Kinetic Inflation in Stringy and Other Cosmologies

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Abstract

An inflationary epoch driven by the kinetic energy density in a dynamical Planck mass is studied. In the conformally related Einstein frame it is easiest to see the demands of successful inflation cannot be satisfied by kinetic inflation alone. Viewed in the original Jordan-Brans-Dicke frame, the obstacle is manifest as a kind of graceful exit problem and/or a kind of flatness problem. These arguments indicate the weakness of only the simplest formulation. From them can be gleaned directions toward successful kinetic inflation.

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I. PRELUDE

Recently, there has been interest in a possible gravity-driven, kinetic inflation. In the standard inflationary picture [1], a potential drives an era of accelerated expansion. A remarkable alternative appears in any theory for which the Planck mass is dynamical [3], such as Jordan-Brans-Dicke theories [2] or string theories. Due to the direct coupling of the Planck field to the metric, an acceleration of the cosmological expansion could result, even in the absence of a potential. The cosmic acceleration is driven by the unique kinetic energy density of the dynamical Planck mass.

In Refs. [3] and [4], general scalar-tensor theories of gravity, i.e. general Jordan-Brans-Dicke (JBD) theories, were investigated. It was shown that the pressure is negative and the expansion of the universe is accelerated if (i) the kinetic coupling parameter evolves with Planck mass subject to a bound or if (ii) the kinetic coupling is a negative constant for a certain branch of cosmological solutions [4]. It was shown independently in Ref. [5] that an accelerated cosmic expansion could be driven by the string dilaton. The low-energy effective action of string theory is equivalent to a JBD theory with negative kinetic coupling parameter. Thus the string dilaton produces an example of this general property of JBD theories of gravity.

Since an acceleration of the cosmological expansion is a fundamental element in inflation, it is natural to wonder if gravity-driven, kinetic inflation could supplant the standard potential-driven inflation [3]. Additional motivation comes from string theories. Previously, string theories were shown to interrupt potential dominated inflation [6]. The kinetic energy in the dilaton field overwhelmed the potential energy. As a result, standard inflation could not proceed unhindered. If the kinetic energy in the Planck field could actually drive inflation, in lieu of the potential, string theory would not only be compatible with inflation but would actually predict an unusual source of inflation.

However kinetic inflation stumbled from the outset. It has been uncovered that inflation could not be exited properly in string theory if the lowest order effective Lagrangian is used
Possible remedies to the graceless exit were suggested and are actively being pursued.

It might be hoped that more general JBD theories would be less problematic. As argued here, this is not the case. A nominal condition for the acceleration to be relevant for the causal physics of inflation was explored in Ref. [4]. In this way, the range of general scalar-tensor theories was restricted. Building from these ideas, the application of the kinetic driven acceleration to the phenomenology of inflation is studied more fully in this paper. The theory is taken to be of the JBD type with a general kinetic coupling parameter $\omega(m_{pl})$ where $m_{pl}$ is the Planck mass. The only metric-dilaton coupling is assumed to be a $\Phi R$ coupling where $\Phi$ is equal to the Planck mass squared, $\Phi = m_{pl}^2$, and $R$ is the Ricci scalar. Both the JBD frame and the conformally related Einstein frame are investigated. Obstacles to exiting inflation are encountered and a brief treatment of flatness is given.

II. INTRODUCTION

The presence of an acceleration alone does not ensure the success of an inflationary model. The universe must inflate enough for a causally connected region to envelop the extent of our observable universe. If this sufficient inflation condition is satisfied, then the horizon problem of the standard cosmology is resolved. The question of sufficient inflation is addressed in the JBD frame and in the conformally related Einstein frame in this paper. It is shown that successful inflation requires a positive acceleration of the Einstein frame scale factor at some time prior to today. Upon conformal transformation to the Einstein frame, no source for such an acceleration is apparent. Unless the acceleration is generated as a result of some subtle unforeseen effects, this implies that a successful execution of the kinetic inflation is impossible in this simplest $\Phi R$ model.

Though it is simpler to view in the Einstein frame, any obstacle to successfully completing inflation can be seen directly in the JBD frame. The difficulty will appear as a kind of graceful exit problem and/or a kind of flatness problem. The cosmological solutions can be broken into two branches. One branch of solutions, later named D-branch solutions, will make the
universe flatter. However, it is also shown that inflation cannot be exited successfully into an expanding phase, as was already noted for the specific string case in Ref [7]. The other branch of solutions, later named X-branch solutions, may be able to exit into an expanding phase. However, this branch encounters a sort of flatness problem. Both the graceless exit and the flatness problem can be understood in terms of the lack of an acceleration in the Einstein frame.

Although the difficulties discussed raise concerns, a kinetic driven inflation is not ruled out. The obstacle applies to the simplest \( \Phi R \) model for which the kinetic driven acceleration can be transformed away. Possible ways of avoiding these obstacles can be suggested. Effects which may be important include higher curvature couplings which result, for example, from the full string action, or massive string modes, or a potential for the Planck field. The inclusion of a potential may return the model to potential dominated inflation or may boost the kinetic energy so that the kinetic inflation is more effective. Perhaps, with some persistence the obstacles may be overcome.

### III. SCALAR-TENSOR GRAVITY

As a starting point, the action and resultant equations of motion for the universe and fields will be presented in this section. Some of the results of Ref. [4] are reproduced in this section for completeness.

The gravitational action for a general Jordan-Brans-Dicke theory is

\[
A[g_{\mu\nu}, \Phi] = \int d^4x \sqrt{-g} \left[ \frac{\Phi}{16\pi} R - \frac{\omega(\Phi)}{16\pi} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right] .
\]  

(3.1)

The metric signature \((- , + , + , +)\) was used and \( R \) is the scalar curvature. Newton’s constant \( G = \Phi^{-1} \). The field \( \Phi \) is related to the Planck mass through \( \Phi = m_{Pl}^2 \). A given theory is specified by choosing the functional form of \( \omega(\Phi) \). The low energy effective string action has the form (3.1) with \( \omega = -1 \) and \( \Phi = \exp(-\phi) \) where \( \phi \) is the dilaton:

\[
A_{\text{string}} = \int d^4x \sqrt{-g} \frac{e^{-\phi}}{16\pi} \left[ R + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] .
\]

(3.2)
The more general action (3.1) will be used throughout.

