process assuming the radiation to be exactly thermal, at the end we are left with a mixed state. In other words, although we know the initial state of the system we cannot predict what the final state would be. This clearly violates the laws of Quantum Mechanics.

Of course in the previous discussion it was assumed that the semiclassical derivation of the emission of particles is valid during the whole process of the black hole evaporation. However, when the mass of the black hole becomes close to the Planck mass the semiclassical description breaks down. The prediction of what happens after that point is impossible without a reliable theory of quantum gravity. One can imagine several possibilities [10]:

- The evaporations produces a naked singularity of negative mass which persists.
- The evaporation slows down and stops leaving a remnant black hole with a mass of the order of the Planck mass.
- The black hole evaporates completely but all the information about the initial state is encoded in the radiation that escapes to infinity.
- The black hole evaporates completely and takes with it all the information about its initial state and conserved quantities, except those coupled to long range fields (mass, charge and angular momentum).
- Evaporation ends with the formation of a wormhole.
- None of the above.

The Hawking temperature can be reobtained using the euclidean continuation of the black hole metric. Let us consider a general metric of the form (we drop the angular part)

$$ds^2 = -\lambda(r)dt^2 + \frac{dr^2}{f(r)} . \quad (4.8)$$

The event horizon is located at the point $r = r_0$ if $\lambda(r_0) = f(r_0) = 0$ and

$$\lambda(r) \sim \lambda'(r_0)\xi, \quad f(r) \sim f'(r_0)\xi, \quad (4.9)$$

with $\xi = r - r_0$. Let us now perform a Wick rotation into euclidean space by performing the substitution $t \rightarrow i\tau$. The line element is now

$$ds^2 \sim \lambda'(r_0)\xi d\tau^2 + \frac{1}{f'(r_0)}d\tau^2 . \quad (4.10)$$

If we define the new coordinate $\rho = 2\sqrt{\xi f'(r_0)}$ we can rewrite the last expression in the following way

$$ds^2 \sim d\rho^2 + \frac{1}{4}\lambda'(r_0)f'(r_0)\rho^2 d\tau^2 . \quad (4.11)$$

In order to avoid singularities we need to identify

$$\theta = \frac{\tau}{2}\sqrt{\lambda'(r_0)f'(r_0)} , \quad (4.12)$$

with an angle ranging between 0 and 2π . This means that we have the equivalence relation

$$\tau \sim \tau + \frac{4\pi}{\sqrt{\lambda'(r_0)f'(r_0)}} . \quad (4.13)$$

It is well known that the thermal partition function $Z(\beta) = \text{Tr} e^{-\beta H}$ in Minkowski space can be represented as an euclidean path integral with the euclidean time coordinate compactified to a circumference of length $\beta = 1/T$. If we use this connection between Minkowski space and the euclidean space $\mathbb{R}^3 \times S^1(\beta)$ in the context of the euclidean black hole, we have that the *temperature* of the black hole is given by

$$T_H = \frac{1}{4\pi} \sqrt{\lambda'(r_0) f'(r_0)}. \quad (4.14)$$

Let us check that this expression works. In the case of the Schwarzschild black hole we have

$$\lambda(r) = f(r) = 1 - \frac{2GM}{r}. \quad (4.15)$$

Applying (4.14) we see that $T_H = 1/(8\pi GM)$, in perfect agreement with Hawking's result. For the Reissner-Nordström black hole with mass M and charge Q

$$\lambda(r) = f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}, \quad (4.16)$$

and the Hawking temperature turns out to be

$$T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi (M + \sqrt{M^2 - Q^2})}, \quad (4.17)$$

Hawking discovery that black holes radiate as though they were black bodies seems to lead to an apparent violation of the laws of Quantum Mechanics. Since the radiation emitted by the black hole is thermal the *out* state is a mixed state described by the density matrix

$$\rho_{out} = N(\beta) e^{-\beta H}, \quad (4.18)$$

with $\beta = 8\pi GM$. The *in* state, on the contrary, is a pure state $|0_{in}\rangle$ with the associated density matrix

$$\rho_{in} = |0_{in}\rangle \langle 0_{in}|. \quad (4.19)$$

Hawking [10] has proposed to use a superscattering operator $\mathbb{S}$ relating the density operators in the two regions

$$\rho_{out}^A{}_B = \mathbb{S}^A{}_B \rho_{in}^C{}_D. \quad (4.20)$$

We have to require the $\mathbb{S}$ -operator to satisfy several conditions: first it has to map initial density operators of unit trace into final density operators which are positive, semi-definite and of unit trace. This implies that

$$\mathbb{S}^C{}_A \mathbb{S}^B{}_A = \delta_A^B. \quad (4.21)$$

Moreover, $\mathbb{S}$ has to map hermitian operators into hermitian operators,

$$\mathbb{S}^A{}_B \mathbb{S}^D{}_C = \mathbb{S}^B{}_A \mathbb{S}^D{}_C. \quad (4.22)$$

In ordinary Minkowskian quantum field theory we have the unitary S -operator which maps states in the asymptotic past into states in the asymptotic future. In this case the $\mathbb{S}$ -operator can be factorized as the tensor product $S \otimes S^\dagger$

$$\mathbb{S}^A{}_B \mathbb{S}^D{}_C = S^A{}_C \bar{S}^D{}_B. \quad (4.23)$$

With this form for the superscattering operator, if the system is in a pure state it will remain in a pure state, as required by the laws of Quantum Mechanics. This is not the case for a general $\mathbb{S}$. In fact it is possible to show that $\mathbb{S}$ -operator has the form (4.23) as long as the Green's functions of the theory verify asymptotic completeness, that is to say, if the asymptotic *in* and *out* states span the Hilbert space of the theory. Hawking has argued that the axiom of asymptotic completeness is not verified in quantum gravity; this has the consequence that quantities that are not coupled to long range fields are not conserved (for example global $U(1)$ charges [10]). Further analysis and criticism of these arguments can be found in [11].

In order to test whether or not Hawking radiation produces a violation of the unitary evolution in Quantum Mechanics we need to study a model in which the approximations made are controlled. We are going to study three models: dilaton gravity, two-dimensional dilaton gravity and the stringy black hole (for a review see [12]).

