A Note on the Symmetries of the Gravitational Field of a Massless Particle

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Abstract

It is shown that the metric of a massless particle obtained from boosting the Schwarzschild metric to the velocity of light, has four Killing vectors corresponding to an $E(2) \times \mathbb{R}$ symmetry-group. This is in agreement with the expectations based on flat-space kinematics but is in contrast to previous statements in the literature [1]. Moreover, it also goes beyond the general Jordan-Ehlers-Kundt-(JEK)-classification of gravitational pp-waves as given in [2].

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Several years ago R. Sexl and one of the authors (P.C.A.) obtained a gravitational impulsive plane-fronted wave with parallel rays (pp-wave) by boosting the Schwarzschild metric to the singular limit [3]. Since then this metric\(^1\) has been widely used to discuss ultra-relativistic scattering processes in both the classical and semiclassical context [4, 5]. Moreover, the method of boosting was applied also to other configurations like the Kerr or the Reissner-Nordstrøm metric [6, 7]. In flat spacetime the trajectory of a free particle of zero rest mass is described by its initial position and 4-momentum which is null. The subgroup of the Poincare group that leaves the trajectory invariant is isomorphic to \(E(2) \times \mathbb{R}\). Its generators act in the null-hyperplane orthogonal to the given vector. On physical grounds one would therefore expect that the gravitational field generated by a massless particle shares the same symmetries. Already in 1962, in a now classical paper [2], Ehlers et al gave a complete classification of symmetries for pp-waves together with their standard form for each symmetry. A comparison of this classification with the metric of a massless particle tells one that the metric should have only two instead of the expected four Killing vectors, namely the translation

\(^1\)denoted AS-metric by some authors
along the given lightlike propagation direction and the corresponding space rotation. This contradiction was emphasized in a paper by Schücker [1] who argued that the full symmetry implies the presence of infinite mass density which general relativity does not allow for. In this note we show that the symmetry of the field in question indeed admits the expected four Killing vectors. The above mentioned contradiction is resolved by noting that the metric, which is an impulsive gravitational wave, has in its standard form a distributional metric coefficient. Thus, allowing for distributions a careful analysis of the Killing equations shows the presence of the full symmetry. It is also straightforward to see that the JEK-classification is based on regular functions only.

Our approach to derive the Killing vectors of the AS-geometry will start from its unboosted counterpart, the Schwarzschild geometry. Since both geometries, Schwarzschild and AS, belong to the Kerr-Schild class they admit a decomposition

$$g_{ab} = \eta_{ab} + f k_a k_b,$$

where \( \eta \) denotes the flat part of the decomposition, \( k \) a geodetic null vector-field and \( f \) a scalar function. The existence of the flat background-part in the
decomposition gives a natural meaning to the concept of boosts as its isometries. Moreover this geometrical aspect will turn out to be of great calculational advantage. Choosing Kerr-Schild coordinates $x^a$ for the Schwarzschild metric brings $\eta$ into its standard form and $f$ and $k$ become respectively

$$f = \frac{2m}{r} \quad k^a = \left( \frac{1}{x^r} \right)$$

(1)

where $x^i$ denotes spatial cartesian coordinates according to the (flat) Minkowski metric and $r$ its corresponding radial distance. With respect to these coordinates the Killing vectors of (1) corresponding to staticity and spherical symmetry can be written as

$$\xi_t^a = (\partial_t)^a \quad (2)$$

$$\xi_\omega^a = \hat{\omega}_b^a x^b \quad \hat{\omega}_b^a \xi_t^b = 0 \quad \hat{\omega}^{ab} = -\hat{\omega}^{ba},$$

where $\hat{\omega}$ denotes an infinitesimal rotation matrix. The index-raising in the last line of (2) was performed with the help of $\eta$, which will be implicitly assumed in the following unless stated otherwise. Interpreting the Kerr-Schild coordinates as referring to a Lorentz-frame associated with an observer that is at rest asymptotically, allows us to find the form of the Killing vectors
with respect to a boosted asymptotic observer