It is assumed that the spatial gradients in \( \Phi \) are negligible and in general the universe is homogeneous and isotropic. The metric is thus Friedmann-Robertson-Walker (FRW). The equation of motion for \( \Phi \) is

\[
\ddot{\Phi} + 3H \dot{\Phi} = -\frac{1}{(3 + 2\omega)} \frac{d\omega}{d\Phi} \dot{\Phi}^2 ,
\]

where \( H = \dot{a}/a \) and \( a \) is the scale factor. For economy of notation define

\[
f(\Phi) \equiv (1 + 2\omega(\Phi)/3)^{1/2} .
\]

The \( \Phi \) equation of motion has as solution

\[
\dot{\Phi} = \frac{-C}{a^3 f} .
\]

The constant of integration, \( C \), can be positive, negative, or zero.

The equation of motion for the scale factor \( a \), obtained from the Einstein-like equations,

\[
G_{\mu\nu} = (8\pi/\Phi)T_{\mu\nu} ,
\]

is

\[
H^2 + \frac{\kappa}{a^2} = -\frac{\dot{\Phi}}{\Phi} H + \frac{\omega}{6} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 .
\]

Eqn (3.6) can be solved for \( H \):

\[
H = -\frac{\dot{\Phi}}{2\Phi} \left( 1 \pm f\sqrt{1 - Z\kappa} \right) ,
\]

where \( Z \) is the ratio of the curvature term to a kinetic piece,

\[
Z = \frac{1/a^2}{f^2/4 \left( \dot{\Phi}/\Phi \right)^2} = \left( \frac{a^2\Phi}{2C} \right)^2 .
\]

When \( \kappa \) is taken to be zero, it will be explicitly stated. Nonzero curvature is considered in §VIB.

If curvature is negligible initially, so that the metric is taken for illustration to be roughly flat, then \( H \) reduces to

\[
H = -\frac{\dot{\Phi}}{2\Phi} \left( 1 \pm f \right) .
\]
There are two branches for $H$. If $f > 1$ ($\omega > 0$), then the Hubble expansion is positive if the upper sign is chosen for $\dot{\Phi} > 0$ and the lower sign is chosen for $\dot{\Phi} < 0$. If instead $f < 1$ ($\omega < 0$), then the universe contracts when $\dot{\Phi} > 0$ for either branch and the universe expands when $\dot{\Phi} < 0$ for either branch.

Some useful relations can be obtained for later reference. First consider the equation of motion (3.6) rewritten as

$$
\left( H + \frac{\dot{\Phi}}{2\Phi} \right)^2 = \frac{1}{4} f^2 \left( \frac{\dot{\Phi}}{\Phi} \right)^2 - \frac{\kappa}{a^2}. \tag{3.10}
$$

Taking the square-root and reexpressing this equation gives

$$
\frac{d \ln(\Phi a^2)}{dt} = \pm \left( -f \frac{\dot{\Phi}}{\Phi} \right) \sqrt{1 - \kappa}. \tag{3.11}
$$

The horizon distance can be related to $\Phi$ and $a$ with the aid of eqn (3.11). Consider the three values of $\kappa$ separately. First, take $\kappa = 0$. Using (3.5) on the left hand side of (3.11) and integrating over $dt$ gives

$$
\Phi a^2(1 - \delta) = \pm C \int \frac{dt'}{a'} \tag{3.12}
$$

the constant of integration is included in

$$
\delta = \frac{a_i^2 \Phi_i}{a^2 \Phi}. \tag{3.13}
$$

The subscript $i$ denotes initial values and $\delta$ is always positive. From (3.12), the particle horizon distance can be deduced. The distance a photon has travelled since the beginning of time is defined by the integral $d_\gamma = \int dt'/a'$. Thus from (3.12), it follows that

$$
d_\gamma = \frac{\pm \Phi a^2(1 - \delta)}{C}, \tag{3.14}
$$

or

$$
d_\gamma = \pm \left( -\frac{\Phi}{f\Phi} \right) (1 - \delta). \tag{3.15}
$$

The same procedure as was followed for the flat case can be used to find the horizon size for $\kappa \neq 0$. For $\kappa = +1$, the horizon size is
\[ d_\gamma = \pm (2a) \arcsin \left( \frac{a^2 \Phi}{2C} \right) \]  

(3.16)

while for \( \kappa = -1 \) the horizon size is

\[ d_\gamma = \pm (2a) \arcsinh \left( \frac{a^2 \Phi}{2C} \right) \]  

(3.17)

Some of these relations will prove useful in the following sections.

**A. Cosmic Acceleration**

The acceleration of the scale factor in a flat universe is given by [4]

\[ \frac{\ddot{a}}{a} = -\frac{1}{2} \left( \frac{\dot{\Phi}}{\dot{\Phi}} \right)^2 f \left( f \pm 1 - \frac{df}{d\ln \Phi} \frac{1}{f^2} \right) . \]  

(3.18)

In order for \( \ddot{a} > 0 \), the following condition must be satisfied:

\[ f \pm 1 - \frac{df}{d\ln \Phi} \frac{1}{f^2} < 0 \]  

(3.19)

The scale factor of the universe will accelerate if:

(i) the functional form of \( f = (1 + 2\omega(\Phi)/3)^{1/2} \) changes as the universe evolves such that it obeys the bound of (3.19). The bound can in principle be satisfied for any value of \( \dot{\Phi} \) and for both branches. Notice also that \( f \) is not constrained to be less than 1.

Alternatively, \( \ddot{a} > 0 \) if

(ii) a constant but negative \( \omega \) \((\dot{f} < 1)\) is combined with the lower sign only. As mentioned below eqn (3.9), this corresponds to an accelerated expansion only if \( \dot{\Phi} < 0 \). The particular combination \( \omega = -1, \dot{\Phi} < 0 \) with the lower sign, was studied in Refs. [5] and [7] in the context of string theory.