5 Four dimensional dilaton gravity

Let us start with four dimensional dilaton gravity. We consider the action describing a dilaton and a $U(1)$ gauge field coupled to gravity

$$S = \int d^4x \sqrt{-g} e^{-2\phi} \left(R + 4g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \right), \quad (5.1)$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength corresponding to the A_α $U(1)$ gauge field. The first thing that we see in (5.1) is that the curvature does not have the same form as in the Einstein-Hilbert action. This problem can be fixed by rescaling the metric according to

$$\tilde{g}_{\alpha\beta} = e^{-2\phi} g_{\mu\nu}. \quad (5.2)$$

In terms of this rescaled metric it is possible to get a spherically symmetric solution with angular momentum equal to zero and magnetic charge Q as [13]

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2GM}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 \left(r - \frac{Q^2}{2GM} e^{-2\phi_0} \right), \\ e^{-2\phi} &= e^{-2\phi_0} \left(1 - \frac{Q^2}{2GM r} e^{-2\phi_0} \right), \\ F &= Q \sin \theta d\theta \wedge d\varphi. \end{aligned} \quad (5.3)$$

Let us compare this solution with the known Reissner-Nordström solution which describe a non-rotating black hole with electric charge Q in Einstein gravity. First we see that instead

of two horizons ($r_{\pm} = GM \pm \sqrt{(GM)^2 - Q^2}$) here we have only one located at $r = 2GM$. Beside this, the curvature singularity is now at

$$r = \frac{Q^2}{2GM} e^{-2\phi}. \quad (5.4)$$

As in the case of the Reissner-Nordström solution we also have here a extremal black hole when $Q^2 = (2GM)^2 e^{2\phi_0}$. Let us concentrate in the study of this extremal case; in the limit in which $Q^2 \rightarrow (2GM)^2 e^{2\phi_0}$ we can introduce the new coordinate $\sigma = r + 2GM \log(r - 2GM)$ in terms of which the metric and the dilaton become

$$\begin{aligned} ds^2 &\rightarrow e^{2\phi_0} (-dt^2 + d\sigma^2 + 2GM d\Omega^2), \\ e^{2\phi} &\rightarrow \frac{e^{-2\phi_0}}{2GM} e^{\sigma/2GM}. \end{aligned} \quad (5.5)$$

This expression for the line element shows that the space-time in the case of the extremal dilatonic black hole is non singular and has two asymptotic regions. One correspond to $r \rightarrow \infty$ (as in the case of the Schwarzschild space-time) and the second is in the limit $\sigma \rightarrow -\infty$ in which the geometry approach a 1 + 1 dimensional Minkowski metric times a two-sphere of radius $2GM e^{\phi_0}$. It is worth stressing that although the resulting space-time is non-singular and has no horizons, this configuration is not very stable in the sense that if we throw a low energy particle into this extremal black hole a horizon and a singularity are immediately produced. By restricting ourselves to s-waves modes (i.e., radial infalling matter) we have an effective two-dimensional problem in which we can forget about the angular dependence. This fact will be exploited when extracting conclusions from the two-dimensional dilatonic black hole to be studied in next section.

6 Dilatonic two-dimensional black hole

We are going to proceed to the study of the two-dimensional dilatonic black hole [14]. We start with the action

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} [R + 4(\nabla\phi)^2 + 4\lambda^2]. \quad (6.1)$$

We can regard this action as the result of the dimensional reduction from the extremal dilatonic black hole studied above; let us notice that the term corresponding to the field strength does not appear since $F_{\mu\nu}$ has no non-zero components in the plane (t, r) . Before going forward let us introduce some concepts that will be useful in what follows. Let us consider a closed space-like two-dimensional surface M in an arbitrary space-time. We will say that this surface is *trapped* if both the outgoing and the ingoing set of future directed null geodesics orthogonal to the surface converge. In the case of the four-dimensional black hole studied in the previous section we can consider the two-spheres at the point labeled by the null coordinates (σ^+, σ^-) whose area is a function of the coordinates $4\pi \tilde{R}^2(\sigma^+, \sigma^-)$ in the metric $\tilde{g}_{\mu\nu}$. In this case the surface will be trapped if its area decreases as we move in any of the two null directions, i.e.,

$$\partial_{\pm} \tilde{R}(\sigma^+, \sigma^-) < 0. \quad (6.2)$$

However since the metric $\tilde{g}_{\mu\nu}$ is related with $g_{\mu\nu}$, the σ -model metric, by a conformal transformation involving the exponential of the dilaton field we have that

$$4\pi \tilde{R}^2(\sigma^+, \sigma^-) = 4\pi r^2 e^{-2\phi}. \quad (6.3)$$

This allow us to define a trapped point in the two-dimensional theory as that at which

$$\partial_{\pm} \phi < 0. \quad (6.4)$$

As an example of a trapped surface in the Schwarzschild black hole we have any surface $r = \text{constant} < 2GM$, since any null geodesic starting inside the horizon will end up in the singularity.

From the action (6.1) we can get the classical equations for the fields

$$\begin{aligned} 4\Box\phi - 4(\nabla\phi)^2 + R + 4\lambda^2 &= 0, \\ \nabla_{\mu} \nabla_{\nu} \phi + g_{\mu\nu} [(\nabla\phi)^2 - \Box\phi - \lambda^2] &= 0. \end{aligned} \quad (6.5)$$

From these equations it is straightforward to see that there is a solution given by $R = \Box\phi = 0$ and $(\nabla\phi)^2 = \lambda^2$ which corresponds to $g_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric, and a dilaton linear in the spatial coordinate; we will call this solution, well known in the context of two-dimensional string theory, the lineal dilaton vacuum (LDV). An interesting feature of this solution is that it is not invariant under translations. The reason is that the effective coupling constant in this theory is e^{ϕ} . Since the dilaton is linear in σ for $\sigma \rightarrow -\infty$ we have a weakly coupled theory whereas for $\sigma \rightarrow \infty$ we are in a strong coupling regime. In the context of two-dimensional dilatonic gravity the LDV is the closest solution to Minkowski space-time

