Two Mode Quantum Systems - I: Invariant Classification of Squeezing Transformations and Squeezed States

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Abstract

A general analysis of squeezing transformations for two mode systems is given based on the four dimensional real symplectic group $Sp(4, \mathbb{R})$. Within the framework of the unitary metaplectic representation of this group, a distinction between compact photon number conserving and noncompact photon number nonconserving squeezing transformations is made. We exploit the $Sp(4, \mathbb{R}) - SO(3, 2)$ local isomorphism and the $U(2)$ invariant squeezing criterion to divide the set of all squeezing transformations into a two parameter family of distinct equivalence classes and representative elements chosen for each class. Familiar two mode squeezing transformations in the literature are recognized in our framework and seen to form a set of measure zero. Examples of squeezed coherent and thermal states are worked out. The need to extend the heterodyne detection scheme to encompass all of $U(2)$ is emphasized.

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I. INTRODUCTION

The theoretical analysis [1] and experimental [2], [3], [4] realization of squeezed states of radiation continue to receive a great deal of attention. While much of the work so far has concerned itself with single mode situations [5], [6], [7], some analysis of two-mode states has also been presented [8], [9]. Other nonclassical effects of radiation beyond second order have also received attention in the literature [10]. More recently, a general invariant squeezing criterion for n-mode systems has been developed by some of us elsewhere [11].

The purpose of the present paper is to study squeezing transformations for two-mode systems, and to develop a classification scheme for them motivated by the above mentioned invariant squeezing criterion. Basic to all such discussions is the four-dimensional real symplectic group Sp(4, R), of real linear homogeneous canonical transformations, and the unitary metaplectic representation of this group acting on the Hilbert space of states for a two-mode quantum system. The structure of the noncompact group Sp(4, R) (and its n-mode counterpart Sp(2n, R)) leads to a natural separation of its elements into passive (compact), and active (noncompact) types. Here the adjectives passive and active mean total photon number conserving and nonconserving respectively. This group theoretical framework gives us an unambiguous way of defining precisely the family of squeezing transformations: they are the active elements of Sp(4, R) and they do not form a subgroup. The action of the maximal compact (passive) subgroup U(2) of Sp(4, R) on the set of squeezing transformations by conjugation leads to a natural equivalence relation, leading to the emergence of equivalence classes and convenient representative elements as well. In studying the physical properties of a state subjected to squeezing, therefore we are able to isolate the dependence on intrinsic squeezing parameters and separate them from other passive factors. As might be expected, the single squeeze factor encountered in the studies of single mode states gets enlarged here to two independent intrinsic squeeze factors; and it turns out that the two mode squeezing transformations so far studied in the literature form a very small subset of all the independent available possibilities.

The material in this paper is arranged as follows: Section II sets up the basic kinematics for two mode systems, the Fock and Schrödinger representations and the actions of Sp(4, R) on the canonical variables and states. The variance matrix for a general state and its change under Sp(4, R) are derived. After identifying the maximal compact or passive U(2) subgroup of Sp(4, R), the U(2) invariant squeezing criterion for two mode systems is discussed. Section III introduces the generators for the metaplectic representation of Sp(4, R) and brings out the connection to the SO(3, 2) Lie algebra. The photon number conserving compact generators and the remaining noncompact ones are clearly identified. Based on the polar decomposition theorem for general elements of Sp(4, R) we are then led to a precise definition of squeezing transformations: These are single exponential of linear combinations of the noncompact metaplectic generators. We then proceed to break up the set of all squeezing transformations into equivalence classes under U(2) action. We find that these classes form a continuous two parameter family describable by points in an octant in a two dimensional plane. The families of Caves-Schumaker transformations and essentially single mode transformations correspond to one dimensional lines bounding this octant and so are measure zero. Section IV applies our formalism to two mode squeezed coherent and thermal states. In Section V we see how the heterodyne detection scheme fits into our analysis. We argue that it is necessary to experimentally realise all elements of the U(2) subgroup of Sp(4, R); the heterodyne scheme only handles a one parameter subset of U(2). Section VI contains some concluding remarks.

II. SYMPLECTIC GROUP FOR TWO MODES AND THE SQUEEZING CRITERION

We consider two orthogonal modes of the radiation field, with annihilation operators $a_j$, $j = 1, 2$, and corresponding creation operators $a_j$. These two modes could, for example, be two different frequencies for the same or different propagation directions and polarizations, two different propagation directions at a common frequency, two different
polarization states of plane waves degenerate in frequency and direction of propagation, etc. To express the familiar canonical commutation relations among $a, a^\dagger$ in a compact manner, we arrange these operators in the form of a four-component column vector:

$$\xi^{(c)} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix},$$

$$a = 1, 2, 3, 4. \quad (2.1)$$

The superscript (c) indicates that the entries here are complex, i.e., nonhermitian, operators. For discussion of quadrature squeezing, however, we need to also deal with the hermitian quadrature components of these operators. Therefore we define another column vector $\bar{\xi}$ with four hermitian entries, related to $\xi^{(c)}$ by a fixed numerical matrix, as follows:

$$\bar{\xi} = (\xi) = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix},$$

$$q_j = \frac{1}{\sqrt{2}}(a_j + a_j^\dagger), \quad p_j = \frac{-i}{\sqrt{2}}(a_j - a_j^\dagger);$$

$$\xi^{(c)} = \Omega \xi, \quad \bar{\xi} = \Omega^{-1} \xi^{(c)}.$$ (2.2)

This transformation is analogous to the one connecting the circular polarization (helicity) basis and the linear polarization basis familiar in classical polarization optics.

The canonical commutation relations can now be written either in terms of $\xi$ or (in two ways) in terms of $\xi^{(c)}$:

$$[\xi, \bar{\xi}] = iJ_{\omega},$$

$$[\xi^{(c)}, \xi^{(c)}] = J_{\omega},$$

$$[\xi^{(c)} \cdot e^{i\Omega}, \xi^{(c)}] = \Sigma_{\omega} J_{\omega}.$$ (2.3)

Linear canonical transformations of the $q_j$ and $p_j$ are those real linear transformations of these operators that preserve their commutation relations. They constitute the four-dimensional real symplectic group $Sp(4, \mathbb{R})$. A general real linear homogeneous transformation on the $q$'s and $p$'s is described by a $4 \times 4$ real matrix $S$ acting as follows:

$$\xi - \xi' = S\xi,$$

$$\xi' = S\bar{\omega}\xi.$$(2.4)

If the $\xi'$ are to satisfy the same commutation relations as the $\xi$, the condition on $S$ is:

$$S\beta S^T = \beta.$$ (2.5)

This is the defining property for the elements of the group $Sp(4, \mathbb{R})$, which is a noncompact group:

$$Sp(4, \mathbb{R}) = \{S = 4 \times 4 \text{ real matrix} \mid S\beta S^T = \beta \}. \quad (2.6)$$

Note that $\beta = -\beta^{-1}$ itself is an element of $Sp(4, \mathbb{R})$ whereas $\Sigma$ is not. Further, $S \in Sp(4, \mathbb{R})$ implies $-S, S^T, S^{-1} = \beta S^T S^T \in Sp(4, \mathbb{R})$ and $\det S = 1$ for every $S \in Sp(4, \mathbb{R})$. 