**IV. SUFFICIENT INFLATION**

An acceleration of the scale factor alone by no means guarantees a resolution of the initial condition problems of cosmology. Consider the horizon problem. In the standard
cosmology, our observable horizon contains many regions which were causally disconnected at earlier times. Consequently, the smoothness of the observed universe would appear to have no causal explanation. In the inflationary scenario, a dynamical explanation of the large scale homogeneity and isotropy is provided. In standard inflation a potential energy density drives an era of accelerated expansion. During the rapid expansion, a causally connected region that was small at the beginning of inflation grew large enough to contain our observed universe. Subsequently, entropy was produced as the energy in the potential converted to radiation.

The question is, if the potential driven acceleration is replaced by a kinetic driven acceleration, can the horizon problem be resolved? In other words, can a kinetic driven acceleration blow up a region causally connected early in the history of the universe large enough to encompass our observable universe today.

This requirement of sufficient inflation can be stated in equations. The comoving size of a causally connected region at some earlier time $t_*$ is defined by the comoving distance a photon has travelled since the beginning of time, $d_\gamma(t_*)/a_*$. Today, the extent of the observable universe is $\sim H_o^{-1}$. A causal explanation for the smoothness of the universe today follows if the comoving size of the observable universe today fits inside a comoving region causally connected at $t_*$,

$$\frac{d_\gamma}{a_*} \gtrsim \frac{1}{H_o a_*}.$$  \hspace{1cm} (4.1)

A detailed study of (4.1) is left to appendix B. In the next section, this expression will be considered in terms of conformally transformed variables.

**V. EINSTEIN FRAME**

The condition of sufficient inflation can be studied under a conformal transformation to the Einstein frame. In the Einstein frame, the theory of gravity is the usual Einstein theory with a fundamental Planck scale $M_p = 1.2 \times 10^{19}$ GeV. In the Einstein picture the FRW
The universe is filled with an ordinary, minimally coupled scalar field. There is no acceleration of the Einstein frame scale factor. However, it is argued in this section that the sufficient inflation condition requires an acceleration some time prior to today in the Einstein frame. This indicates that the kinetic acceleration felt in the Jordan-Brans-Dicke (JBD) frame cannot lead to a successful inflation model unless additional effects are invoked.

The conformal transformation on the metric
\[ g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \]  
defined by \( \Omega = M_0/\Phi^{1/2} \) takes the JBD action into an Einstein theory. Under the conformal transformation the action becomes (see for instance Ref. [8]),
\[ A = \int d^4 x \sqrt{-\tilde{g}} \left[ \frac{M^2}{16\pi} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi \right] \]  
(5.2) 
A redefinition of the fields was also performed:
\[ \Psi \equiv M_0 \sqrt{\frac{3}{16\pi}} \int \frac{(1 + 2\omega/3)^{1/2}}{\Phi} d\Phi. \]  
(5.3) 
Notice that \( \Psi \) is real and the energy density in the \( \Psi \)-field, \( \rho_\Psi = \dot{\Psi}^2/2 \) is positive if \( \omega \geq -3/2 \).

The momentum associated with the field, \( p_\Psi = p_\Psi \), is always positive.

In addition to the conformal transformation, perform the coordinate transformation
\[ d\tilde{t} = \Omega^{-1} dt \]  
\[ \tilde{a} = \Omega^{-1} a \]  
(5.4) 
(5.5) 
so that the spacetime interval can be written in the usual FRW form,
\[ d\tilde{s}^2 = \Omega^{-2} ds^2 \]  
(5.6)
\[ = -(\Omega^{-1} dt)^2 + (\Omega^{-1} a)^2 \left[ \frac{d\varphi^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]  
(5.7)
\[ = -\dot{a}^2 + \ddot{a}^2 \left[ \frac{d\varphi^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \]  
(5.8) 
The metric is the usual FRW metric with scale factor \( \ddot{a} \propto \Phi^{1/2} a \). Incidentally, Einstein time always increases with increasing JBD time. To see this, notice that the conformal
transformation $\Omega = M_\phi / \Phi^{1/2}$ is always positive. If $dt > 0$, so that time ticks forward in the JBD frame, then $\tilde{d}t$ is always greater than zero. Since $\tilde{d}t / dt = \Omega^{-1} > 0$, the slope of $\tilde{t}(t)$ is positive. Therefore Einstein time always ticks forward with JBD time.

The evolution of the scale factor $\tilde{a}$ and of $\Psi$ can be found directly in the Einstein frame in a flat cosmology. The Hubble equation is

$$\tilde{H}^2 = \left( \frac{8\pi}{3M_o^2} \right) \tilde{\rho}_\Psi ,$$

where the kinetic energy density in $\Psi$ is

$$\tilde{\rho}_\Psi = \frac{1}{2} \left( \frac{d\Psi}{d\tilde{t}} \right)^2 .$$

As always, an overdot will be used to denote a derivative with respect to JBD time. A derivative with respect to Einstein time will be written out explicitly. The $\Psi$ equation of motion is

$$\frac{d^2\Psi}{d\tilde{t}^2} + 3\tilde{H} \frac{d\Psi}{d\tilde{t}} = 0 ,$$

which has solution $d\Psi / d\tilde{t} = -B / \tilde{a}^3$ where $B$ is an arbitrary constant of integration. Using eqn (5.3), $B$ can be related to the arbitrary constant $C$ of eqn (3.5),

$$B = \frac{C}{M_o} \sqrt{\frac{3}{16\pi}} .$$

From the arguments of appendix A, only $\dot{\Phi} < 0$ is relevant and so only $d\Psi / d\tilde{t} < 0$ is relevant. Therefore we shall hereafter assume $B > 0$ ($C > 0$). Using the solution to the $\Psi$ equation of motion in $\tilde{H}$ of eqn (5.9) gives

$$\tilde{H} = \pm \frac{C}{2M_o^2} \frac{1}{\tilde{a}^3} .$$

The upper sign corresponds to an expansion and will hereafter be called X-branch solutions. The lower sign corresponds to a decrease in the Einstein frame scale factor and will hereafter be called D-branch solutions.

