Conformal Properties of Primary Fields in a $q$-Deformed Theory

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Abstract

We examine some of the standard features of primary fields in the framework of a $q$-deformed conformal field theory. By introducing a $q$-OPE between the energy momentum tensor and a primary field, we derive the $q$-analog of the conformal Ward identities for correlation functions of primary fields. We also obtain solutions to these identities for the two-point function.


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In recent years, there has been growing interest in the study of quantized universal enveloping algebras. Loosely called quantum groups, they first appeared in the study of the Quantum Yang-Baxter Equations related to the inverse scattering problem [1]. Subsequently, it was shown that they can be obtained from representations of mathematical structures called quasi-triangular Hopf algebras [2]. These structures which often depend on a parameter $q$ can be regarded as $q$-deformations of Lie algebras in the sense that as $q \rightarrow 1$ the algebra reduces to the usual Lie algebra.

Explicit realizations of some of these quantum groups have been obtained by many authors [2-4]. For instance the Jordan-Schwinger approach often used in the study of angular momentum algebra has been suitably generalized to give bosonic ($q$-oscillator) representations of the quantum group $SU_q(2)$ [4]. More recently, Curtright and Zachos [5] have constructed a $q$-analogue of the centreless Virasoro algebra by using a differential realization of $SU_q(1,1)$ (see also Ref.[6]). The central extension to this algebra has been furnished by Aizawa and Sato [7]. In fact, they have also found a $q$-deformed operator product expansion (OPE) between two energy momentum tensors which realizes this algebra. This naturally paves the way for a $q$-deformed conformal field theory.

In this letter we study the properties of primary fields in the spirit of Ref.[7]. Here we reexamine some of the well known issues pertaining to standard conformal field theory (CFT) [8] in the context of such a $q$-deformed theory. In particular, we introduce a $q$-OPE of the energy momentum tensor with a primary field which extends the $q$-OPE of Ref.[7] to primary fields of arbitrary conformal weights. The deformation reflected in this $q$-OPE is shown to be equivalent to the one used by Chaichan et.al. [9]. Using arguments paralleling those used in standard CFT, we obtain the $q$- analog of the conformal Ward identity and the projective Ward identities for correlation functions of primary fields. In particular, for the two-point function, it is shown that these Ward identities do not uniquely determine it when $|q| = 1$. Using the $q$-OPE we also realize the algebra between the modes of the energy momentum tensor and those of a holomorphic primary field.

We begin by summarizing some basic features of standard CFTs [8,10] that will be used or modified later. Consider a primary field $\Phi(z, \overline{z})$ with conformal weights $\ell, \overline{\ell}$. It is defined by its transformation under $z \rightarrow z' = f(z)$, $\overline{z} \rightarrow \overline{z'} = \overline{f(\overline{z})}$:

$$
\Phi(z, \overline{z}) \rightarrow \Phi'(z, \overline{z}) = (\partial f)^\ell(\overline{\partial f})^{\overline{\ell}}\Phi(f(z), \overline{f(\overline{z})})
$$

(1)

where $f(z)$ and $\overline{f(\overline{z})}$ are arbitrary holomorphic and antiholomorphic functions respectively. When the transformation is infinitesimal, i.e., $f(z) = z + \epsilon(z)$ and $\overline{f(\overline{z})} = \overline{z} + \overline{\epsilon(\overline{z})}$, then

$$
\Phi'(z, \overline{z}) = \Phi(z, \overline{z}) + \Delta_{\epsilon, \overline{\epsilon}}\Phi(z, \overline{z})
$$

(2a)

with

$$
\Delta_{\epsilon, \overline{\epsilon}}\Phi(z, \overline{z}) = (\ell \partial \epsilon + \epsilon \partial)\Phi(z, \overline{z}) + (\overline{\ell} \overline{\partial} \overline{\epsilon} + \overline{\epsilon} \overline{\partial})\Phi(z, \overline{z})
$$

(2b)
In particular when \( \epsilon(z) = \epsilon_n z^{-n+1} \) and \( \overline{\epsilon(z)} = \overline{\epsilon_n} z^{-n+1} \) where \( \epsilon_n \) and \( \overline{\epsilon_n} \) are small constants, we have

\[
\Delta_n \Phi(z, \overline{z}) = \epsilon_n \delta_n \Phi(z, \overline{z}) + \overline{\epsilon_n} \delta_n \Phi(z, \overline{z}) \tag{3a}
\]

where

\[
\delta_n \Phi(z, \overline{z}) = (z \partial + h(n + 1) - n)z^n \Phi(z, \overline{z}) \tag{3b}
\]

\[
\overline{\delta_n} \Phi(z, \overline{z}) = (\overline{z \partial} + \overline{h}(n + 1) - n)\overline{z^n} \Phi(z, \overline{z}) \tag{3c}
\]

In the following, we will only consider the holomorphic terms with similar results holding for the antiholomorphic ones.