In order to find non-trivial solutions to (6.5) it is convenient to introduce light cone-coordinates $x^{\pm} = x^0 \pm x^1$ and choose a conformal gauge $g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu}$. The line element now can be written as

$$ds^2 = -e^{2\phi} dx^+ dx^-. \quad (6.6)$$

Using these coordinates and imposing the gauge conditions $g_{++} = g_{--} = 0$ as constraints it is possible to find the solution

$$e^{-2\phi} = e^{-2\rho} = \frac{M}{\lambda} - \lambda^2 x^+ x^-, \quad (6.7)$$

so the line element and the curvature are given by

$$\begin{aligned} ds^2 &= \frac{dx^+ dx^-}{\frac{M}{\lambda} - \lambda^2 x^+ x^-}, \\ R &= \frac{4M\lambda}{\frac{M}{\lambda} - \lambda^2 x^+ x^-}, \end{aligned} \quad (6.8)$$

with M a free parameter. We see that there is a singularity located at $x^+ x^- = M\lambda^{-3}$ and that the horizons corresponds to the surface $x^{\pm} = 0$. The study of the Penrose diagram reveals us that the solution found is indeed a black hole.

Up to now we have described a classical eternal black hole. However in order to address problems such that the information loss or the final fate of the black hole we need to study the quantum aspects of the formation and the evaporation of the black hole. In order to perform such a study we will couple the *geometrical* action (6.1) to a matter action of the form [14]

$$S_M = -\frac{1}{4\pi} \sum_{i=1}^N \int d^2x \sqrt{-g} g^{\alpha\beta} \nabla_\alpha f_i \nabla_\beta f_i, \quad (6.9)$$

with f_i a set of massless scalar fields. Using light-cone coordinates we see that the matter fields obeys the equation of motion $\partial_+ \partial_- f_i = 0$. Let us suppose that we send a pulse of energy that for simplicity we consider to be a shock wave described by the energy-momentum tensor

$$T_{++} = \frac{1}{2} \partial_+ f \partial_+ f = a \delta(x^+ - x_0^+). \quad (6.10)$$

Physically this describes a pulse of energy sent from $x_0 = \infty$ at $t_0 = -\infty$ (such that $x_0^+ = t_0 - x_0$) which travels at the speed of light, i.e. along null geodesics. The field equations are exactly the same than in the eternal black hole except for the fact that the constraint $g_{++} = 0$ gets modified for the presence of the component T_{++} of the energy-momentum tensor of the matter fields. Let us analyze the resulting geometry. For $x^+ < x_0^+$ the pulse has not been emitted yet and the space time correspond to the LDV. However for $x^+ > x_0^+$ the space-time is given by the black hole solution described earlier (let us notice that the energy-momentum tensor vanishes for $x^+ \neq x_0^+$). We only need to match the two solutions along the discontinuity surface $x^+ = x_0^+$. The resulting solution is

$$e^{-2\phi} = e^{-2\phi} = -a(x^+ - x_0^+) \theta(x^+ - x_0^+) - \lambda^2 x^+ x^-, \quad (6.11)$$

where $\theta(x)$ is the step function. For $x^+ > x_0^+$ we recover the black hole solution (6.7) with $M = ax_0^+ \lambda$ after performing a shift in the x^- coordinate (the corresponding Penrose diagram is depicted in fig. 4).

We have successfully described the formation of a two-dimensional black hole by collapsing matter. It is possible to connect the result obtained with the discussion at the end of section 5. The LDV solution found in two dimensions corresponds to the dimensional reduction of the four-dimensional extremal black hole. As mentioned, this configuration is not stable in the sense that if we throw a low energy particle inside the extremal black hole we end up with a non-extremal singular black hole. The latter correspond (in two dimensions) to the black hole solution for $x^+ > x_0^+$.

After having fulfilled the first part of the task (to describe the formation of the black hole) we go on trying to understand in a deeper way the second (and more interesting part), the evaporation of the black hole. In a two-dimensional black-hole there is a relation between the Hawking radiation and the trace anomaly of the energy-momentum tensor (see [6] and references therein). It is a well know fact that for any conformal field theory the trace of the energy-momentum tensor vanishes. However, after quantization, this is no longer true and we have an anomaly associated with the trace of $T_{\mu\nu}$

$$\langle T^\mu{}_\mu \rangle = \frac{c}{24} R^{(2)}, \quad (6.12)$$

Figure 4: Penrose diagram describing the formation of a black hole by a shock wave.

where c is the central charge for the theory and $R^{(2)}$ is the scalar curvature in two dimensions. If we consider N scalar fields with $c = 1$ and take the conformal gauge $g_{\mu\nu} = e^{2\rho} \eta_{\mu\nu}$, equation (6.12) reduces to

$$\langle T_{+-} \rangle = -\frac{N}{12} \partial_+ \partial_- \rho. \quad (6.13)$$

Now it is possible to obtain the expectation value for the other components of $T_{\mu\nu}$ using the local conservation of this tensor; for example for the T_{--} component we will have

$$\partial_+ \langle T_{--} \rangle + \partial_- \langle T_{+-} \rangle - \Gamma_{--}^+ \langle T_{+-} \rangle = 0, \quad (6.14)$$

with Γ_{--}^+ the affine connection. Any solution of this equation will involve an arbitrary function which has to be fixed by choosing appropriate boundary conditions. We can perform this computation for the case of the shock wave studied early. The result is that the value of $\langle T_{--} \rangle$ near the horizon when $x^+ \rightarrow \infty$, which represents the flux of particles at I_R^+ , is given by

$$\langle T_{--} \rangle|_{horizon} = \frac{N}{48} \lambda^2, \quad (6.15)$$

The most interesting point of this way of getting the Hawking radiation is that we only used the trace anomaly of the energy-momentum tensor together with its local conservation.