The Einstein frame scale factor can be found as a function of Einstein time by integrating (5.13),
\[ \tilde{a} = \left[ \tilde{a}_i^3 \pm \frac{3C}{2M_i^2} (\tilde{t} - \tilde{t}(t_i)) \right]^{1/3}. \]  

(5.14)

For X-branch solutions \( \tilde{a} > \tilde{a}_i \) and the scale factor grows. For the D-branch solutions \( \tilde{a} < \tilde{a}_i \) and the scale factor drops.

For completeness, \( \Psi \) can be found as a function of Einstein time,

\[ \Psi = \Psi_i \mp \frac{M_i}{\sqrt{12\pi}} \ln \left[ \tilde{a}_i^3 \pm \frac{3C}{2M_i^2} (\tilde{t} - \tilde{t}(t_i)) \right]. \]  

(5.15)

Again, the upper sign refers to X-branch solutions and the lower sign refers to D-branch solutions.

Clearly, the second time derivative of the scale factor is always negative,

\[ \frac{d^2 \tilde{a}}{dt^2} = -\frac{1}{2} \left( \frac{C}{M_i^2} \right)^2 \frac{1}{\tilde{a}^5}. \]  

(5.16)

For X-branch solutions the universe expands at a decelerating rate. For D-branch solutions, (5.16) gets more and more negative. The universe contracts at an accelerated rate.

The Einstein frame cosmology seems to know nothing about \( \omega(\Phi) \) and thus does not appear to distinguish between different scalar-tensor theories. However, an observer who carries rulers and clocks must be included in order to compare events in one frame with events in a conformally related frame. Conformally related observers do agree on the occurrence of events though they disagree on the interpretation of the physics. Once an observer is included, the rulers and clocks of that observer can be shown to scale as functions of the conformal factor. The different scalar-tensor theories can then be distinguished [9].

Consider a scalar-tensor theory which leads to an acceleration of the cosmic expansion in the JBD frame. The acceleration is conformally transformed away in the Einstein frame. Still, a kinetic driven acceleration of the scale factor in one frame and a deceleration of the scale factor in a conformally related frame can be made consistent. The acceleration of the JBD scale factor is attributed to the rate of change of rulers in the Einstein frame [1].

While it is true that the effect can be viewed without contradiction in either the JBD or the Einstein picture, the sufficient inflation condition imposes an additional requirement.
The equivalence of the two pictures means that if eqn (4.1) is satisfied in one frame, it must be satisfied in all frames. If, on the other hand, eqn (4.1) cannot be satisfied in one frame, then it cannot be satisfied in a conformally related frame. One could try case by case in the JBD frame to hunt for a solution to the sufficient inflation condition. However, by looking in the Einstein frame we can immediately show that expression (4.1), transformed accordingly, cannot be satisfied by a kinetic-driven acceleration alone, at least not in a $\Phi R$ model. By considering the Einstein picture, it can be predicted that any attempt to meet condition (4.1) will fail unless additional effects are conjured up.

Consider the sufficient inflation condition (4.1)

$$\frac{d_{x*}}{a_*} \geq \frac{1}{H_o a_o} .$$

(5.17)

This expression can be rewritten in terms of Einstein variables by conformally transforming both sides. Consider first the left hand side. For all times, the comoving horizon size is the same in both frames

$$\frac{d_{x}}{a} = \int dt/a = \int d\tilde{t}/\tilde{a} = \frac{d_{x}}{\tilde{a}} .$$

(5.18)

To transform the right hand side of (5.17) notice that in general

$$\tilde{a} \ddot{H} = a(H - \dot{\Phi}/2\Phi) .$$

(5.19)

The right hand side of (5.17) can be found by evaluating (5.19) today, $\tilde{a}_o \ddot{H}_o = a_o(H_o - \dot{\Phi}_o/2\Phi_o)$. By today, the Planck mass should be anchored in the vicinity of $M_* = 1.2 \times 10^{19}$ GeV and the conformal factor should be nearly one. In other words, by today, $\dot{\Phi}_o \sim 0$. So, today, Einstein and JBD quantities are equivalent. Therefore (5.19) reduces to $\ddot{H}_o \tilde{a}_o = H_o a_o$.

It follows that in terms of Einstein quantities, the sufficient inflation condition requires

$$\frac{d_{x*}}{a_*} \geq \frac{1}{H_o a_o} .$$

(5.20)

To clarify, eqn (5.20) is the conformally transformed version of the sufficient inflation condition imposed in the JBD frame. If the Planck mass today was not near $M_*$, then eqn (5.20) would not look so similar to the JBD form.
To evaluate (5.20) further, one would need to know the particle horizon distance. The distance a photon has travelled since $\tilde{t}(t_i)$ is

$$d_\gamma(\tilde{t}) = \frac{\dot{a}}{a} \int_{\tilde{t}(t_i)}^{\tilde{t}(t)} \frac{d\tilde{t}}{\hat{a}(\tilde{t})} = \frac{M^2}{C} \tilde{a}^3 (|1 - \delta|), \quad (5.21)$$

where, as in eqn (3.13), $\delta \equiv \frac{\dot{a}^2}{\tilde{a}^2} = \frac{(a^3 \Phi_1)}{(a^2 \Phi)}$. For X-branch solutions $\delta < 1$ while for D-branch solutions $\delta > 1$. Therefore, in terms of $\hat{H}$,

$$d_\gamma(\tilde{t}) = \frac{1}{2} \left| (1 - \delta) \hat{H}^{-1}(\tilde{t}) \right| \quad (5.22)$$

Consider for now solutions for which the Einstein universe expands, i.e. X-branch solutions. With (5.22) in eqn (5.20), sufficient inflation demands

$$\frac{1}{\hat{H} \tilde{a}_c} \geq \frac{1}{\hat{H}_0 \tilde{a}_h}, \quad (5.23)$$

where factors of order 1 were dropped. Using the definition of $\hat{H}$, the above equation is equivalent to

$$\frac{d\tilde{a}}{dt} \bigg|_c \geq \frac{d\tilde{a}}{dt} \bigg|_h. \quad (5.24)$$

Eqn (5.24) states that the expansion rate is greater today than in the past; that is, the expansion increases at some time prior to today and thus the Einstein frame scale factor accelerates at some time prior to today.