In a quantum theory, the variation in \( \Phi(z, \overline{z}) \) is implemented by the "equal-time" commutator:

\[
\delta_n \Phi(w, \overline{w}) = \left[ \oint_{C_0} \frac{dz}{2 \pi i} z^{n+1} T(z), \Phi(w, \overline{w}) \right] \tag{4}
\]

where \( T(z) \) is the holomorphic component of the energy momentum tensor. On the \( z \)-plane, different times correspond to concentric circles of different radii and the notion of time ordering is replaced by that of radial ordering:

\[
R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w| \end{cases} \tag{5}
\]

In this scheme the "equal-time" commutator is given by [10]

\[
\oint_{C_0} \frac{dz}{2 \pi i} z^{n+1}[T(z), \Phi(w, \overline{w})] = \left( \oint_{|z| > |w|} - \oint_{|z| < |w|} \right) \frac{dz}{2 \pi i} z^{n+1} R(T(z)\Phi(w, \overline{w}))
\]

\[
= \oint_{C_p} \frac{dz}{2 \pi i} z^{n+1} R(T(z)\Phi(w, \overline{w})) \tag{6}
\]

where the last integral is taken around all the poles in the OPE of \( T(z) \Phi(w, \overline{w}) \) which we assume are located on the \( |z| = |w| \) contour. Indeed, by comparison with (3b) one can infer that

\[
T(z) \Phi(w, \overline{w}) = \frac{h \Phi(w, \overline{w})}{(z-w)^2} + \frac{\partial \Phi(w, \overline{w})}{(z-w)} + \text{regular terms.} \tag{7}
\]

Now a \( q \)-deformation of the theory is achieved by replacing eqns. (3b) and (3c) by the corresponding \( q \)-anlogs. For this purpose we consider the deformation as defined by Chaichan et. al. [9]:

\[
\delta_n \Phi(z, \overline{z}) \rightarrow \delta_n^q \Phi(z, \overline{z}) = [z \partial + h(n + 1) - n]z^n \Phi(z, \overline{z}) \tag{8}
\]

which essentially replaces the bracket in (3b) by a \( q \)-bracket defined by

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{9}
\]
Eqn. (8) thus, serves as a definition of a primary field in a $q$-deformed theory. We will now like to implement this variation as an "equal-time" commutator as in eqn. (6). To this end, we introduce a $q$-OPE of $T(z)$ with $\Phi(w, \bar{w})$, which we write as,

$$
(T(z)\Phi(w, \bar{w}))_q = \frac{[h/2]}{(z-w)} \left\{ \frac{\Phi(wq^{-1}, \bar{w})}{zq^{h/2} - wq^{-h/2}} + \frac{\Phi(wq, \bar{w})}{zq^{-h/2} - wq^{h/2}} \right\} + \frac{1}{(z-w)} \partial^q_w \Phi(w, \bar{w}) + \text{regular terms}
$$

(10)

where $\partial^q_w$ is the $q$-analog of the derivative:

$$
\partial^q_w f(w) = \frac{f(wq) - f(wq^{-1})}{w(q - q^{-1})}.
$$

(11)

Using this definition for the $q$-derivative, we can also rewrite the $q$-OPE as

$$
(T(z)\Phi(w, \bar{w}))_q = \frac{1}{w(q - q^{-1})} \left\{ \frac{\Phi(wq, \bar{w})}{z - wq^h} - \frac{\Phi(wq^{-1}, \bar{w})}{z - wq^{-h}} \right\} + \text{regular terms}
$$

(12)

which shows that it is singular at the points $z = wq^\pm h$. It is easy to verify that the above OPE leads to the correct variation in $\Phi$, by evaluating the integral in eqn. (6) with $Cp$ taken as a contour encircling the points $wq^h$ and $wq^{-h}$. Before proceeding farther, let us make a few observations:

1. It is evident from the above expression that there are poles present at two points rather than one. These poles are both of order 1, unlike the undeformed case where the $z = w$ pole is of order 2. It is interesting to note, however, that in the limit $q \to 1$ these poles will coalesce to form a pole of order 2 at $z = w$. In fact, in the limit $q \to 1$, our $q$-OPE reduces to the standard one (eqn. (7)).