5
The action of \( Sp(4,\mathbb{R}) \) on the nonhermitian operators \( \xi^{(i)} \) is by a (generally) complex matrix \( S^{(i)} \) related to \( S \) by conjugation with \( \Omega \):

\[
S \in Sp(4,\mathbb{R}) : \quad \xi = S \xi \Leftrightarrow \\
\xi^{(i)} = S^{(i)} \xi^{(i)} \\
S^{(i)} = \Omega \cdot S \cdot \Omega^t. \tag{2.7}
\]

The matrices \( S^{(i)} \) constitute a representation of \( Sp(4,\mathbb{R}) \) equivalent in the group theoretical sense to the defining real representation (2.6). This is analogous to passing from the real matrix description of three-dimensional rotations using cartesian components to a complex description in terms of spherical components of vectors.

Let us at this point recall the two familiar and convenient descriptions of the Hilbert space \( \mathcal{H} \) on which \( \xi \) and \( \xi^{(i)} \) act in an irreducible manner. In the Fock representation the orthonormal basis vectors are simultaneous eigenvectors of the number operators for the two independent orthogonal modes:

\[
|n_1, n_2\rangle = \frac{a_1^\dagger n_1 a_2^\dagger n_2}{\sqrt{n_1!n_2!}} |0, 0\rangle, \\
a_1 |0, 0\rangle = 0, \quad a_1 a_2 |n_1, n_2\rangle = \delta_{n_1 n_2} |n_1, n_2\rangle, \\
(n_1^*, n_2^* | n_1, n_2\rangle = \xi_{n_1 n_2} \xi^*_{n_1 n_2}. \tag{2.8}
\]

The actions of \( a_1 \) and \( a_1^\dagger \) on these vectors are standard:

\[
a_1 |n_1, n_2\rangle = \sqrt{n_1 + 1} |n_1 + 1, n_2\rangle, \\
a_1^\dagger |n_1, n_2\rangle = \sqrt{n_2 + 1} |n_1, n_2 + 1\rangle, \\
a_1 |n_1, n_2\rangle = \sqrt{n_1} |n_1 - 1, n_2\rangle, \\
a_1^\dagger a_2 |n_1, n_2\rangle = \sqrt{n_2} |n_1, n_2 - 1\rangle. \tag{2.9}
\]

In the Schrödinger representation the Hilbert space \( \mathcal{H} \) is realised as the space \( L^2(\mathbb{R}^2) \) of square integrable functions of two real variables \( q_1 \) and \( q_2 \):

\[
\mathcal{H} = \left\{ \psi(q_1, q_2) : \int d q_1 d q_2 \psi(q_1, q_2)^* \psi(q_1, q_2) < \infty \right\}. \tag{2.10}
\]

Now \( q_1 \) act multiplicatively and \( p_1 \) as operators of differentiation:

\[
(q_1 \psi)(q_1', q_2) = q_1^* \psi(q_1, q_2), \\
(p_1 \psi)(q_1, q_2) = -i \frac{\partial}{\partial q_1} \psi(q_1, q_2). \tag{2.11}
\]

The connection between the Fock and the Schrödinger representations is established by the Hermite-Gaussian functions familiar from the theory of the harmonic oscillator, extended to two modes.

Now we turn to the action of \( Sp(4,\mathbb{R}) \) on \( \mathcal{H} \). Since the hermiticity properties and commutation relations of the \( \xi \) are maintained by the transformation (2.4) for any \( S \in Sp(4,\mathbb{R}) \), and since the \( \xi \) act irreducibly on \( \mathcal{H} \), it follows from the Stone-von Neumann theorem [12] that it should be possible to construct a unitary operator \( U(S) \) on \( \mathcal{H} \) implementing (2.4) via conjugation:

\[
S \in Sp(4,\mathbb{R}) : \quad S \xi \xi^{(i)} = U(S)^{-1} \xi U(S), \\
U(S)^* U(S) = 1 \text{ on } \mathcal{H}. \tag{2.12}
\]

This \( U(S) \), for each \( S \), is clearly arbitrary up to (and only up to) virtue of irreducibility an \( S \)-dependent phase. Even after making use of this freedom, however, it turns out that we cannot choose the operators \( U(S) \) for various \( S \) so as to give us a true unitary representation of \( Sp(4,\mathbb{R}) \) on \( \mathcal{H} \). Rather, after maximum simplification, they give us a two-valued representation of this group [13]:

\[
S_1, S_2 \in Sp(4,\mathbb{R}) : \quad U(S_1) U(S_2) = \pm U(S_1 S_2). \tag{2.13}
\]

Alternatively we can regard the \( U(S) \), chosen so as to obey this composition law, as providing a true and faithful unitary representation of the four-dimensional metaplectic group \( Mp(4) \), which in turn is a two-fold cover of \( Sp(4,\mathbb{R}) \). In the literature the operators \( U(S) \) are often said to provide the metaplectic representation of \( Sp(4,\mathbb{R}) \). The connection between \( Mp(4) \) and \( Sp(4,\mathbb{R}) \) has some of similarity with the one between \( SU(2) \) and \( SO(3) \) familiar in angular momentum theory.
Now we consider physical states of the two-mode system, the action of $\text{Sp}(4, \mathbb{R})$ on them, and the statement of a suitable squeezing criterion. Let $\rho$ be the density operator of any (pure or mixed) state of the two-mode radiation field. With no loss of generality we may assume that the means $\text{Tr}(\rho \xi_n)$ of $\xi_n$ vanish in this state. (Any non-zero values for these means can always be reinstated by a suitable phase space displacement which had no effect on the squeezing properties.) Squeezing involves the set of all second order noise moments of the quadrature operators $q_i$ and $p_j$. To handle them collectively we define the variance or noise matrix $V$ for the state $\rho$ as follows:

$$V = (V_{\alpha\beta}) = \frac{1}{2} \text{Tr}(\rho (\xi_n \xi_n^T)) = V_{\xi \xi} = \frac{1}{2} \text{Tr}(\rho (\xi_n \xi_n^T)).$$

This definition is valid for a system with any number of modes. For a two-mode system it can be written explicitly in terms of $q_j$ and $p_j$ as:

$$V = \begin{pmatrix}
q_1^2 & q_1 p_1 & \frac{1}{2} (q_1 p_1) & \frac{1}{2} (q_1 p_1) \\
q_1 p_1 & p_1^2 & \frac{1}{2} (q_1 p_1) & \frac{1}{2} (q_1 p_1) \\
\frac{1}{2} (q_1 p_1) & \frac{1}{2} (q_1 p_1) & p_1^2 & p_1 p_1 \\
\frac{1}{2} (q_1 p_1) & \frac{1}{2} (q_1 p_1) & p_1 p_1 & \frac{1}{2} (p_1^2)
\end{pmatrix}$$

This matrix is real symmetric positive definite and obeys additional inequalities expressing the Heisenberg uncertainty principle of quantum mechanics [11].

When the state $\rho$ is transformed to a new state $\rho'$ by the unitary operator $U(S)$ for some $S \in \text{Sp}(4, \mathbb{R})$, we see easily from eqs (2.12, 2.14) that the variance matrix $V$ undergoes a symmetric symplectic transformation:

$$S \in \text{Sp}(4, \mathbb{R}): \rho = U(S) \rho U(S)^{-1} \Rightarrow V' = S V S^T.$$

This transformation law for $V$ preserves all the properties mentioned at eq. (2.15).