For D-branch solutions the universe contracts. It is even clearer here that an acceleration is ultimately needed. Today we live in the Einstein frame and today the universe expands, $\frac{d\tilde{a}}{dt} \bigg|_c > 0$. At some time between today and the epoch of unusual behavior, during which the Einstein frame scale factor contracted, an acceleration is needed.

If the sufficient inflation condition is met, even the conformally transformed scale factor must accelerate. No acceleration of the Einstein scale factor is manifest in kinetic-driven inflation (see eqn (5.16)). If the effect is to be meaningful for inflation, this contradiction in the Einstein picture must be resolved.

Perhaps there is some way around this obstacle. It may be that there is some unforseen subtlety which rectifies this. Most likely, additional effects will need to be posited. If such
additional effects are capable of accelerating the Einstein frame scale factor, the kinetic
acceleration may be altogether replaced. It should be stressed that the above argument is
limited to $\Phi R$ theories. Higher order curvature terms which cannot be transformed away,
for example, will not be tainted by these results.

The Einstein frame results reveal in one swoop the obstacle to completing kinetic in-
flation. Although this alone is convincing, it is instructive to consider the problems as they
appear in the JBD frame. For completeness then, we finish the paper by considering the
JBD frame. The obstacle will appear as a graceful exit problem and/or a flatness problem.
We will see that the shortcomings in the JBD frame can again be attributed to the absence
of an acceleration in the Einstein picture.

VI. JORDAN-BRANS-DICKE FRAME

There seem to be so many different possible cases to study. However, the set of possibilites
relevant for inflation can be paired down considerably. While the expansion of the universe
may be accelerated by a variety of forms for $\omega$ and values of $\dot{\Phi}$, only cosmologies for which
$\dot{\Phi} < 0$ and $f < 1 \ (\omega < 0)$ can be relevant for inflation. For a justification of this statement
the reader is refered to appendix A or Ref. [4] where the argument first appears. There
are still two branches of cosmological solutions: X- and D-branches. That nomenclature is
carried over into the JBD frame discussion.\textsuperscript{1}

\textsuperscript{1}To connect with the terminology of Ref. [7], note the X-branch was there called the minus branch
and the D-branch was there called the plus branch.
The content of the two different branches can be summarized in the following chart:

<table>
<thead>
<tr>
<th>$\Phi &lt; 0, \omega(\Phi) &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X-branch (upper sign)</td>
</tr>
<tr>
<td>$H &gt; 0$</td>
</tr>
<tr>
<td>$\Phi a^2$ grows</td>
</tr>
<tr>
<td>$\ddot{a}$ expands</td>
</tr>
<tr>
<td>D-branch (lower sign)</td>
</tr>
<tr>
<td>$H &gt; 0$</td>
</tr>
<tr>
<td>$\Phi a^2$ drops</td>
</tr>
<tr>
<td>$\ddot{a}$ Drops</td>
</tr>
</tbody>
</table>

Only the range of parameters represented in the above chart are considered. This should make discussion of the JBD frame more manageable.

A. Graceful Exit and the D-Branch

Any obstacle to successfully completing inflation should be encountered directly in the JBD frame, without reference to the Einstein frame. For D-branch solutions the obstacle takes the form of a graceful exit problem. The graceless exit is discussed in this section.

It was argued in Ref. [7], in the context of string theory, that the accelerated inflation could not be exited gracefully. As it happens, the string model is an example of the more general acceleration of §III. Specifically, the low energy effective string action corresponds to the D-branch solution for $\omega = -1$ in the language of this paper. It is shown here that all accelerations in a D-branch solution will suffer from a kind of graceful exit problem, regardless of the form of $\omega(\Phi)$. For variable $\omega$, the graceful exit problem is of a new sort. The accelerated expansion can be turned off by tuning $\omega(\Phi)$. However, there is another behavior which cannot be exited.

As can be seen from quick inspection of eqn (3.19), the accelerated expansion can easily be exited if $\omega(\Phi)$ is allowed to vary. However, for the D-branch, a branch change is needed if the product $a\Phi^{1/2}$ is not to drop forever. Consider the evolution of the product $a\Phi^{1/2}$ given by eqn (3.11). For $\dot{\Phi} < 0$ and the D-branch, notice eqn (3.11) becomes
\[
d \frac{d \ln(a \Phi^{1/2})}{dt} = - \frac{f |\dot{\Phi}|}{2\Phi} = - \frac{C}{2a^3 \Phi} \tag{6.1}
\]

The quantity \(a \Phi^{1/2}\) always decreases. Today by contrast, no variation in the strength of gravity has been observed. It follows that the field \(\Phi\) must be nearly constant today. Since the universe still expands, the product \(a \Phi^{1/2}\) grows today. A smooth transition to our universe appears to be prohibited.

To see this another way, consider eqn (6.1) rewritten in terms of \(H\),

\[
H = \frac{|\dot{\Phi}|}{2\Phi} (1 - f) . \tag{6.2}
\]

If \(f\) were to grow in excess of 1, the epoch of accelerated growth of the scale factor would be exited, as eqn (3.19) shows. However, as \(f\) exceeds 1, \(H\) will pass through zero and the universe will undergo collapse.\(^2\) Again, the D-branch does not connect smoothly onto our expanding cosmology.