2. Recall that the "equal-time" commutator in eqn. (6) was evaluated as a difference of two integrals with contours which are concentric and close to the $|z| = |w|$ contour but one having radius $|z| > |w|$ and the other $|z| < |w|$. They combine into a single contour which is taken to be a small circle centred around the singular point ($z = w$). For this scheme to be applicable here, we must require that the two poles lie on the $|z| = |w|$ contour, since otherwise this poles will not make any contributions to the integral. This means that we must restrict ourselves to the case when $|q| = 1$, i.e. $q$ should be taken as a pure phase ($q = e^{i\alpha \pi}$).

3. It is worth noting that the variation in $\Phi$ obtained by our $q$-OPE is similar to the one used by Chaichan et. al. only for the case $\epsilon(z) = z^{n+1}$. For arbitrary $\epsilon(z)$, their variation is assumed to be of the form

$$
\delta^q \Phi(z, \bar{z}) = \epsilon(z)^{1-h} \partial^q_z (\epsilon(z)^h \Phi(z, \bar{z}))
$$

(13)
while ours is given by
\[
\delta^q \Phi(z, \overline{z}) = \oint_{C_p} \frac{d\xi}{2\pi i} \epsilon(\xi) R(T(\xi)\Phi(z, \overline{z}))_q
\]
\[
= \epsilon(z q^h) \partial^q \Phi(z, \overline{z}) + [\hbar]\partial_{\xi}^h \epsilon(z) \Phi(z q^{-1}, \overline{z}).
\] (14)

Both, however, reduces to (8) when \( \epsilon(z) \) is taken to be \( z^{n+1} \).

(4) When \( h = 2 \), our expression is similar to the one given by Aizawa and Sato [7] for the OPE of two energy momentum tensors when the central charge in their expression is taken as zero.

With the \( \Phi \)-OPE defined as above, we can write down the \( \Phi \)-Ward identities for the correlation functions of primary fields. We begin by considering the action of the generator of infinitesimal conformal transformations on the correlation of \( n \) primary fields \( \{\Phi_i(w, \overline{w})\} \) with corresponding conformal weights \( h_i, \overline{h}_i (i = 1, 2, \ldots, n) \):.
\[
< \oint_{C_0} \frac{dz}{2\pi i} \epsilon(z) T(z) \Phi_1(w_1, \overline{w}_1) \ldots \Phi_n(w_n, \overline{w}_n) >_q .
\] (15)

(The contour \( C_0 \) encircles all the points \( \{w_i q^{h_k} | k = \pm 1\}_{i=1,2,\ldots,n} \) in the above expression, the correlation function \( < \ldots >_q \) is taken relative to the "in" (\( [0 >_q \)) and the "out" (\( q < 0 \)) vacuums which are defined by requiring that
\[
L_m|0 >_q = 0 \quad m > -1
\]
\[
q < 0|L_m = 0 \quad m < 1
\] (16a)
\[
(16b)
\]
where
\[
L_m = \oint_{C_0} \frac{dz}{2\pi i} z^{m+1} T(z), \quad m \in \mathbb{Z},
\] (17)
are modes in the expansion,
\[
T(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}.
\] (18)

Note that conditions (16a) and (16b) ensures the regularity of \( T(z)|0 >_q \) and its adjoint at \( z = 0 \) and \( z = \infty \).

By analyticity, the contour \( C_0 \) in expression (15) can be deformed to a sum of \( n \) contours with each contour \( C_i \) surrounding the points, \( \{w_i q^h, w_i q^{-h}\} \). Then as a consequence of the \( \Phi \)-OPE, we have
\[
\oint_{C_0} \frac{dz}{2\pi i} \epsilon(z) < T(z) \Phi_1(w_1, \overline{w}_1) \ldots \Phi_n(w_n, \overline{w}_n) >_q
\]
\[
\sum_{i=1}^n < \Phi_1(w_1, \overline{w}_1) \ldots \oint_{C_i} \frac{dz}{2\pi i} \epsilon(z) \Phi_i(w_i, \overline{w}_i) \ldots \Phi_n(w_n, \overline{w}_n) >_q
\]
\[
= \sum_{i=1}^n \oint_{C_i} \frac{dz}{2\pi i} \epsilon(z) c^h_{z;w_i} \Phi_1(w_1, \overline{w}_1) \ldots \Phi_n(w_n, \overline{w}_n) >_q.
\] (19)
where the differential operator \( L^h_{z; w_i} \) is given by

\[
L^h_{z; w_i} = \frac{1}{z - w_i} \left\{ \frac{q^{-w_i \partial w_i}}{zq^{h_i/2} - w_iq^{-h_i/2}} + \frac{q^{w_i \partial w_i}}{zq^{h_i/2} - w_iq^{-h_i/2}} \right\}. \tag{20}
\]