Towards setting up a squeezing criterion, we identify an important subgroup of $\text{Sp}(4, \mathbb{R})$, namely its maximal compact subgroup $K \equiv U(2)$. This consists of matrices $S \in \text{Sp}(4, \mathbb{R})$ having a specific block form determined by two-dimensional unitary matrices belonging to $U(2)$:

$$K = U(2) = \{ S(X, Y) \in \text{Sp}(4, \mathbb{R}) : U = X - i Y \in U(2) \} \subset \text{Sp}(4, \mathbb{R}).$$

$$S(X, Y) = \begin{pmatrix} X & Y \\
- Y & X \end{pmatrix}.$$

(2.17)

Here $X$ and $Y$ are the real and imaginary parts of $U$, and the unitary condition $U^* U = 1$ on $U$ ensures that $S(X, Y)$ obeys the symplectic condition (2.5). We also recognise that this subgroup $K$ is the intersection of the symplectic and the orthogonal groups in four real dimensions, namely

$$K = U(2) = \text{Sp}(4, \mathbb{R}) \cap O(4) :$$

$$S(X, Y) \{ J \text{ or } 1 \} \quad S(X, Y)^T = J \text{ or } 1.$$

(2.18)

The complex form $S^{(0)}(X, Y)$ corresponding to $S(X, Y)$, and which acts on $\xi^{(0)}$, is rather simple; it is given by

$$S^{(0)}(X, Y) \equiv S^{(0)}(1)$$

$$= \begin{pmatrix} X - i Y & 0 \\
0 & X + i Y \end{pmatrix}$$

$$= \begin{pmatrix} U & 0 \\
0 & U^* \end{pmatrix}.$$  

(2.19)

Thus the maximal compact subgroup of $\text{Sp}(4, \mathbb{R})$ mixes $a_j$ and $a_j$ unitarily, but does not mix the $a_j$ with the $a_j$. This fact may be explained by saying that the subgroup $U(2)$ of $\text{Sp}(4, \mathbb{R})$ consists of passive, or total photon number conserving, transformations.

In group theory one talks of a maximal compact subgroup and not the maximal compact subgroup of a noncompact group. In the present case the physical requirement that the total number of photons (a particular generator of the group) be conserved (left invariant under adjoint action) singles out a unique maximal compact subgroup and hence
we talk of the maximal compact subgroup. In contrast, elements of $Sp(4, \mathbb{R})$ outside this subgroup $U(2)$ are non-compact elements which do not conserve total photon number, and so describe active transformations. (These properties of compact and non-compact elements of $Sp(4, \mathbb{R})$ will become transparent when we identify their generators in the following Section).

As has been discussed in detail elsewhere [11], [14], for a multimode system it is physically reasonable to set up a definition of squeezing which is invariant under the subgroup of passive transformations of the full symplectic group. For the present case of two-mode systems, we evidently need a $U(2)$-invariant squeezing criterion. That is, our definition must be such that if a state $\rho$ with variance matrix $V$ is found to be squeezed, then the state $\rho(U(S(X,Y)) \rho U(S(X,Y))^{-1}$ with variance matrix $V'' = S(X,Y) V S(X,Y)^T$ must also be squeezed, for any $U = X - iY \in U(2)$.

Conventionally a state is said to be squeezed if any one of the diagonal elements of $V$ is less than $1/2$. The diagonal elements correspond, of course, to fluctuations in the "chosen" set of quadrature components of the system. The $U(2)$-invariant definition is as follows: the state $\rho$ is a quadrature squeezed state if either some diagonal element of $V$ is less than $1/2$ (and then we say that the state is manifestly squeezed), or some diagonal element of $V'' = S(X,Y) V S(X,Y)^T$ for some $U = X - iY \in U(2)$, is less than $1/2$:

$$\rho \text{ is a squeezed state } \iff \exists \text{ a } a \text{ and some } X - iY \in U(2).$$

That is, running over $S(X,Y) \in U(2)$ is the same as running over all possible sets of quadrature components. We may say that since any element of $U(2)$ passively mixes the two modes, the appropriate $S(X,Y) \in U(2)$ which achieves the above inequality (assuming the given $V$ permits the same) must choose the right combination of quadratures to make the otherwise possibly hidden squeezing manifest.

To implement this definition in practice, it would appear that even if a state is intrinsically squeezed, we may have to explicitly find a suitable $U(2)$ transformation which when applied to $V$ makes the squeezing manifest. This however could be complicated. Here the point to be noticed and appreciated is that diagonalisation of a noise matrix $V$ generally requires a real orthogonal transformation belonging to $SO(4)$ which may not lie in $U(2) = O(4) \cap Sp(4, \mathbb{R})$. It is therefore remarkable that, as shown in [11], the $U(2)$-invariant squeezing criterion (2.20) can be expressed in terms of the spectrum of eigenvalues of $V$, namely:

$$\rho \text{ is a squeezed state } \iff \lambda(V) \leq \frac{1}{2}.$$  

That is, while the diagonalization of $V$ is in general not possible within $U(2)$ which is a proper subgroup of $O(4)$ any one particular (and hence the smallest) eigenvalue of $V$ can be made to become one of the diagonal elements of $V$ transformed by an appropriate $S(X,Y) \in U(2)$. In other words any quadrature component can be taken to any other quadrature component by a suitable element of $U(2)$. We shall in the sequel work with the $U(2)$-invariant squeezing criterion (2.20).

III. THE $Sp(4, \mathbb{R}) - SO(2, 2)$ CONNECTION AND CLASSIFICATION OF TWO-MODE SQUEEZING TRANSFORMATIONS

We have shown in the previous Section that the group $Sp(4, \mathbb{R})$ of linear canonical transformations contains two kinds of elements: passive total photon-number conserving elements belonging to the maximal compact subgroup $K = U(2)$; and active noncompact elements lying outside this subgroup, and which do not conserve total photon number. It is clear from the $U(2)$-invariant squeezing criterion (2.20;2.21) that the former elements cannot produce squeezing. This is because the corresponding changes in the variance matrix $V'' = S(X,Y) V S(X,Y)^T$, being similarity transformations (recall that $S(X,Y)$ corresponds to canonical rotations in phase space), preserve the eigenvalue spectrum of $V$, hence $\forall \lambda(V) \geq 1/2$ implies $\forall \lambda(V'') \geq 1/2$ and conversely for every $S(X,Y) \in U(2)$. The
noncompact elements of $Sp(4, R)$, on the other hand, while they do not form a subgroup, have the potential to produce a squeezed state starting from a non-squeezed state. Thus they may be called squeezing transformations. The following questions then naturally arise: what are the really distinct squeezing transformations which are not related to each other by just passive transformations, and how can they be invariantly labelled or parametrised?

To answer these questions it is useful to work at the level of the Lie algebra $Sp(4, R)$ of $Sp(4, R)$, and the hermitian generators of the unitary metaplectic representation $U(2)$ acting on $\mathcal{H}$. As is well known, these generators are basically all possible hermitian symmetric quadratic expressions in the canonical variables $q$ and $p$. They may be expressed more transparently for our purposes in terms of $a$'s and $a^\dagger$'s. To begin with we list the independent generators in a uniform manner valid for a system with any number of modes or degrees of freedom:

$$W_+ = W_+ = \frac{1}{2} (q_0, p_0) = \frac{1}{2} (q^0, p^0, a^0, a^0, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger) .$$

$$V_+ = V_+ = \frac{1}{2} (a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger) .$$

$$Z_0 = Z_0 = \frac{1}{2} (a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger, a_0^\dagger) .$$

$W_{\pm} = W_{\pm} = a_{\pm}^\dagger a_{\pm} - a_{\pm} a_{\pm} \pm \delta_{\pm}$.

For a general $n$-mode system we have $n^2$ independent $W$'s, $n(n+1)/2$ independent $V$'s and $n(n+1)/2$ independent $Z$'s; we can see that we have accounted for all possible independent hermitian symmetric quadratic expressions in the $q$'s and $p$'s (equivalently in $a$'s and $a^\dagger$'s).