If other energy densities which do not couple directly to \(\Phi\) are included, then

\[
\frac{d \ln(a \Phi^{1/2})}{dt} = - \sqrt{\left(\frac{f |\dot{\Phi}|}{2\Phi}\right)^2 + \frac{8\pi \rho}{3\Phi}} . \tag{6.3}
\]

Regardless of the heating mechanism, if energy is transferred in a simple way from kinetic into a hot particle bath, the behavior will not turn around. In order to turn this around, negative quantities must appear on the right hand side of (6.3). Attempts were made in [7] to include negative potential energy densities which arise naturally in supersymmetry. Although the authors found for their case that a branch change could be induced by a negative potential, they also found the branch change occurred in pairs. If the universe began with the D-branch it would insist on ending up in the D-branch.

In the conformally related Einstein frame, \(\Phi^{1/2} \propto \dot{a}\) dropping corresponds to a contracting universe. The difficulty in effecting the branch change can be seen in the Einstein frame

\(^2\)When \(\omega\) is allowed to be negative, the universe can undergo collapse even for \(\kappa = 0\). In a Brans-Dicke universe, for which \(\omega\) is a positive constant, this is not possible [10].
as the difficulty in turning the evolution from a contraction into an expansion. As argued in §V, an acceleration of the Einstein frame scale factor is needed.

This trap appears at first glance to affect only D-branch solutions. There are entire families of X-branch solutions which do not immediately run into this obstacle. For X-branch solutions, $a \Phi^{1/2}$ grows and a smooth connection onto our universe may be possible. However, when curvature is included, the X-branch will tend to turn into a D-branch if $\kappa = +1$. An attempt to avoid this leads to a kind of flatness problem.

**B. Flatness**

In the standard cosmology, the universe would quickly veer away from a flat appearance, unless extraordinary initial conditions are imposed. Initially, curvature is unimportant in determining the dynamics of the scale factor and the universe looks roughly flat. As the scale factor grows, the curvature term should quickly come to dominate in the determination of the standard cosmological evolution since the curvature term in the equation of motion scales as $1/a^2$ while the standard radiation density term scales as $\rho \sim 1/a^4$. If $\rho$ is to continue to influence the cosmic dynamics today, then the entropy in radiation must be enormous. In terms of the dimensionless entropy, $S \propto a^3 T^3$, the standard cosmology requires $S \gtrsim 10^{90}$. The need for such a huge value of the otherwise arbitrary constant entropy is the famed flatness problem.

During any epoch of acceleration, by contrast, the universe is made flatter. This can be demonstrated by considering the equation of motion

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi}{3\Phi} \rho .$$  \hspace{1cm} (6.4)

Define the curvature radius

$$R_{\text{curv}} = \frac{R(t)}{|\kappa|^{1/2}} .$$  \hspace{1cm} (6.5)

Comparing the scales $R_{\text{curv}}^{-1}$ and $H$ shows
As the universe accelerates, $\dot{a}$ grows. The importance of curvature will diminish as $\dot{a}$ grows, thus rendering the universe flatter. In inflation, the huge entropy is generated dynamically at the end of the accelerating phase.

While it is true for a kinetic driven acceleration that the ratio (6.6) drops, closer inspection reveals a subtle kind of flatness problem for X-branch solutions. Consider expression (3.11) with curvature included:

$$\frac{R_{\text{curv}}^{-1}}{H} = \frac{|\kappa|^{1/2}}{\dot{a}} \quad (6.6)$$

where again $Z = \left(\frac{a^2 \Phi}{\mathcal{V}}\right)^2$. For X-branch solutions, the right hand side of (6.7) is positive and $\Phi a^2$ grows. Consequently, $Z$ grows and the curvature term gains in importance relative to the kinetic term in the square root. [Although curvature is less important than $H$ (eqn (6.6)), curvature may be more important than this piece of $H$.]

Consider the X-branch with different curvatures. For $\kappa = +1$, the right hand side of (6.7) passes through zero when $Z$ reaches 1. A branch change is induced as the X-branch evolves into a D-branch solution. Subsequently, the right hand side of (6.7) is negative, $Z$ drops, and $a^2 \dot{\Phi}$ drops forever. The usual graceful exit problem of the D-branch solutions is encountered.

To avoid this fate, one could require that inflation ends and the entropy is released before $Z$ gets near 1. However, this requirement amounts to a kind of flatness problem. In order to see this, consider the sufficient inflation condition (4.1). We make two assumptions. Firstly, it is assumed that at some time $t_{\text{end}}$ entropy is produced and inflation ends. Secondly, it is assumed that the universe has evolved adiabatically since the end of inflation so that $a_{\text{end}} T_{\text{end}} = a_s T_s$. The Hubble constant today can be expressed as $H_s = a_s^{1/2} T_s^2 / M_s$ where, again, $M_s = 1.2 \times 10^{19}$ GeV is the Planck mass today and $a_s = \gamma(t_s) \eta_s = (8\pi/3)(\pi^2/30)g_s(t_s) \eta_s$ where $\eta_s \sim 10^4 - 10^5$ is the ratio today of the energy density in matter to that in radiation. In a model of kinetic inflation the constraint (4.1) becomes
\[
\frac{a_{\text{end}}}{a_s} \gtrsim a_s^{-1/2} \frac{M_s}{T_s} \left[ \frac{\dot{a}_s^{-1}}{T_{\text{end}}} \right], \quad (6.8)
\]

If \( Z < 1 \) then \( a_{\text{end}} < \frac{C^{1/2}}{\Phi_{\text{end}}^{1/2}} \). Notice that for X-branch solutions, \( a_s \Phi_s^{1/2} < a_{\text{end}} \Phi_{\text{end}}^{1/2} \). Using these inequalities in the sufficient inflation condition, along with eqn (3.14), gives (with \( \delta = 0 \) for simplicity)

\[
C^{1/2} \gtrsim 10^{30} \frac{\Phi_{\text{end}}^{1/2}}{T_{\text{end}}}, \quad (6.9)
\]

If the Planck mass at the end of inflation is \( \sim M_s \) and \( T_{\text{end}} \) is presumably much less than this, the above condition can only be satisfied if \( C \) is huge; at least \( C \gtrsim 10^{60} \). This represents a kind of fine tuning, a kind of flatness problem.