\[
\xi_t^a = \frac{1}{m} P^a, \\
\xi_\omega^a = \tilde{\omega}^a b x^b, \\
\xi_\tilde{a}^a = \frac{1}{m} ((\tilde{a} x) Q^a - (Q x) \tilde{a}^a), \\
\dot{\omega}^a b = \tilde{\omega}^a b + \frac{1}{m} (Q^a \tilde{a}_b - \tilde{a}^a Q_b),
\]

where \(P\) and \(Q\) denote the timelike and spacelike vectors spanning the boost 2-plane and vectors in the orthogonal 2-plane are tagged by a tilde (i.e. \(P^a \tilde{a}_a = Q^a \tilde{a}_a = 0\)). \(P\) and \(Q\) are conveniently normalized to \(\pm m^2\) and tend to a null vector \(p\) in the limit \(m \to 0\). To take \(m \to 0\), we multiply \(\xi_t\) and \(\xi_\tilde{a}\) with \(m\) thereby obtaining the limit

\[
(m \xi_t^a) \to p^a =: \zeta_t^a, \\
\xi_\omega^a \to \tilde{\omega}^a b x^b =: \zeta_\omega^a, \\
(m \xi_\tilde{a}^a) \to (\tilde{a} x) p^a - (p x) \tilde{a}^a =: \zeta_\tilde{a}^a.
\]

From their mutual commutation relations

\[
[\zeta_t, \zeta_\omega] = 0 \quad [\zeta_t, \zeta_\tilde{a}] = 0 \\
[\zeta_\tilde{a}, \zeta_\tilde{a}] = 0 \quad [\zeta_\omega, \zeta_\tilde{a}] = \zeta_\omega \tilde{a}
\]
one sees that the $\zeta$ form a representation of the Lie-algebra of $E(2) \times IR$.

The corresponding limit of the Schwarzschild geometry

$$g_{ab} = \eta_{ab} - 8\delta(px) \log \rho \, p_a p_b,$$

(5)

may either be obtained by employing a "singular" boost [3] or by solving the Einstein equations with a boosted source [8]. The result is a pp-wave with profile function $f = -8\delta(px) \log \rho$, where $\rho$ denotes the radial distance in the orthogonal 2-plane. It is now easy to verify that all the $\zeta$ of (4) have the Killing-property

$$L_\zeta g_{ab} = (\zeta \partial) f \, p_a p_b + \partial_a \zeta_b + \partial_b \zeta_a + f (p_a \partial_b(p\zeta) + p_b \partial_a(p\zeta)) =$$

(6)

$$= (\zeta \partial) f \, p_a p_b + \partial_a \zeta_b + \partial_b \zeta_a = 0,$$

where $\partial$ is the covariant derivative with respect to $\eta$. Let us however, for the sake of explicitness, exhibit the only non-trivial term of (6) relative to $\zeta_\tilde{a}$

$$(\zeta_\tilde{a} \partial) f = -8(\tilde{a} x) p^2 \delta'(px) + 8(px) \delta(px) \frac{1}{\rho} e^a_{\rho} \tilde{a}_a,$$

which vanishes due to the product $px \delta(px)$ and the fact that $p$ is null.

This shows that the Killing vectors of the Schwarzschild metric survive the ultrarelativistic boost in the sense that the resulting vector fields are Killing with respect to the boosted metric. Comparing (5) with the general
classification of Killing vectors for pp-waves [2] one would only expect the presence of $\zeta_t$ and $\zeta_\tilde{\omega}$. The fact that the 2-parameter family of vector fields $\zeta_\tilde{a}$ is also Killing depends crucially upon the presence of the $\delta(px)$-factor in $f$, which also explains the discrepancy with [2], where no distributional profile functions were considered. The implications of this aspect for the general classification of pp-waves will be investigated in a forthcoming publication.

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References


*The Ultrarelativistic Kerr-Geometry and its Energy-Momentum Tensor.*