For two-mode systems, we have four $W$'s, and three each of the $V$'s and $Z$'s, adding up to a total of ten, which is the dimension of $Sp(4, R)$. The combinations that are respectively passive photon number conserving generators and active nonconserving generators are:

$$W_+ = W_+ = a_{\dagger} a_{\dagger} - a_{\dagger} a_{\dagger} ,$$

$$V_+ = V_+ = a_{\dagger} a_{\dagger} - a_{\dagger} a_{\dagger} .$$

The former (3.2a) are generators of $U(2)$, and their passive nature is conveyed by the property

$$[N, W_{\pm} - W_{\mp}] = [N, V_{\pm} - V_{\mp}] = 0 .$$

$$N = a_{\dagger} a_{\dagger} + a_{\dagger} a_{\dagger} .$$

It can be checked that the ten hermitian operators in (3.1) are closed under commutation. They give a representation of the Lie algebra of $Sp(4, R)$ in a particular basis, and the presence of the zero-point energy terms $i\delta_{\pm}$ in $V_{\pm}$ and $Z_{\pm}$ is essential. While, as we have stated, the expressions above have a uniform pattern for any number of degrees of freedom, in the two-mode case we can exploit the fact that the group $Sp(4, R)$ is locally isomorphic to the de Sitter group $SO(3, 2)$ [15], as they share the same Lie algebra. We therefore choose the basis for $Sp(4, R)$ to make this explicit: it helps us visualise geometrically the analysis to follow. We define combinations $Q, J_1, J_2, L_1, r = 1, 2, 3$ of the $W$'s, $V$'s and $Z$'s as:

$$Q = \frac{1}{4} (V_{11} + V_{12} + Z_{11} + Z_{12}) = \frac{1}{4} (N + 1)$$

$$= \frac{1}{2} (a_{\dagger} a_{\dagger} + a_{\dagger} a_{\dagger}) .$$

$$J_1 = \frac{1}{4} (V_{11} + Z_{11}) = \frac{1}{2} (a_{\dagger} a_{\dagger} + a_{\dagger} a_{\dagger}),$$

$$J_2 = \frac{1}{2} (W_{11} - W_{11}) = \frac{1}{2} (a_{\dagger} a_{\dagger} - a_{\dagger} a_{\dagger}),$$

$$J_3 = \frac{1}{4} (V_{11} - V_{12} + Z_{11} - Z_{12}) = \frac{1}{2} (a_{\dagger} a_{\dagger} - a_{\dagger} a_{\dagger}),$$

$$K_1 = \frac{1}{4} (V_{11} - V_{12} - Z_{11} + Z_{12}) = \frac{1}{2} (a_{\dagger} a_{\dagger} + a_{\dagger} a_{\dagger}),$$

$$K_2 = \frac{1}{2} (W_{11} + W_{11}) = \frac{1}{2} (a_{\dagger} a_{\dagger} - a_{\dagger} a_{\dagger}),$$

$$K_3 = \frac{1}{2} (Z_{11} - Z_{12}) = -\frac{1}{2} (a_{\dagger} a_{\dagger} + a_{\dagger} a_{\dagger}).$$

13
\[
L_1 = \frac{1}{2}(W_1 - W_3) = \frac{i}{4}(a_1^2 - a_1^1 - a_2^2 + a_2^1),
\]
\[
L_2 = \frac{1}{2}(V_1 + V_3 - Z_1 + Z_3) = \frac{1}{4}(a_1^2 + a_1^1 + a_2^2 + a_2^1),
\]
\[
L_3 = \frac{1}{2}(W_1 + W_3) = -\frac{i}{2}(a_1^2 - a_1^1 - a_2^2 + a_2^1).
\]

While at first sight these expressions may appear complicated, the commutation relations have a suggestive simple form:

\[
[J_+, J_-] = i\epsilon_{\alpha\beta\gamma}J_{\gamma},
\]
\[
[Q, J_\alpha] = 0;
\]
\[
[J_+, K_\alpha, L_\beta] = i\epsilon_{\alpha\beta\gamma}K_{\gamma},
\]
\[
[Q, K_\alpha, L_\beta] = \pm (K_\alpha = iL_\beta);
\]
\[
[K_+, K_\alpha, L_\beta] = [L_+, L_\beta] = -i\epsilon_{\alpha\beta\gamma}J_{\gamma}.
\]

Comparing and combining the above two sets of equations we see that the four operators \(Q\) and \(J\) are the photon-number conserving generators of \(U(2)\) (being respectively the generators of the \(U(1) \) and \(SU(2)\) parts of \(U(2)\)): while the six operators \(K\) and \(L\) are the noncompact generators of squeezing transformations.

The connection to \(SO(5,2)\) becomes evident by regarding these generators as the various components of an antisymmetric set \(M_{AB} = -M_{BA}\), where the indices \(A, B\) go over the range \(1, 2, \ldots, 5\):

\[
Q = M_{45}, \quad J_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}M_{\beta\gamma},
\]
\[
K_\alpha = M_{4\alpha}, \quad L_\beta = M_{5\beta}.
\]

Then the commutation relations (3.5) take on the de Sitter form

\[
[M_{AB}, M_{CD}] = i\epsilon_{ABCE}M_{CE} = g_{BE}M_{AC} + g_{AE}M_{CB} - g_{EC}M_{BA},
\]
\[
I_{AB} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}.
\]

In this picture, the compact generators \(Q, J\) are generators of rotations in the 4-5 plane and in the 1-2-3 subspace respectively, together making up the maximal compact \(SO(2) \times SO(3)\) subgroup of \(SO(3,2)\). With respect to the Fock basis \(|n_1, n_2\rangle\) for \(\mathcal{H}\) given in eq. (2.8), the situation is that all these vectors with a fixed total number \(n = n_1 + n_2, (n + 1)\) in all, are eigenvectors of \(Q\) with a common eigenvalue \((n + 1)/2\); and they simultaneously provide the spin \(n/2\) representation of the \(SO(3)\) subgroup generated by \(J\). On the other hand, the \(K_\alpha\) and \(L_{\beta}\) are noncompact Lorentz boost generators in the \(r = 4\) and \(r = 5\) planes respectively.

With this algebraic background, we go on to consider two-mode squeezing transformations. It is a fact [16] that each matrix \(S \in Sp(4, \mathbb{R})\) can be decomposed, globally and uniquely, into the product of two particular kinds of \(Sp(4, \mathbb{R})\) matrices: one factor belongs to the subgroup \(K\), the other to a subset \(\Pi\) defined in the following way:

\[
\Pi = \{ S \in Sp(4, \mathbb{R}) \mid S^T = S = \text{positive definite} \} \subset Sp(4, \mathbb{R}).
\]

We shall hereafter denote elements in \(\Pi\) by \(P, \tilde{P}, \ldots\). The decomposition mentioned above is then:

\[
S \in Sp(4, \mathbb{R}) : \quad S = P \cdot S(X, Y),
\]
\[
P \in \Pi,
\]
\[
X - \tilde{Y} \in U(2),
\]

with both factors being uniquely determined by \(S\). This is the form in the present context of the familiar polar decomposition formula [17] for a general matrix. For the metaplectic operators \(U(S)\) we have the corresponding statement

\[
U(S) = U(P) \cdot U(S(X, Y)),
\]
\[
U(P) = \exp i(\text{real linear combination of } \bar{F} \text{ and } \bar{J}^T),
\]
\[
U(S(X, Y)) = \exp i(\text{real linear combination of } Q \text{ and } \bar{J}^T).
\]