As we have already pointed out when quantizing fields in non-trivial backgrounds we have to take into account the back-reaction of the fields over the background space-time. In the case of the black hole we are neglecting the modifications to the metric derived from the energy contained in the Hawking radiation. When we are dealing with a very massive black

hole this is a good approximation since the Hawking temperature is low. However at the final stages of the black hole evaporation, when the mass is low, the flux of energy radiated substantially modifies the geometry of the black hole solution. An immediate consequence of not having taken into account this back reaction is the fact that if we integrate the energy radiated by the black hole over all times we find that it is infinite. This is an unphysical result since it is impossible for a black hole to radiate more energy than it contains. One way in which the back reaction can be incorporated into the equations is by introducing the quantum energy-momentum tensor $\langle T_{\mu\nu} \rangle$ as a source for the classical gravitational field. Let us recall that this quantum energy-momentum tensor differs from the classical one because of the conformal anomaly associated with $\langle T_{+-} \rangle$. Doing so one finds that the resulting equations can be derived from the non-local action functional

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} \left[R + 4(\nabla\phi)^2 + 4\lambda^2 \right] - \frac{N}{96\pi} \int d^2x \sqrt{-g} R \square^{-1} R + S_M. \quad (6.16)$$

This theory can be easily studied in the limit in which $N \rightarrow \infty$ while $N e^{2\phi}$ is held fixed. However when studying the semiclassical equations for ϕ and ρ (the conformal factor) one finds that a singularity is met when reaching the critical line defined by

$$e^{-2\phi_{cr}(t,x)} = \frac{N}{12}. \quad (6.17)$$

This line defines the quantum version of the classical singularity of the black hole. Besides this singularity we find a set of points for which $\partial_\pm \phi < 0$; as we discussed earlier these are trapped points which defines the so-called apparent black hole. Unfortunately this approach breaks down when we get close to the singularity and we are not able to decide what the final fate of the black hole will be. However we have gone a long way and we have been able to address the problem of the back reaction and to describe both the formation and the first stages of the evaporation of a two dimensional dilatonic black hole.

7 Stringy black holes

As it is repetitively stressed in every article on the subject, string theory is the most promising candidate for a consistent theory of quantum gravity. Being so one is tempted to test its predictions in extreme situations that cannot be addressed using classical General Relativity, such as the Universe in the Planck era or the black hole scenario. Before plunging into the study of stringy black holes let us recall some general features about string theory [15].

The basic assumption of string theory is that the elementary constituents of Nature are not point particles but one dimensional objects. In the same way that, when evolving in time, a point particle describes a one-dimensional trajectory in a D dimensional space, a string will describe a two-dimensional surface, its worldsheet. In order to construct an action functional for the free string we can mimic the procedure in the case of the point particle and define it as the area swept by the worldsheet during its motion. It is convenient to write this action in a second order form introducing the field $h_{\alpha\beta}$ ($\alpha, \beta = 0, 1$), which represent the intrinsic metric in the worldsheet, to get

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} G_{ij}(X) \partial_\alpha X^i \partial_\beta X^j h^{\alpha\beta} + \dots, \quad (7.1)$$

where the dots stand for other massless modes such as the dilaton or the antisymmetric tensor. In the following we will take $\alpha' = 2$.

It can be seen that in order to quantize the theory in a consistent way the underlying two-dimensional field theory described by (7.1) has to be conformal also at the quantum level. To be concrete, let us consider the propagation of a free closed string. Its worldsheet would be a cylinder over which we introduce the spatial coordinate $\sigma \in [0, 2\pi)$ and the time coordinate τ along its axis. From these coordinates we can define a pair of complex coordinates $w = \tau + i\sigma$, $\bar{w} = \tau - i\sigma$ in terms of which the flat metric becomes

$$ds^2 = dw d\bar{w}. \quad (7.2)$$

It is easy to see that conformal transformations correspond to holomorphic changes of coordinates

$$w \rightarrow f(w), \quad \bar{w} \rightarrow \bar{f}(\bar{w}). \quad (7.3)$$

It is extremely convenient to pass from the cylinder with complex coordinates $\{w, \bar{w}\}$ to the complex plane; this can be accomplished using the conformal transformation

$$z = e^w, \quad \bar{z} = e^{\bar{w}}. \quad (7.4)$$

Since the theory is conformal the energy momentum tensor is traceless. In complex coordinates this means that the mixed components vanishes, i.e., $T_{z\bar{z}} = 0$. If we now apply the local conservation of $T_{\mu\nu}$ we find

$$\partial_{\bar{z}} T_{zz} = 0, \quad \partial_z T_{\bar{z}\bar{z}} = 0. \quad (7.5)$$

This means that $T_{zz}(z) = T(z)$ [resp. $\bar{T}(\bar{z})$] is a holomorphic function (resp. antiholomorphic). Being so, we can decompose both $T(z)$ and $\bar{T}(\bar{z})$ in Laurent series around $z = 0$

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \quad (7.6)$$

Since the conformal group is generated by the energy-momentum tensor we see that in two dimensions this group is infinite-dimensional; the generators L_n will satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \quad (7.7)$$

and similarly for the antiholomorphic generators $\bar{L}_n$. The central term appearing in (7.7) is a quantum anomaly parametrized by the central charge c .

The requirement of conformal invariance puts very strong constraints on the string theories that can be consistently quantized. When we consider the string moving in a general space with metric $g_{ij}(X)$ the action (7.1) describe a two-dimensional non-linear sigma model. In general this will not be a conformal theory for arbitrary values of the couplings $g_{ij}(X)$ and the condition that the β -functions vanish will impose restriction over the metric. In the general case this can only be done in perturbation theory in the loop expansion parameter and we have corrections to all orders in α' . At lowest level the vanishing of the β -functions gives the Einstein equations for the metric

$$\beta_{ij}^g \equiv c R_{ij} + O(\alpha') = 0. \quad (7.8)$$

Of course this generalizes to the case in which we have included other massless modes such as the dilaton and the antisymmetric tensor.

The simplest case of a conformal field theory is a set of D free scalar fields in two-dimensions. From the point of view of string theory this corresponds to a free string moving in a D -dimensional Minkowski space-time

$$S = \frac{1}{8\pi} \int d^2 \sigma \eta_{ij} \partial_\alpha X^i \partial^\alpha X^j. \quad (7.9)$$

The central charge in the matter sector is D (each free scalar field contribute with a unit of central charge). The cancellation of the conformal anomaly requires the dimension to be 26; this can be seen in several ways, for example computing the contribution to the central charge from the reparametrization ghosts that we need to introduce in order to perform a covariant gauge fixing (see [15] for details).