Furthermore, since \( \epsilon(z) \) is arbitrary, we can write,

\[
<T(z)\Phi_1(w_1, \overline{w}_1) \ldots \Phi_n(w_n, \overline{w}_n) >_q = \sum_{i=1}^{n} L^h_{z; w_i} < \Phi_1(w_1, \overline{w}_1) \ldots \Phi_n(w_n, \overline{w}_n) >_q, \tag{21}
\]

which is the unintegrated form of the \( q \)-Ward identity. It is easy to see that it reduces to the usual one as \( q \to 1 \).

Next let us consider the \( q \)-analog of the projective Ward Identities. From (16a) and (16b) it is easy to see that the generators \( L_{0, \pm 1} \) annihilate both the "in" and the "out" vacuums. On substituting \( \epsilon(z) = z^{m+1} \) for \( m = -1, 0, 1 \) into eqn. (19) and integrating, we have

\[
\sum_{i=1}^{n} w_i^{-1} [w_i \partial w_i] < \Phi_1(w_1, \overline{w}_1) \ldots \Phi_n(w_n, \overline{w}_n) >_q = 0 \tag{22a}
\]

\[
\sum_{i=1}^{n} [w_i \partial w_i + h_i] < \Phi_1(w_1, \overline{w}_1) \ldots \Phi_n(w_n, \overline{w}_n) >_q = 0 \tag{22b}
\]

\[
\sum_{i=1}^{n} w_i [w_i \partial w_i + 2h_i] < \Phi_1(w_1, \overline{w}_1) \ldots \Phi_n(w_n, \overline{w}_n) >_q = 0 \tag{22c}
\]

for any \( n \)-point function. These are the \( q \)-analogs of the projective Ward identities. Now, it is well known that in standard CFT the two-point and three-point functions are severely constrained by the Ward identities. In fact, they are uniquely determined up to a normalization constant. The situation for the \( q \)-deformed case is not quite the same. Here when \( |q| = 1 \) the \( q \)-Ward identities do not uniquely specify them as we will illustrate below.

For this purpose assume an ansatz for the correlation function of two primary fields \( \Phi_1(w_1, \overline{w}_1), \Phi_2(w_2, \overline{w}_2) \) with conformal weight \( h_1, h_2 \) respectively to be of the form

\[
<T \Phi_1(w_1, \overline{w}_1) \Phi_2(w_2, \overline{w}_2) >_q = \frac{1}{(w_1 - w_2)^n} \frac{(\overline{w}_1 - \overline{w}_2)^n}{|w_1| > |w_2|}, \tag{23}
\]

where

\[
(w_1 - w_2)^n_q = \prod_{k=1}^{n} (w_1 - w_2q^{n-2k+1}) = \sum_{k=1}^{n} \left[ \frac{n!}{n-k} \right] \frac{[n]!}{[k]!} w_1^{n-k} (-w_2)^k \tag{24}
\]
is the $q$-analog of the distance function $(w_1 - w_2)^n$ \cite{7}. On substitution into (22a), (22b) and (22c) we obtain the following conditions:

\begin{align}
[h_1 - n] + [h_2] &= 0 \quad (25a) \\
[h_2 - n] + [h_1] &= 0 \quad (25b) \\
[2h_1 - n] &= 0 \quad (25c) \\
[2h_2 - n] &= 0 \quad (25d) \\
[2h_2] - [2h_1] &= 0. \quad (25e)
\end{align}

Apart from the obvious solution

\[ h_1 = h_2 = n/2, \quad (26) \]

we also have for $q = e^{i\pi\alpha}$,

\begin{align}
&h_1 = n/2 + k/\alpha \quad (27a) \\
&h_2 = n/2 + l/\alpha, \quad (27b)
\end{align}

where $k$ and $l$ are arbitrary integers which are either both even or both odd. Adding the two we have

\[ n = h_1 + h_2 - (k + l)/\alpha \quad (28) \]

and this means that $n$ which characterizes the solution is not unique by virtue of the fact that $k$ and $l$ are arbitrary.