We may now identify precisely the most general squeezing transformation within.
the $\text{Sp}(4,\mathbb{R})$ framework, as the operator $\mathcal{U}(P)$, characterised by two numerical three-dimensional vectors $\vec{K}$, $\vec{L}$ appearing as coefficients of $\mathcal{K}$ and $\mathcal{L}$ in the exponent:

$$\mathcal{U}(\vec{K}, \vec{L}) = \exp \left( i \vec{K} \cdot \mathcal{K} - i \vec{L} \cdot \mathcal{L} \right). \quad (3.11)$$

Thus we reserve the name squeezing transformations for these elements of $\Pi$ within $\text{Sp}(4,\mathbb{R})$, represented in the metaplectic representation by a single exponential factor. (It is well to keep in mind that $\Pi$ is not a subgroup, so the product of two such single exponential squeezing transformations is in general not a similar single exponential. This is analogous to the well-known fact that the product of two $\text{SO}(3,1)$ Lorentz boosts is not just a boost but a boost followed or preceded by a rotation called Wigner rotation [18]).

We may relate the decomposition (3.10) of a general metaplectic transformation to a general quadratic Hamiltonian quite directly. Any such Hamiltonian containing both photon conserving and nonconserving terms with possibly time dependent coefficients would lead via the Schrödinger equation to a unitary finite time evolution operator which can be uniquely decomposed into the product form (3.10). Thus, integration of the Schrödinger equation leads in general to a specific passive factor and another specific squeezing transformation. In case the Hamiltonian is time independent and a combination only of the generators $\mathcal{K}$ and $\mathcal{L}$, this evolution operator is already of the form $\mathcal{U}(\mathcal{P})$.

Since we have a $U(2)$-invariant squeezing criterion, as we have seen, elements of $U(2)$ have no effect on the squeezed or nonsqueezed status of any given state. This means that the $U(2)$ transform, by conjugation, of a squeezing transformation is another squeezing transformation which should be regarded as equivalent to the first one. It is clear that, in any case, any equivalence relation among set of squeezing transformations as defined by us above should be based on processes which take one squeezing transformation to another.

Now from the commutation relations (3.5b) we can see that the generators $\mathcal{K}$, $\mathcal{L}$, and the squeezing transformations $\mathcal{U}(\mathcal{K}, \mathcal{L})$ defined in eq. (3.11), behave as follows under conjugation by elements of $U(2)$:

$$e^{i\theta \vec{K} \cdot \mathcal{K}} e^{-i\theta \mathcal{L} \cdot \mathcal{L}} = e^{i\theta \mathcal{K} \cdot \mathcal{K}} e^{-i\theta \mathcal{L} \cdot \mathcal{L}}, \quad (3.12a)$$

$$e^{i\theta \cdot \mathcal{J} \mathcal{K} \cdot \mathcal{K}} e^{-i\theta \cdot \mathcal{L} \cdot \mathcal{L}} = e^{i\theta \cdot \mathcal{K} \cdot \mathcal{K}} e^{-i\theta \cdot \mathcal{L} \cdot \mathcal{L}}, \quad (3.12b)$$

$$e^{i\theta \cdot \mathcal{J} \mathcal{K} \cdot \mathcal{K}} e^{-i\theta \cdot \mathcal{L} \cdot \mathcal{L}} = e^{i\theta \cdot \mathcal{K} \cdot \mathcal{K}} e^{-i\theta \cdot \mathcal{L} \cdot \mathcal{L}}, \quad (3.12c)$$

$$R_{\alpha}(\theta) = \delta_{\alpha \alpha} \cos \alpha + \alpha \cdot \sin \alpha \frac{1 - \cos \alpha}{\alpha^2} - \epsilon_{\alpha} \epsilon \alpha \frac{\sin \alpha}{\alpha}, \quad (3.12d)$$

Here we have listed separately the effects of $U(1)$ and $SU(2)$ within $U(2)$ on the generators and the squeezing transformations. The questions raised at the start of this Section can now be posed more precisely: If the set of squeezing transformations $\mathcal{U}(\mathcal{K}, \mathcal{L})$ is separated into distinct nonoverlapping equivalence classes based on the $U(2)$ action (3.12), how can we conveniently choose $U(2)$-invariant parameters to label these classes, and then pick a convenient representative element from each class? We answer these questions in this sequence.

It is clear that we need to construct a complete set of independent expressions in $\mathcal{K}$ and $\mathcal{L}$, invariant under both $U(1)$ and $SU(2)$ actions (3.12b, 3.12c). We begin by defining the matrix $M$ of scalar products among $\mathcal{K}$ and $\mathcal{L}$, which is then $SU(2)$ invariant:

$$M(\mathcal{K}, \mathcal{L}) = \begin{pmatrix} \mathcal{K} \cdot \mathcal{K} & \mathcal{K} \cdot \mathcal{L} \\ \mathcal{K} \cdot \mathcal{L} & \mathcal{L} \cdot \mathcal{L} \end{pmatrix}. \quad (3.13)$$

This is a real, symmetric positive semi-definite matrix. With respect to $U(1)$ action, we see from eq. (3.12b) that $M(\mathcal{K}, \mathcal{L})$ undergoes a similarity transformation by the rotation matrix of angle $\theta$:

$$M(\mathcal{K}, \mathcal{L}) = R(\theta) M(\mathcal{K}, \mathcal{L}) R(\theta)^{-1},$$

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (3.14)$$
One now sees that there are two independent $U(2)$ invariants that can be formed:

\[
\begin{align*}
\mathcal{O}_1(\vec{k}, \vec{l}) &= \det M(\vec{k}, \vec{l}) = |\vec{k} \wedge \vec{l}|^2, \\
\mathcal{O}_2(\vec{k}, \vec{l}) &= \text{Tr} M(\vec{k}, \vec{l}) = |\vec{k}|^2 + |\vec{l}|^2,
\end{align*}
\]

(3.15)

and it is easily checked that there are no other invariants independent of these.

Next let us tackle the problems of finding convenient parameters and representative squeezing transformations for the $U(2)$ equivalence classes, one for each class. We see from eq. (3.15) that if $\mathcal{O}_1 > 0$ (i.e., $\mathcal{O}_1 \neq 0$) then the two vectors $\vec{k}$ and $\vec{l}$ are both nonzero and nonparallel; while if $\mathcal{O}_1 = 0$ they are parallel (and one of them could vanish). These are therefore clearly different geometrical situations. Starting with the matrix $M(\vec{k}, \vec{l})$, we see from its $U(1)$ transformation law (3.14) that by a suitable choice of the angle $\theta$ we can arrange the transformed matrix $M(\vec{k}, \vec{l})$ to be diagonal and in the case of unequal eigenvalues to place the longer eigenvalue in the first position. This means that in each equivalence class of squeezing transformations there certainly are elements $U(\vec{k}, \vec{l})$ for which $\vec{k} \cdot \vec{l} = 0$ and $|\vec{k}| \geq |\vec{l}|$. This still leaves us the freedom of action by $SU(2)$. We may now exploit this freedom to put the (mutually perpendicular) vectors $\vec{k}$ and $\vec{l}$ into a convenient geometrical configuration. A look at the forms of the noncompact generators $\hat{K}$ and $\hat{L}$ in eq. (3.4) suggests that we choose $\vec{k}$ and $\vec{l}$ as follows:

\[
\begin{align*}
\vec{k} &= (0, a, 0), \quad \vec{l} = (b, 0, 0), \quad a \geq b.
\end{align*}
\]