There are other examples of exact string backgrounds which results in more interesting physics. This is the case of the stringy black holes [16]. It has been known for some time that sigma models defined on group manifolds G give rise to an exact conformal field theory [17]. The theory is described by the Wess-Zumino-Witten (WZW) action

$$S_{WZW} = \frac{k}{8\pi} \int d^2 \sigma \text{Tr} (g^{-1} \partial_\mu g g^{-1} \partial^\mu g) + \frac{k}{12\pi} \int_B \text{Tr} (g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg), \quad (7.10)$$

where $g : S \rightarrow G$, S is a two-dimensional surface and B is a three-dimensional manifold such that $S = \partial B$. The second part in (7.10) is the Wess-Zumino term whose presence is crucial to cancel the conformal anomaly [18]. The resulting conformal field theory is that associated with a Kač-Moody algebra $\hat{G}_k$ with central charge

$$c = \frac{2k \dim G}{C_2 + 2k}, \quad (7.11)$$

where k is the coupling constant (the level of the Kač-Moody algebra) and C_2 is the second Casimir of the group G . A lot of interesting physics can be extracted from the study of gauged WZW, i.e., a WZW model defined over a coset manifold G/H where H is an anomaly free subgroup of G . In [16] Witten studied two WZW models defined over the coset manifolds $SL(2, \mathbb{R})/SO(1, 1)$ and $SL(2, \mathbb{R})/SO(2)$; in the classical limit $k \rightarrow \infty$ he found that the first one corresponded to a two-dimensional black hole in Minkowski space-time whereas in the second case he got the euclidean version of the black hole. Let us give a rapid overview of the details involved in the computation.

Let us consider a WZW model defined over $SU(2)$. It is an elementary result that any element $g \in SU(2)$ can be written as

$$g = u_0 \mathbf{I} + i \vec{\sigma} \cdot \vec{u}, \quad (7.12)$$

where $\mathbf{I}$ is the 2×2 identity matrix and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. The condition that $g \in SU(2)$, in particular that $\det g = 1$ implies

$$u_0^2 + \vec{u}^2 = 1, \quad (7.13)$$

which means that the $SU(2)$ manifold is the three-sphere S^3 . In the case of $SL(2, \mathbb{R})$ the situation is a little bit different since now the group is not compact. We can write any element $g \in SL(2, \mathbb{R})$ in the form

$$g = a_0 + a_1 \sigma_1 + i a_2 \sigma_2 + a_3 \sigma_3 = \begin{pmatrix} a_0 + a_3 & a_1 + a_2 \\ a_1 - a_2 & a_0 - a_3 \end{pmatrix}. \quad (7.14)$$

Now, if we implement the unimodularity condition $\det g = 1$ we find

$$a_0^2 - a_3^2 - a_1^2 + a_2^2 = 1, \quad (7.15)$$

namely, the $SL(2, \mathbb{R})$ manifold is the three-dimensional Anti-de-Sitter space. The central charge of the conformal field theory associated with the $SL(2, \mathbb{R})$ WZW model is

$$c = \frac{3k}{k-2}. \quad (7.16)$$

The action (7.10) is invariant under the following action of $G_L \otimes G_R$

$$g \longrightarrow h_1 g h_2 \quad g, h_1, h_2 \in G. \quad (7.17)$$

In order to get the black hole geometry we have to gauge either a $SO(2) \sim U(1)$ or a $SO(1, 1)$ subgroup. The action of these subgroups over an element of G is as follows

$$\begin{aligned} SO(2) : \quad & g \rightarrow e^{i\alpha\sigma_3} g e^{i\alpha\sigma_3}, \\ SO(1, 1) : \quad & g \rightarrow e^{\alpha\sigma_3} g e^{\pm\alpha\sigma_3}. \end{aligned} \quad (7.18)$$

To gauge these subgroups we introduce a gauge field A_i ($i = 0, 1$) which is non-dynamical. Now the theory is invariant under the local action of the gauged subgroup and we have to fix the gauge ambiguity; in doing this we eliminate one of the dimensions of the original model. Since the action is quadratic in the gauge fields we can integrate them out and we end up with a sigma model over a two-dimensional target space with the metric

$$ds^2 = k \frac{dx^+ dx^-}{M - x^+ x^-}, \quad (7.19)$$

for the case when we have gauged a $SO(1, 1)$ subgroup. An important point is that in the integration of the gauge fields there is a Jacobian which results in a shift of the original dilaton field. The mass M of the black hole results as the value of this dilaton field at the horizon.

The computation outlined above is only valid in the semiclassical approximation, i.e., in the limit $k \rightarrow \infty$. The reason is that in integrating out the gauge fields we have neglected higher loop corrections, or terms in order $1/k^2$. The central charge of the original gauged WZW model is

$$c = \frac{3k}{k-2} - 1. \quad (7.20)$$

Since we want our theory to be critical, we demand $c = 26$. This implies that $k = 9/4$ and then corrections of order $1/k^2$ are not negligible. That this is so can be easily seen, for example, in computing the central charge from the semiclassical metric which is $c = 2 + 6/k = 14/3$. In [19] properties of the exact conformal field theory associated with the stringy black hole were studied both for the euclidean and the Minkowskian signature.

8 Duality and topology change in string theory

As we have indicated above the local geometry of the target space is determined by the condition that the two-dimensional field theory defined by the sigma model action is a conformal field theory with total central charge in the matter sector equal to 26 (or 15 in the case of the superstrings). It is possible, nevertheless, to find that two different sigma models defined on two completely different target manifolds are equivalent, in the sense that they define the same two-dimensional conformal field theory. This is the case of the theories related by a duality or a mirror transformation in string theory [20, 21, 22].