In the Einstein frame, the flatness problem is seen clearly. Curvature gains in importance in that picture since there is no acceleration; \( \tilde{R}_{\text{curv}}/\tilde{H} = |\kappa|^{1/2}/(d\tilde{a}/d\tilde{t}) \). The Einstein expansion, \( d\bar{a}/d\bar{t} \), is positive and drops for the X-branch. As the universe decelerates, curvature becomes more important. In order to tilt the energy balance in favor of the kinetic energy density in the \( \Psi \) field, a large value of \( C \) is needed. This is analogous to requiring a large value of \( S \) in the standard cosmology to tilt the energy balance in favor of the radiation energy density.

**VII. DISCUSSION**

Conformally related pictures represent no more than different interpretations of the same universe. While these interpretations may disagree wildly, the results of all measurements in terms of a given observer’s rulers and clocks must be the same. In the simplest \( \Phi R \) model, the gravity-driven acceleration of the cosmic expansion can be absorbed under a conformal transformation to the Einstein frame. An acceleration of the cosmic expansion in one frame and a deceleration in a conformally related frame can be attributed to different interpretations of the laws of physics and does not represent an inconsistency in the JBD and the Einstein frame.
However, the condition of sufficient inflation imposes an additional requirement. Implicit in this condition is the demand that today's universe can be reproduced. It is easiest to view inflation upon conformal transformation to the Einstein frame. By doing so, it was shown that the demands of sufficient inflation require an acceleration of the scale factor in the Einstein frame. Since no source for an acceleration of the Einstein expansion exists, this argues that a kinetic inflation alone will be unable to lead to our smooth observable universe today.

It must be that any obstacle to completing inflation can be understood entirely in the JBD frame, without reference to the Einstein frame. This is in fact the case. For all D-branch solutions, the obstacle is manifest in a graceful exit problem. This quandry was found in Ref [7] for the specific case of $\omega = -1$ which describes the low energy effective action of string theory. It is argued in this paper that for general $\omega(\Phi)$, the D-branch always encounters a similar graceful exit problem.

For X-branch solutions, a general no-go has been more difficult to formulate directly in the JBD frame. For $\kappa = +1$, the obstacle does surface in general. For $\kappa = +1$, X-branch solutions evolve into D-branch solutions. As for all D-branch solutions, graceful exit is a problem. A fine tuning of the arbitrary constants is needed to avoid the induced branch change. Such fine tuning was shown to be akin to the flatness problem. It is precisely such unnatural tuning which inflation tries to avoid. For $\kappa = -1$ and $\kappa = 0$, a no-go in the JBD frame is not immediately obvious. However, the results of the Einstein frame predict that any such attempt will be thwarted. It can thus be conjectured that similar obstacles would apply.

The results here indicate the woes of the simplest attempt. A kinetic-driven inflation is not ruled out. Modifications which lead to a kinetic acceleration which cannot be transformed away would be exempt from the arguments in this paper. An attempt to obviate these concerns points, for example, to higher order curvature terms or additional effects from higher order string corrections. Perhaps with some tenacity, successful kinetic inflation can be executed.
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Appendix A: Constraining $\omega(\Phi)$

A much weaker condition than that of sufficient inflation can be used to severely restrict the range of $\omega(\Phi)$ pertinent to inflation. In Ref [4] it was shown that $\omega$ must be negative and the Planck mass must decrease in order for the acceleration to be even nominally relevant for inflation. That discussion is reproduced in brief here. A nominal condition for the acceleration to be relevant for inflation is simply that

$$d_{\gamma} > H^{-1}.$$  \hspace{1cm} (7.1)

If this condition is not met, then the scales affected during the acceleration were never causally connected. This is much weaker than the sufficient inflation condition (4.1).

Use can be made of the flat space results of §III, eqn (3.9) and (3.15). When $\Phi$ grows, the condition $H > 0$ can only be met when $f > 1$ and the upper sign in eqn (3.9) holds. For growing $\Phi$ eqn (7.1) becomes

$$f < -\frac{(1 - \delta)}{(1 + \delta)}$$  \hspace{1cm} (7.2)

where $\delta < 1$. Since $f = (1 + 2\omega/3)^{1/2}$ is always positive, condition (7.2) is impossible to meet if $\Phi$ grows. Therefore, accelerations driven by a growing $\Phi$ cannot be relevant for inflation.

When $\Phi$ drops and $f < 1$, both branches are allowed in eqn (3.11). Using the above two expressions eqn (7.1) becomes

$$f < \pm \frac{(1 - \delta)}{(1 + \delta)}$$  \hspace{1cm} (7.3)

The upper sign again corresponds to X-branch solutions while the lower sign corresponds to D-branch solutions. For the X-branch, $\delta < 1$ and for the D-branch, $\delta > 1$. The weakest requirement of (7.3) is that $f$ be less than 1. If $f < 1$, then $\omega < 0$. The acceleration can therefore only be relevant for inflation if $\Phi$ drops and $\omega < 0$. Although negative $\omega$ leads to a kinetic term in the action with the wrong sign, the net kinetic energy density can still be positive. Classically, the kinetic energy density is positive in an FRW metric if the kinetic coupling is $\omega \geq -3/2$. The energy density must be positive in order to banish ghost modes.
Appendix B: Sufficient Inflation Revisited

It is instructive to pursue the sufficient inflation condition. The material presented in this appendix in no way circumvents the trauma discussed in the body of the paper. It is only intended to lend some intuition for the demands made of the scale factor, the Planck mass and the heating mechanism. It is implicit in the requirements of successful inflation that today’s universe results at the end of the day. While the left hand side and the right hand side of eqn (4.1) can be compared with sheer brute force, that alone does not guarantee that our universe results. We will be reminded of this as we come to it.

The sufficient inflation condition (4.1) can be expressed as a condition on the growth of the scale factor. As in §VIB two assumptions are made. Firstly, it is assumed that at some time $t_{\text{end}}$ entropy is produced and inflation ends. Secondly, it is assumed that the universe has evolved adiabatically since the end of inflation so that $a_{\text{end}} T_{\text{end}} = a_s T_s$.