(3.16)

(A further reason for this choice will emerge shortly). $\mathcal{O}_1$ and $\mathcal{O}_2$ can now be evaluated in terms of $a, b$ to obtain the relations

\[
\begin{align*}
\mathcal{O}_1(\vec{k}, \vec{l}) &= a^2 b^2, \\
\mathcal{O}_2(\vec{k}, \vec{l}) &= a^2 + b^2, \\
& a \geq b \geq 0, \quad (a, b) \neq (0, 0).
\end{align*}
\]

(3.17)

With this parametrisation we can now say: there is a two-fold infinity of distinct equivalence classes of squeezing transformations for two-mode systems, each class corresponding uniquely and unambiguously to a point $(a, b)$ in the octant $a \geq b \geq 0$ in the real $a - b$ plane, excluding the origin. Different points in the octant correspond to intrinsically distinct equivalence classes. Within an equivalence class determined by a point $(a, b)$, of course, one can connect different squeezing transformations $U(\vec{k}, \vec{l})$ by conjugation with suitable $U(2)$ elements. Given a squeezing transformation $P \in SU(2)$ its class $(a, b)$ is determined by solving the equations

\[
\begin{align*}
\text{Tr}(P) &= 2(\cosh \frac{a-b}{2} + \cosh \frac{a+b}{2}), \\
\text{Tr}(P^2) &= 2(\cosh(a-b) + \cosh(a+b)).
\end{align*}
\]

(3.18)

subject to the conditions on $a$ and $b$ appearing in eq. (3.17).

Then we have the following convenient two-mode squeezing transformation representing the equivalence class $(a, b)$:

\[
\begin{align*}
U(0)(a, b) &= U(0)(a, 0) U(0)(0, b), \\
U(0)(a, 0) &= \exp(iaK_3), \\
U(0)(0, b) &= \exp(ibL_1).
\end{align*}
\]

(3.19)

The two factors $U(0)(a, 0)$ and $U(0)(0, b)$ commute and may be written in either order, since according to eq. (3.5c) the noncompact generators $K_3$ and $L_1$ commute.

Finally one can easily calculate the symplectic matrix $S(\mathfrak{a}, b) \in Sp(4, \mathbb{R})$, corresponding to the metapiectic operator $\mathcal{U}(a, b)$, by using eq. (2.12). The result is:

\[
\begin{align*}
\mathcal{U}(a, b)^{-1} \xi U(0)(a, b) &= S(\mathfrak{a}, b) \xi, \\
S(\mathfrak{a}, b) &= \text{diag}(\{e^{a-b}/2, e^{a+b}/2, e^{-(a+b)/2}, e^{-(a-b)/2}\}).
\end{align*}
\]

(3.20)

Now we can clarify that the particular choice (3.16) was dictated by the desire to have $S(\mathfrak{a}, b)$ diagonal. This element of $Sp(4, \mathbb{R})$ describes independent reciprocal scaling of the
standard quadrature components of each mode. This amounts to showing geometrically that it is possible to diagonalise every $P \in \Pi$ using conjugation by $U(2)$.

We illustrate our classification scheme of two-mode squeezing transformations by giving two examples. The extensively studied Caves-Schumaker \cite{9} transformation uses the operator

$$U^{(C-S)}(z) = \exp \left( z \hat{a}_d^\dagger \hat{a}_d^\dagger - z^* \hat{a}_d \hat{a}_d \right).$$

(3.21)

By appearance, this attempts to involve or entangle the two modes maximally. In our notation this squeezing transformation corresponds to the generator combination

$$\begin{align*}
\hat{a}_d^\dagger \hat{a}_d - \hat{a}_d \hat{a}_d^\dagger &= i(\hat{\kappa} \cdot \hat{\vec{R}} + \hat{\vec{I}} \cdot \hat{\vec{L}}), \\
\hat{\kappa} &= -2(0,0, \text{Im} z), \\
\hat{\vec{I}} &= (0,0, \text{Re} z).
\end{align*}$$

(3.22)

Thus the invariant parameters $a$ and $b$ have values

$$a = 2 \text{Im} z, \quad b = 0.$$  

(3.23)

The Caves-Schumaker squeezing transformations, and their $U(2)$ conjugates, all taken together, form a one-parameter family or one-dimensional line, in the $a-b$ octant. In that sense they are a set of measure zero.

Another interesting case is a squeezing transformation that refers essentially to a single mode but masquerades as a two-mode transformation:

$$U(x; \alpha, \beta) = \exp \left[ x(\alpha \hat{a}_d^\dagger + \beta \hat{a}_d) + x^*(\alpha^* \hat{a}_d^\dagger + \beta^* \hat{a}_d) \right].$$

(3.24)

After some simple algebra we find

$$U(x; \alpha, \beta) = \exp \left[ i(\hat{\kappa} \cdot \hat{\vec{R}} + \hat{\vec{I}} \cdot \hat{\vec{L}}) \right],$$

$$\hat{\kappa} + i \hat{\vec{I}} = 2x \left( -i(\alpha^2 - \beta^2), (\alpha^2 + \beta^2), 2i\alpha^* \beta^* \right).$$

(3.25)

The associated invariants and parameters are

$$\begin{align*}
\alpha &= 16 \text{Im} z^2, \quad \beta = 8 \text{Re} z, \\
\alpha^* &= 2(0,0), \quad \beta^* = 0.
\end{align*}$$

(3.26)

These equivalence classes thus lie along the line $a = b$ in the octant, again a one-parameter family of zero measure. Our results are depicted in Figure 1. We note here that the size of an equivalence class depends sensitively upon the values of $(a,b)$. Since none of the generators of $U(2)$ commute with both $K_3$ and $L_1$, for $a \neq 0$ and $b \neq 0$ we have a full four parameter equivalence class. For the case $a \neq 0$ and $b = 0$ the generator $J_3$ of $U(2)$ commutes with $K_3$ and hence we have only a three parameter family of transformations in the equivalence classes.

We conclude this Section with a few comments. Each point $(a,b)$ in the octant denotes an equivalence class of squeezing transformations, whose dependences on $a$ and $b$ would be of physical significance and would show up in a variety of $U(2)$-invariant properties.

The two-mode transformations so far discussed in the literature are basically along the two lines shown in Figure 1. In this sense, most of the intrinsically distinct two-mode transformations, their effects on various states, etc., remain to be explored. Those equivalence classes $(a,b)$ for which $a > b$ involve the two modes in an essential way. We may say purely qualitatively that the distance of the point $(a,b)$ from the line $a = b$, or perhaps better the expression $|(a^2 - b^2)/(a^2 + b^2)|^2$, is a measure of the extent to which both modes are involved in the transformation. In this sense, as remarked earlier, the Caves-Schumaker transformations involve both modes maximally.