One may wonder about the interest of these symmetries in string theory. In fact their interest is two-fold; on the one hand these symmetries may help us to understand to what extent string theory modifies the classical geometrical concepts that we are used to in General Relativity. On the other hand these symmetries play an important role in string phenomenology. Standard reasoning says that at low energy string theory is well described by a *minimal* $N=1$ supergravity lagrangian. These theories depend on three functions: the Kähler potential $\mathcal{K}$, which determines the kinetic terms for the scalar fields, the superpotential W , which gives the Yukawa couplings and the mass matrix, and the gauge kinetic function f which determines the gauge coupling constants. String phenomenology can be summarized in the computation of these three functions $\mathcal{K}$, W and f . In string theory these are computable directly in terms of the two-dimensional conformal field theory associated with a concrete vacuum. However there are problems in trying to put forward this program. For example, the string coupling constant g_s is determined by the vacuum expectation value of the real part of the dilaton field

$$\langle \text{Re } S \rangle = \frac{1}{g_s^2}. \quad (8.1)$$

The value of g_s will tell us if the string theory under consideration is either in a weak or in a strong coupling regime, and this is a very important information in order to decide the reliability of the string perturbative expansion. However, perturbation theory does not generate any potential for $\text{Re } S$ and this means that the determination of g_s lies entirely in non-perturbative effects. Together with this problem we need also to find a mechanism for supersymmetry breaking respecting the energy hierarchy. Duality and mirror symmetry can play an important rôle in the solution for these questions (see, for example, [23]).

9 An example of duality symmetry: string on a torus

In this section we illustrate the duality symmetry by considering the simplest example: a closed bosonic string moving on a torus. Let us consider a d -dimensional lattice Λ_d spanned by a set of vectors $\{\vec{e}_1, \dots, \vec{e}_d\}$. A d -dimensional torus is obtained by dividing $\mathbf{R}^d$ by the lattice Λ_d

$$\mathbf{T}_\Lambda^d = \mathbf{R}^d / (2\pi\Lambda_d). \quad (9.1)$$

This means that two points $A, B \in \mathbf{R}^d$ will be identified if and only if $\vec{A}B$ is a vector in Λ_d . The next step will be to consider a string moving in such a space. After taking a conformal gauge and choosing complex coordinates $z, \bar{z}$ in the worldsheet we can write the action for

the string as

$$S = \frac{1}{2\pi} \int d^2z \partial \vec{X} \cdot \bar{\partial} \vec{X}. \quad (9.2)$$

The resulting equations of motion are $\partial \bar{\partial} \vec{X} = 0$ and a general solution will be of the form

$$\vec{X} = \vec{X}_L(z) + \vec{X}_R(\bar{z}), \quad (9.3)$$

with $\vec{X}_L, \vec{X}_R$ two arbitrary functions. However, because of the structure of the target space (9.1) we must be careful with the boundary condition for $\vec{X}(z, \bar{z})$. This is because now the string can also wrap around the non-contractible circles in the target space, and then we have

$$\vec{X}(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = \vec{X}(z, \bar{z}) + 2\pi \sum_{a=1}^d n^a \vec{e}_a, \quad (9.4)$$

where $n^a \in \mathbf{Z}$ is the number of times that the string wraps along the circle in the $\vec{e}_a$ direction. With these boundary conditions we can write the mode expansion for both $X_L(z)$ and $X_R(\bar{z})$

$$\begin{aligned} \vec{X}_L(z) &= \vec{q}_L - i\vec{p}_L \ln z + i \sum_{n \neq 0} \frac{\vec{\alpha}_n}{n} z^{-n}, \\ \vec{X}_R(\bar{z}) &= \vec{q}_R - i\vec{p}_R \ln \bar{z} + i \sum_{n \neq 0} \frac{\vec{\alpha}_n^*}{n} \bar{z}^{-n}. \end{aligned} \quad (9.5)$$

A very interesting object to compute is the partition function which counts the number of states per mass level

$$\chi(\tau, \bar{\tau}) = \text{Tr } q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \quad (9.6)$$

where $q = \exp(2\pi i \tau)$. From the definition of the energy-momentum tensor

$$T(z) = -\frac{1}{2} \partial \vec{X} \cdot \partial \vec{X}, \quad (9.7)$$

(and correspondingly for the antiholomorphic component) we can write both L_0 and $\bar{L}_0$ in terms of $\vec{p}_L, \vec{p}_R$ and the oscillators $\vec{\alpha}_n$ and $\vec{\alpha}_n^*$ as

$$\begin{aligned} L_0 &= \frac{1}{2} \vec{p}_L^2 + \sum_{n>0} \vec{\alpha}_{-n} \cdot \vec{\alpha}_n, \\ \bar{L}_0 &= \frac{1}{2} \vec{p}_R^2 + \sum_{n>0} \vec{\alpha}_{-n}^* \cdot \vec{\alpha}_n^*. \end{aligned} \quad (9.8)$$

Using these expressions the partition function is

$$\chi(\tau, \bar{\tau}) = \frac{1}{[\eta(\tau)\bar{\eta}(\bar{\tau})]^d} \sum_{(\vec{p}_L, \vec{p}_R)} q^{\frac{1}{2} \vec{p}_L^2} \bar{q}^{\frac{1}{2} \vec{p}_R^2}, \quad (9.9)$$

where $\eta(\tau)$ is the Dedekind eta-function, $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, and the prefactor in the sum in the right hand side of eq. (9.9) correspond to the contribution of the oscillators. In order to compute $\chi(\tau, \bar{\tau})$ we need to sum over all vectors $(\vec{p}_L, \vec{p}_R)$, so we must know where these vectors live.

The dual lattice Λ_d^* is defined as the lattice span by vectors $\{\vec{\theta}^1, \dots, \vec{\theta}^a\}$ such that

$$\vec{\theta}^a \cdot \vec{e}_b = \delta_b^a. \quad (9.10)$$

The vectors $\vec{\theta}^a$ defines the so-called Wigner-Seitz cell. After quantizing the theory we are going to deal with vertex operators which create physical states out of the vacuum. The vertex operators are in general of the form

$$V_{\vec{k}_L, \vec{k}_R}(z, \bar{z}) = : P(\partial \vec{X}, \bar{\partial} \vec{X}) e^{i\vec{k}_L \cdot \vec{X}_L(z)} e^{i\vec{k}_R \cdot \vec{X}_R(\bar{z})} : \quad (9.11)$$

where $(\vec{k}_L, \vec{k}_R)$ are the eigenvalues of the operators $(\vec{p}_L, \vec{p}_R)$ for the state created by (9.11) and $P(\partial \vec{X}, \bar{\partial} \vec{X})$ is a polynomial. Since we want (9.11) to be single-valued when the center of mass $\vec{q}$ is displaced by a vector of Λ_d we have that