It is also interesting to study the commutator algebra (or rather "quocommutator") of the generators \{\(L_n\)\} with the modes \{\(\phi_m\)\} of a primary field. Consider a holomorphic primary field with conformal weights \((h, 0)\),

\[ \Phi(w) = \sum_{m\in \mathbb{Z} - h} \phi_m w^{-m-h}, \quad (29) \]

with the modes \{\(\phi_m\)\} satisfying

\[ \dot{\phi}_m = \oint_{C_0} \frac{dw}{2\pi i} w^{m+k-1} \Phi(w). \quad (30) \]

Here we would like to evaluate the bracket

\[ [L_n, \phi_m] \equiv (L_n \phi_m)_q - (\phi_m L_n)_q \quad (31) \]

where the terms \((\ )_q\) are defined via the $q$-product of two field operators $A(z)$ and $B(w)$ \cite{7}:

\[ (A(z)B(w))_q \equiv A(zq)B(wq^{-1}). \quad (32) \]
For instance, we have (following Ref.[7])

\[
(L_n \phi_m)_q = \oint_{C_1} \frac{dz}{2\pi i} \oint_{C_2} \frac{dw}{2\pi i} z^{n+1} w^{m+h-1} (T(z) \Phi(w))_q \\
= \oint_{C_1} \frac{dz}{2\pi i} \oint_{C_2} \frac{dw}{2\pi i} z^{n+1} w^{m+h-1} T(wq^{-1}) \\
= \oint_{C_1} \frac{dz}{2\pi i} \oint_{C_2} \frac{dw}{2\pi i} z^{n+1} w^{m+h-1} \sum_k L_k(zq)^{-k-2} \sum_l \phi_l(wq^{-1})^{-l-k} \\
= q^{m-n+h-2} L_n \phi_m
\]  

(33)

where \( C_1 \) and \( C_2 \) are contours about the origin such that \( C_2 \subset C_1 \). Similarly

\[
(\phi_m L_n)_q = q^{-(m-n+h-2)} \phi_m L_n.
\]  

(34)

Then by combining (33) and (34), the bracket in (31) can be reexpressed as

\[
[L_n, \phi_m] \equiv q^{m-n+h-2} L_n \phi_m - q^{-(m-n+h-2)} \phi_m L_n.
\]  

(35)

To evaluate this bracket, we use the \( \Phi \)-OPE:

\[
[L_n, \phi_m] = \oint_{C_0} \frac{dz}{2\pi i} \oint_{C_p} \frac{dw}{2\pi i} z^{n+1} w^{m+h-1} R(T(z) \Phi(w))_q \\
= [n(h-1) - m] \phi_{n+m},
\]  

(36)

which gives the "quonmutator" of \( L_n \) with \( \phi_m \).

\[
q^{m-n+h-2} L_n \phi_m - q^{-(m-n+h-2)} \phi_m L_n = [n(h-1) - m] \phi_{n+m}.
\]  

(37)

Again, we can see that this reduces to the standard result as \( q \rightarrow 1 \). It is also interesting to note that if we identify \( \phi_m \) with \( L_m \) with \( h = 2 \) then the above algebra corresponds to the \( q \)-deformed centreless Virasoro algebra,

\[
q^{m-n} L_n L_m - q^{-(m-n)} L_m L_n = [n - m] L_{n+m}
\]  

(38)

proposed by Curtright and Zachos [5].

Finally a few comments on the primary and descendant states. We define the primary state corresponding to a primary field \( \Phi(w, \overline{w}) \) of weights \((h, \overline{h})\) as

\[
|h, \overline{h} > = \lim_{w, \overline{w} -> 0} \Phi(w, \overline{w}) |0 >_q
\]  

(39)
in close analogy with standard case. In particular, for a holomorphic field with weights \( (h, 0) \) the primary state \( |h >_q \equiv |h, 0 >_q \) can also be defined as

\[
|h >_q = \phi^{-h} |0 >_q .
\]  

(40)

Note that the modes \( \{ \phi_m \} \) for \( m \geq -h + 1 \) must annihilate the "in" vacuum as a requirement for the regularity of \( \Phi(w)|0 >_q \) at \( w = 0 \). Using this fact together with (16a) and (37) we have

\[
L_n|h >_q = 0 \quad \text{for} \quad n > 0
\]

(41)

and

\[
L_0|h >_q = q^2 |h||h >_q .
\]

(42)

The \( q \)-descendant states are then constructed by subjecting the primary states to operations of \( L_n \)'s for \( n < 0 \):

\[
|h; k_1, k_2 \ldots k_m >_q = L_{-k_1} L_{-k_2} \ldots L_{-k_m} |h >_q .
\]

(43)

In passing, we would like to remark that it would also be interesting to study the conformal properties of secondary fields which give rise to the above \( q \)-descendant states. These together with the primary fields would then constitute a basis for the study of \( q \)-string theory.
REFERENCES

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