IV. SQUEEZED COHERENT AND THERMAL STATES FOR TWO MODES

The general two mode coherent state with complex two-component displacement $\vec{a} = (a_1, a_2)$ is defined by

$$|\vec{a}\rangle = \exp \left( \vec{a} \cdot \hat{a}_d^\dagger - \vec{a}^* \cdot \hat{a}_d \right) |0,0\rangle.$$
\[
\exp \left( -\frac{1}{2} |\alpha_1|^2 - \frac{1}{2} |\alpha_2|^2 \right) \exp \left( \alpha_1 \alpha_1^* - \alpha_2 \alpha_2^* \right) |0,0\rangle.
\] (4.1)

For this state the means of the quadrature components \(\xi\) do not vanish in general:
\[
(\delta |\xi\rangle \langle \xi|) = \sqrt{2} (\Re \alpha_1, \Re \alpha_2, \Im \alpha_1, \Im \alpha_2)^T.
\] (4.2)

The variance matrix is however independent of \(\delta\):
\[
V (|\xi\rangle \langle \xi|) = V (|\bar{\xi}\rangle \langle \bar{\xi}|) = \frac{1}{2} 1_{4\times4}.
\] (4.3)

The most general squeezed coherent state is obtained by applying \(U(P)\) for some \(P \in \Pi \subset Sp(4,\mathbb{R})\) to \(|\delta\rangle\) for some \(\delta\). This \(U(P)\) is conjugate, via some \(U(2)\) element, to \(U^{\otimes}(a,b)\) for some \(a, b\). Now the effect of a \(U(2)\) transformation on \(|\delta\rangle\) is to give us another coherent state \(|\delta'\rangle\), \(\delta'\) being the \(U(2)\) transform of \(\delta\). But the variance matrix is in any case \(\delta\)-independent. To examine the \(U(2)\)-invariant squeezing condition, therefore, it suffices to examine the particular class of squeezed coherent states
\[
|\delta; a, b\rangle = U^{\otimes}(a,b) |\delta\rangle.
\] (4.4)

From eqs (2.16.3.20), the calculation of the variance matrix for this state is trivial, and it is in fact diagonal:
\[
V (|\delta; a, b\rangle \langle \delta; a, b|) = S^{\otimes}(a, b) V (|\delta\rangle \langle \delta|) S^{\otimes}(a, b)^T
\]
\[
= \frac{1}{2} S^{\otimes}(2a, 2b)^T \\
= \frac{1}{2} \begin{pmatrix} 1 & e^{-(a-b)} & e^{(a-b)} & e^{-(a+b)} \end{pmatrix}.
\] (4.5)

Since \(a\) and \(b\) are nonnegative, and in addition \(a + b > 0\), we see that the least eigenvalue of this variance matrix is
\[
(\lambda (V (|\delta; a, b\rangle \langle \delta; a, b|))) = \frac{1}{2} e^{-(a+b)} < \frac{1}{2}
\] (4.6)

These states are thus always squeezed.

If we apply any passive \(U(2)\) transformation \(S(X,Y)\) to any one of the states \(|\delta; a, b\rangle\) defined above, the variance matrix will in general change as \(V \rightarrow V' = S(X,Y) V S(X,Y)^\dagger\):

but its eigenvalue spectrum, and in particular \(\lambda(V)\), remains unaltered. Thus all the states symbolically written as \(U(U(2)) |\delta; a, b\rangle\), for various \(U(2)\) elements, are squeezed to the same extent as \(|\delta; a, b\rangle\), and have \(\lambda(V)\) given by eq. (4.6).

The Schrödinger wave functions for the subfamily of squeezed coherent states (4.4) are particularly simple, since they are products of single mode squeezed coherent state wavefunctions:
\[
\psi^{(a)}(q_1; \alpha_1, a - b) = \psi^{(a)}(q_1; \alpha_1, a) \psi^{(a)}(q_1; \alpha_2, a + b)
\]
\[
\psi^{(a)}(q_1; \alpha_1, a) = \frac{\exp \left[ \text{ia} (\alpha_1 + \frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_2^2) \right]}{\sqrt{\text{ia} (\alpha_1 + \frac{1}{2} \alpha_1^2 - \frac{1}{2} \alpha_2^2)}}
\] (4.7)

(For a general state \(U(U(2)) |\delta; a, b\rangle\), we do not expect such a product form). When we set \(b = 0\) (Caves-Schumaker limit), both factors show the same amount of squeezing; while when we set \(a = b\) (essentially single mode situation) we see squeezing only in the factor referring to the second mode. These features are as we would have expected.

The next example we look at is the case of a two-mode thermal state subjected to squeezing. The motivation in making this choice is that the starting density operator is explicitly \(U(2)\)-invariant. The normalized density operator corresponding to inverse temperature \(\beta = \frac{1}{kT}\) is described in the Fock and Schrödinger representations by:
\[
\rho_S(\beta) = (1 - e^{-\beta})^2 \exp \left[ -\beta |\psi_1; \alpha, -\alpha_1^*\rangle \langle \psi_1; \alpha, -\alpha_1^*| \right]
\]
\[
= (1 - e^{-\beta})^2 \sum_{n_1,n_2=0}^\infty e^{-\beta(n_1+n_2)} |n_1, n_2\rangle \langle n_1, n_2|.
\] (4.5a)

\[
\rho_S(q_1^2, q_2^2; \beta) = \frac{2}{\pi} \tanh^2 \frac{\beta}{2} \exp \left[ -\frac{1}{2} \left( \cosh \frac{\beta}{2} + \cosh \frac{\beta}{2} \right) (q_1^2 + q_2^2 - q_1^2 - q_2^2) \right]
\]
\[
- \left( \cosh \frac{\beta}{2} - \cosh \frac{\beta}{2} \right) (q_1 q_2^* + q_2 q_1^*) \right];
\] (4.5b)

with \(\Gamma(2)\) invariance expressed by
\[
e^{i\beta Q_1} \rho_S(\beta) e^{-i\beta Q_1} = e^{i\beta \cdot J} \rho_S(\beta) e^{-i\beta \cdot J} = \rho_S(\beta).
\] (4.9)

Therefore it suffices to examine the properties of the density operator obtained by conjugating \(\rho_S(\beta)\) with \(U(U(2)) |\delta; a, b\rangle\).
\[ \rho(\beta, a, b) = U^{(b)}(a, b) \rho_0(\beta) U^{(b)}(a, b)^\dagger. \]  
\[ \text{The most general squeezed thermal state is evidently} \]
\[ U(U(2)) \rho(\beta, a, b) U(U(2))^\dagger. \]
\[ \text{but this has the same squeezing properties as } \rho(\beta, a, b). \]

For the thermal state \( \rho_0(\beta) \) the variance matrix is well known [11]:
\[ V(\rho_0(\beta)) = \frac{1}{2} \coth \frac{\beta}{2} \mathbf{1}. \]
\[ \text{Therefore for the particular set of squeezed thermal states (4.10), we have diagonal variance matrices:} \]
\[ V(\rho(\beta, a, b)) = S^{(b)}(a, b) V(\rho_0(\beta)) S^{(b)}(a, b)^T \]
\[ = \frac{1}{2} \coth \frac{\beta}{2} S^{(b)}(a, b) \]
\[ = \frac{1}{2} \coth \frac{\beta}{2} \operatorname{diag} \left( e^{-a+b}, e^{a-b}, e^{a+b}, e^{-a+b} \right). \]
\[ \text{The least eigenvalue is evidently} \]
\[ \lambda(V) = \frac{1}{2} \coth \frac{\beta}{2} e^{-a+b}. \]
\[ \text{so for a given temperature, squeezing sets in when} \]
\[ a + b > \ln \coth \frac{\beta}{2}. \]

In Figure 1, this region consists of all points in the \( a - b \) octant to the right of the line \( a + b = \ln \coth \beta/2 \), which is a line perpendicular to the line \( a = b \) and at a distance \( \ln \coth(\beta/2)/\sqrt{2} \) from the origin.

\[ \text{V. THE HETERODYNE DETECTION SCHEME} \]