$$\frac{1}{2}(\vec{p}_L + \vec{p}_R) \in \Lambda_d^*. \quad (9.12)$$

On the other hand, from the boundary conditions (9.4) we have

$$\vec{p}_L - \vec{p}_R \in \Lambda_d. \quad (9.13)$$

Eqs. (9.10)-(9.12) implies that $(\vec{p}_L, \vec{p}_R)$ is a vector lying in an even, Lorentzian selfdual lattice. In fact we can expand $\vec{p}_L$ and $\vec{p}_R$ in terms of $\vec{e}$ and $\vec{\theta}^b$ as

$$\begin{aligned} \vec{p}_L &= m_a \vec{\theta}^a + \frac{1}{2} n^a \vec{e}_a, \\ \vec{p}_R &= m_a \vec{\theta}^a - \frac{1}{2} n^a \vec{e}_a, \end{aligned} \quad (9.14)$$

with n_a, m^a integers. Thus we can write explicitly the solitonic contribution the the partition function (9.9). The most interesting thing is that $\chi(\tau, \bar{\tau})$ is invariant under the duality transformation

$$\vec{e}_a \longleftrightarrow 2\vec{\theta}^a, \quad n^a \longleftrightarrow m_a. \quad (9.15)$$

This symmetry of the partition function is in fact a symmetry of the whole theory in the sense that the conformal field theories associated with the string moving in $\mathbf{R}^d/\Lambda_d$ and in $\mathbf{R}^d/\Lambda_d^*$ are the same theory (all its correlation functions are equal). We have checked the invariance of the one-loop partition function (9.9) under duality. The interchange of Λ_d and Λ_d^* is however a symmetry of the genus g partition function. However, when $g > 1$ there is an induced non-zero value for the dilaton field in the dual theory which results in a transformation of the string coupling constant [24, 25]. In the case in which we have only one dimension compactified to a circle of radius R , duality symmetry is reduced to the known fact that the theory is invariant under the replacement

$$R \longleftrightarrow \frac{2}{R}, \quad m \longleftrightarrow n \quad (9.16)$$

where n and m are respectively the momentum and the winding number, together with the transformation of the string coupling constant

$$g_{st} \longleftrightarrow \frac{\sqrt{2}}{R} g_{st}. \quad (9.17)$$

Duality symmetry can be generalized to non-flat manifolds. This can be seen using Buscher's transformation [26]. Let us consider the non-linear σ -model described by the following action

$$S = \frac{1}{2\pi} \int d^2 z G_{ij}(X) \partial X^i \bar{\partial} X^j, \quad (9.18)$$

and let us assume that the metric $G_{ij}(X)$ has an isometry. This means that there exists a vector field (Killing vector) $\xi^i(X)$ such that

$$\nabla_i \xi_j + \nabla_j \xi_i = 0. \quad (9.19)$$

As a consequence of the existence of this isometry, the original action (9.18) is invariant under the global transformation

$$\delta_\epsilon X^i = \epsilon \xi^i(X). \quad (9.20)$$

Buscher's construction allow us to perform a duality transformation for every isometry of the original model. To get the dual model we gauge the isometry by substituting the ordinary derivative ∂ by the covariant derivative

$$DX^i = \partial X^i + A \xi^i, \quad (9.21)$$

and similarly for $\bar{\partial}$, where A is a gauge field that transform under the gauge transformation as $\delta A = -\partial \epsilon$. The starting point to obtain the dual theory will be the gauge invariant action

$$S = \frac{1}{2\pi} \int d^2 z \left[G_{ij}(X) DX^i \bar{D}X^j + \chi(\partial \bar{A} - \bar{\partial} A) \right]. \quad (9.22)$$

If we integrate out the lagrangian multiplier χ , we have the constraint $\partial \bar{A} = \bar{\partial} A$ which, global issues aside, means that $(A, \bar{A})$ is pure gauge and therefore (9.22) is equivalent to (9.18) after a coordinate transformation. This leave us with the original theory described by (9.18). Since the action is quadratic in $A, \bar{A}$, we could first integrate out the gauge fields. If we take adapted coordinates, in which the isometry acts by translations in the X^0 coordinate, and integrating out the $A, \bar{A}$ fields, we arrive at the following action

$$S_{dual} = \frac{1}{2\pi} \int d^2 z \left[\tilde{G}_{00} \partial \chi \bar{\partial} \chi + \tilde{G}_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu + \tilde{B}_{0\mu} (\partial \chi \bar{\partial} X^\mu - \partial X^\mu \bar{\partial} \chi) \right], \quad (9.23)$$

where $\mu, \nu = 1, \dots, d$ and the new metric and antisymmetric tensor are given by

$$\begin{aligned} \tilde{G}_{00} &= \frac{1}{G_{00}}, \\ \tilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{1}{G_{00}} G_{0\mu} G_{0\nu}, \\ \tilde{B}_{0\mu} &= \frac{1}{G_{00}} G_{0\mu}, \end{aligned} \quad (9.24)$$

and $\tilde{G}_{0\mu} = \tilde{B}_{\mu\nu} = 0$. As an example of this construction let us consider the three-dimensional sphere S^3 with line element

$$ds^2 = a^2 (d\psi + \cos \theta d\phi)^2 + a^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (9.25)$$

The translations $\psi \rightarrow \psi + \text{constant}$ is an isometry which act without fixed points. Using (9.24) the dual metric becomes:

$$d\tilde{s}^2 = a^{-2} d\chi^2 + a^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (9.26)$$

and a corresponding antisymmetric tensor field. We see that the dual manifold is $S^1 \times S^2$, and then duality symmetry relates two field theories defined on spaces with different topology.

The construction outlined above can be generalized to the case in which we have D commuting Killing vectors ξ_a ($a = 1, \dots, D$), $[\xi_a, \xi_b] = 0$. In the non-abelian case the study of duality symmetry requires more care. The extension of Buscher's construction for manifolds with a non-abelian group of isometries has been carried out in [27].