As before, the Hubble constant today can be expressed as $H_o = a_{o}^{1/2} T_s^2 / M_o$ where $a_{o} = \gamma (t_o) \eta_o = (8\pi / 3) (\pi^2 / 30) g_s (t_o) \eta_o$ and $\eta_o \sim 10^4 - 10^5$ is the ratio today of the energy density in matter to that in radiation. Finally, using (3.15) for $d_{\gamma s}$, in a model of kinetic inflation the constraint (4.1) becomes

$$\frac{a_{\text{end}}}{a_s} \gtrsim a_{o}^{-1/2} M_o \left[ \frac{f \dot{\Phi} / \Phi (1 - \delta)^{-1}}{T_{\text{end}}} \right].$$

(7.4)

The flat space results have been used since curvature cannot aid in a solution to the horizon problem. In fact, curvature leads to a problem of its own, namely the flatness problem (see §VIB). The amount of inflation needed depends on the efficiency in converting the kinetic energy into temperature and thus on the specifics of a heating mechanism.

Eqn (7.4) can be pushed further if some simple conjectures about the heating mechanism are made. Let $T_{\text{end}}$ be given by

$$T_{\text{end}} = \epsilon E_{\text{end}}$$

(7.5)

where $\epsilon$ is the efficiency with which the kinetic energy density is converted into entropy and $E_{\text{end}}$ is the net available kinetic energy density. Suppose the energy available for conversion
into particles is the full energy density in the $\Phi$ field times a unit volume, $E_{\text{end}} = \rho_{\text{end}} a_{\text{end}}^3$.

In an FRW cosmology, the energy density in the $\Phi$-field can be expressed as

$$\rho_\Phi = \frac{3}{32\pi} \frac{\dot{\Phi}^2}{\Phi} (f \pm 1)^2 \geq 0$$

(7.6)

[4]. So, $T_{\text{end}}$ becomes

$$T_{\text{end}} = \epsilon \left( \frac{C^2}{a_{\text{end}}^3 \Phi_{\text{end}}} \frac{(f_{\text{end}} \pm 1)^2}{f_{\text{end}}^2} \right).$$

(7.7)

Using this input into the sufficient inflation condition (7.4), along with eqn (3.14) gives the condition

$$\frac{a_{\text{end}}^2 \Phi_{\text{end}}}{a_{\text{end}}^2 \Phi_{\text{end}}} \geq a_o^{-1/2} M_o \left( 1 - \delta \right)^{-1} \frac{f^2}{f (f \pm 1)^2} \frac{1}{\epsilon C}.$$  

(7.8)

Take $(1 - \delta) \sim O(1)$ and $C \sim O(1)$ and $f_{\text{end}} \sim O(1)$ for now. Using $M_o = 1.2 \times 10^{19}$ GeV and $T_o = 2.3 \times 10^{13}$ GeV, eqn (7.8) becomes roughly

$$\frac{a_{\text{end}}^2 \Phi_{\text{end}}}{a_{\text{end}}^2 \Phi_{\text{end}}} \geq \frac{10^{99}}{\epsilon}.$$  

(7.9)

Even if the efficiency $\epsilon$ is 1, condition (7.9) would require $a^2 \Phi$ to be bigger in the past. From eqn (3.11), only for D-branch solutions will $a^2 \Phi$ decrease. For the above choice of parameters, it follows that only D-branch solutions will satiate the requirements of sufficient inflation.\(^3\)

If the universe always expands then $a_{\text{end}} > a_*$. This simple inequality can be used in conjunction with (7.9) to estimate the minimum change in the Planck mass,

---

\(^3\)For D-branch solution, $\delta > 1$. If $\delta_o$ is very large, then, using the definition of $\delta$, eqn (7.8) becomes

$$\frac{a_{\text{end}}^2 \Phi_{\text{end}}}{a_{\text{end}}^2 \Phi_{\text{end}}} \geq a_o^{-1/2} M_o \frac{f_{\text{end}}^2}{T_o} \frac{1}{(f_{\text{end}} \pm 1)^2} \epsilon C.$$  

(7.10)

where subscript $i$ denotes initial values. A large $\delta$ means the onset of inflation is postponed until $a^2 \Phi$ drops substantially from its initial value. It does not make sufficient inflation any easier to satisfy.
The Planck mass must drop by roughly 15 orders of magnitude, or more, during inflation. It is not enough that this condition is satisfied. As discussed in §VI A, a mechanism is needed to induce a branch change. If no branch change is induced, inspection of the equations of motion indicate that $H$ would likely pass through zero and the universe would enter an age of collapse. Thus the D-branch would not lead to our expanding universe today.

Consider instead a different choice of parameters. For instance $f_{end} = 10^{-15}$ so that $\omega$ is near $-3/2$ to 1 part in $10^{15}$. Keep $(1 - \epsilon) \sim \mathcal{O}(1)$ and $C \sim \mathcal{O}(1)$. Then

$$\frac{a_i^2}{a_{end}^2 \Phi_{end}} \geq \frac{1}{\epsilon}$$

and $a^3 \Phi$ need not drop. Thus if $f_{end}$ is fantastically small, then X-branch solutions may be able to meet condition (4.1). Alternatively, $C$ would need to be unnaturally large. In order for X-branch solutions to address the sufficient inflation condition, it appears some fine tuning would be involved.

Before ending, one last comment can be made. To take a different perspective, the arguments in this paper reveal that kinetic driven acceleration alone is not enough to remedy the initial condition problems of cosmology. To some extent this is obvious. An inflationary model combines the acceleration with some prescription for heating the universe. The black box remains the heating mechanism. Two possible mechanisms are (1) Hawking-Unruh radiation generated as a consequence of the accelerated expansion or (2) particle production generated from oscillations in the Planck field. The Planck field could oscillate as a result of the changing kinetic coupling. Just as with oscillations induced by a potential, the oscillating field can decay into particles. If the heating mechanism involved unusual physical processes, as it often does, it might be possible to introduce the effects needed to amend the problems enumerated in this paper. While it is highly unsatisfying to place such unresolved issues into a black box, the possibility could not go without mention.
REFERENCES