These generalized symmetries play an important rôle in model building, as we indicated above, because they help to lift flat directions and to give a vacuum expectation value to the dilaton field. They can also be used to give mechanism of supersymmetry breaking, although they in general lead to Anti-de-Sitter spaces.

10 Mirror Symmetry

In order to make contact with the observed low energy world a basic fact in need of explanation is why it contains four open dimensions. For the bosonic string the number of open dimensions can go up to 26, while in the fermionic strings this number is reduced to 10, still too far from 4. Since we have no understanding of this basic fact one assumes that six of the spatial dimensions are compactified in an internal manifold K with a typical length of the order of the Planck length, so the string moves in $M_4 \times K$ where M_4 is the four dimensional Minkowski space.

The choice of the internal manifold K has very important consequences with respect to the spectrum of massless states in M_4 . In fact the massless modes are related with the zero modes of some operators in K , and the number of these zero modes depend in general on the topology of K . For example, the number of generations with respect of the gauge group E_6 is

$$N_g = \frac{|\chi(K)|}{2}, \quad (10.1)$$

where $\chi(K)$ is the Euler number of K . A very important requirement to be fulfilled for the string perturbative expansion to be consistent is that the spectrum of the theory must be supersymmetric. This can be accomplished if and only if the conformal field theory associated with the internal manifold has a global $N = 2$ superconformal symmetry and the charge of the $U(1)$ current entering in the $N = 2$ algebra is an integer [28]. This fact imposes a very strong constraint over the structure of the internal manifold. The condition for a $N = 1$ supersymmetric σ -model to admit a second supersymmetry is that the target space is a Kähler manifold [29]. Besides this condition we need also the σ -model to define a conformal field theory; this can only be accomplished when K has vanishing first Chern class, or in other words, when it admits a Ricci-flat Kähler metric [30]. These Ricci-flat Kähler manifolds are known as Calabi-Yau manifolds (for a review see, for example, [31]).

$N = 2$ superconformal field theories are therefore the building blocks for superstring compactifications. In these theories we have, besides the energy-momentum tensor $T(z)$, two

fermionic spin-3/2 supercurrents $G^\pm(z)$ and a $U(1)$ bosonic current $J(z)$. We can expand these operators in modes as we did in the case of the bosonic string ($N = 0$). Since now we have spinor fields we have two different types of boundary conditions for the fermionic fields when $z \rightarrow e^{2\pi i} z$, periodic or antiperiodic. In the first case we are in the so-called Neveu-Schwarz (NS) sector whereas in the second one we have the Ramond (R) sector. The zero modes in the (R) sector reproduce the $N = 2$ global supersymmetry algebra

$$\{G_0^+, G_0^-\} = 2 \left(L_0 - \frac{c}{24} \right), \quad (G_0^\pm)^2 = 0. \quad (10.2)$$

On the other hand we have realizations of these theories as supersymmetric σ -models defined on Calabi-Yau manifolds. As it is the case for bosonic σ -models, there can be more than one realization of a given $N = 2$ superconformal field theory in terms of supersymmetric σ -models defined on different Calabi-Yau manifolds. These two manifolds have to be regarded as equivalent from the point of view of a string propagating on them.

Given two manifolds K and K^* of real dimension d , we will say that they form a mirror pair if they satisfy the following relation [33, 32]

$$h^{p,q}(K) = h^{\frac{d}{2}-p,q}(K^*), \quad (10.3)$$

where $h^{p,q}(K)$ are the Hodge numbers of the manifold K . Given an $N = 2$ superconformal field theory it is possible to show that there exists in the NS sector a ring of chiral primary states, the chiral ring [33], which can be shown to be isomorphic to the deformed cohomology ring of the manifold. This isomorphism allow us to identify $h^{p,q}(K)$ with the dimension of the Hilbert space consisting of ground states with $U(1)_L \times U(1)_R$ charges (q_L, q_R) given by

$$(q_L, q_R) = \left(-\frac{d}{2} + p, -\frac{d}{2} + q \right). \quad (10.4)$$

The correspondence (10.4) together with (10.3) implies that mirror transformation, from the point of the associated $N = 2$ superconformal field theory, only amounts to a trivial change in the relative sign of the two components of the chiral current

$$J(z) \longrightarrow -J(z), \quad \bar{J}(\bar{z}) \longrightarrow \bar{J}(\bar{z}), \quad (10.5)$$

and then the final conformal field theory is identical to the original one. This equivalence of the associated conformal field theories implies a relation between the deformed cohomology rings of K and K^* . The reader should not be misled in believing that the mirror transformation is trivial. The very existence of such a Calabi-Yau manifold satisfying (10.3) for a given K is highly non-trivial, and although much progress has been made, there is so far no general proof of this conjecture. When considering three-fold ($d = 6$) Calabi-Yau manifolds in connection with string compactifications the relationship between the deformed cohomology rings of the mirror pair has far-reaching consequences in the computation of the Yukawa couplings between low energy fermions and scalars. Gauge invariance under the E_6 group implies that there can be only two types of such couplings, $\langle 27^3 \rangle$ and $\langle \bar{27}^3 \rangle$. Because of the Kaluza-Klein ansatz the strength of these couplings can be expressed as an integral over K_6 of a $(3, 3)$ form. What is really interesting is that the $\langle 27^3 \rangle$ coupling is a purely topological

quantity, while the $\langle \overline{27}^3 \rangle$ depends on the complex structure. In the conformal field theory the mirror map (10.5) implies that the (*chiral, chiral*) and the (*antichiral, chiral*) rings are interchanged. The equivalence between these rings and the deformed cohomology ring of the manifold implies that the mirror map interchanges Kähler moduli with the complex moduli. A consequence of this is that in the mirror manifold K_g^* the Yukawa coupling $\langle \overline{27}^3 \rangle$ is now a relatively easy to compute topological quantity. Since we know that the mirror map is a symmetry of the string theory, the Yukawa couplings computed in K_g and in K_g^* are exactly the same.

Duality symmetry and mirror symmetry are probably a reflection of a deeper symmetry underlying string theory that would be a generalization of classical general covariance. The exploration of general $N = 2$ superconformal theories will probably lead to generic low energy consequences of string theory which should be *signatures* of their presence.

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