We have so far not specified in any detail the two orthogonal modes of radiation being subjected to squeezing. Let us at this point consider a situation well studied experimentally by the heterodyne detection scheme [19]. Here the two modes differ only slightly in frequency, but are otherwise similar. In this kind of experimental arrangement, what is actually measured is the fluctuation of a certain photocurrent, and this in turn gives the fluctuation or variance of the \( \rho \)-quadrature component of a particular passive combination of the original modes. The combinations of \( a_1 \) and \( a_2 \) that are involved form the one-parameter family
\[ a(u) = \frac{1}{\sqrt{2}} (a_1 - a_2) e^{-iu}. \quad 0 \leq u < 2\pi. \]

This can be regarded as the first component of a \( U(2) \)-transformed pair \( a_1, a_2 \):
\[ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-iu/2} & e^{-iu/2} \\ -e^{iu/2} & e^{iu/2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \]
\[ a_1 \equiv a(u). \]

The hermitian quadrature component whose fluctuation is measured is
\[ q(u) = \frac{1}{\sqrt{2}} (a_1 + a_2) \]
\[ = \frac{1}{\sqrt{2}} (a_1 - a_2) \cos \frac{u}{2} - \frac{1}{\sqrt{2}} (a_1 + a_2) \sin \frac{u}{2}. \]

The only experimentally adjustable parameter here is the angle \( u \). The family of \( U(2) \) elements realised in the heterodyne scheme is thus only the one-parameter set given in eq. (5.2) parametrised by \( u \), and belonging to \( SU(2) \):
\[ U_{\rho}(u) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-iu/2} & e^{-iu/2} \\ -e^{iu/2} & e^{iu/2} \end{pmatrix} \in SU(2). \]

We notice that this is not a one-parameter subgroup of \( SU(2) \); in particular even the identity element of the group is not contained here.

With this description of the heterodyne setup in our framework, let us see to what extent it can be used to detect \( U(2) \)-invariant squeezing. Now a general two-mode state \( \rho \) with variance matrix \( V \), even if it is squeezed in the intrinsic sense of eq. (2.21), may not be manifestly squeezed. That is, it may happen that \( V_{\rho} \geq 1/2 \) for all \( a = 1, \ldots, 4 \). As our discussion in Section (II) shows, we need to be able to experimentally realise
a general $U(2)$-transformation applied to the state $\rho$, and change its variance matrix to a form where one of its diagonal entries (say the leading one) becomes less than 1/2. However the heterodyne method is generally unable to do this job for us, as it can only realise the one-parameter subset of $SU(2)$ transformations $U_H(\psi)$ for $0 \leq \psi < 4\pi$.

In the two examples of squeezed coherent states and squeezed thermal states studied in the previous Section, we have a family of states related to each other by conjugation with $U(2)$ for each point in the $a - b$ plane. Each equivalence class has appropriate dimensionality of the equivalence class depending upon values of $(a, b)$ as explained in Section III. It turns out that for each $(a, b)$ the heterodyning scheme can detect squeezing in only a one parameter subset of the family of states. Although heterodyning detection covers the whole $a - b$ plane it does not reach all the states corresponding to each point in the $a - b$ plane. For example, in the representative chosen in eqs (4.4, 4.10), for which the variance matrix is already diagonal, the squeezing cannot be detected by this scheme because of the absence of the identity in $U_H$. It should be possible to detect squeezing in these states by a suitably modified scheme. We wish to emphasize that there is a definite need to be able to experimentally implement entire $U(2)$. This would allow the experimenter to detect the degree of squeezing unambiguously without any prior knowledge of the elements of the initial variance matrix.

These remarks show on the one hand the way the heterodyning scheme fits into our general analysis, and on the other hand the need to devise new schemes capable of realising all elements of $U(2)$.

VI. CONCLUDING REMARKS

We have presented a classification scheme for two-mode squeezing transformations, based on the structure of the real four dimensional symplectic group $Sp(4, R)$, and the separation of its elements into passive (compact) and active (noncompact) types. The structure and action of the maximal compact subgroup $U^c(2)$ in $Sp(4, R)$, and the $U^c$-invariant squeezing criterion formulated elsewhere for a general $n$-mode system, have guided our considerations. All our work is in the metaplectic unitary representation of $Sp(4, R)$; and the local isomorphism $Sp(4, R) \approx SO(3, 2)$ has led to a convenient geometric picturization of the situation.

As emphasized in Section III the squeezing transformations $U(P), P \in \Pi \subset Sp(4, R)$, do not form a subgroup of $Sp(4, R)$. The breakup of these transformations into equivalence classes, based on the effect of conjugation by elements of $U(2)$ is the only natural available classification procedure. This is because the definition of equivalence classes for any set of objects has to be based on an equivalence relation defined on that set. Thus, we have treated two elements $P, P' \in \Pi$ as intrinsically equivalent if

$$P = S(X, Y) P S(X, Y)^T \text{ for some } X - iY \in U(2).$$

(6.1)

It should however be realised that the detailed effects of action by $U(P)$ and $U(P')$ on a general initial two mode state $\rho_x$, as seen in the changes caused in the variance matrix $V(\rho_x)$, need not be identical. Since this is a subtle and important point we spell it out in detail. Starting from a general state $\rho_x$ action by a squeezing transformation leads to a new state

$$\rho = U(P); \rho_x U(P)^{-1}.$$  

(6.2)

As seen in Section III, any $U(P)$ is expressible in terms of a representative element $U^{(0)}(a, b)$ as

$$U(P) = U(S(X, Y)) U^{(0)}(a, b) U(S(X, Y))^{-1}$$  

(6.3)

for suitable $X - iY \in U(2)$. Therefore we have

$$\rho = U(S(X, Y)) U^{(0)}(a, b) U(S(X, Y))^{-1}; \rho_x U(S(X, Y)) U^{(0)}(a, b)^{-1} U(S(X, Y))^{-1}. \text{ (6.4)}$$

This leads to eq. (6.16) to the relation

$$V(\rho') = S(X, Y) \ S^T(a, b) S(X, Y)^T V(\rho) S(X, Y) S^T(a, b) S(X, Y)^T.$$  

(6.5)
between the two variance matrices. Now the first and last factors on the right hand side here have no influence on the spectrum, and so on the least eigenvalue, of \( V(\rho) \). Therefore the squeezed or nonsqueezed nature of \( \rho \) is actually determined by the least eigenvalue of the matrix

\[
S(X,Y)^T V(\rho) S(X,Y) = S^{0}(a,b) S(X,Y)^T V(\rho) S(X,Y) S^{0}(a,b). \tag{6.6}
\]

But now the right hand side is in general dependent not only on the invariant parameters \( a, b \) but also on \( X, Y \). In the examples studied in Section IV namely where \( \rho \) is a coherent state or an isotropic thermal state, \( V(\rho) \) happens to be a multiple of the identity matrix, so that on the right hand side of eq. (6.6) the dependence on \( X, Y \) cancels. But this need not happen in general. Thus for instance if we take for \( \rho \) an anisotropic thermal state with unequal temperatures for the two modes, we have only \( U(1) \times U(1) \), rather than \( U(2) \), invariance for this \( \rho \), so the least eigenvalue \( \lambda(\rho) \) of \( V(\rho) \) will depend on \( a, b \) and on two out of the four \( U(2) \) parameters present in \( X - iY \). One can easily convince oneself that the only situation where \( S(X,Y)^T V(\rho) S(X,Y) = V(\rho_0) \), independent of \( X \) and \( Y \) is when \( V(\rho_0) \) is a multiple of the unit matrix; and the isotropic thermal states do reproduce all such cases. Therefore a more detailed study of the effect of a general squeezing transformation on initial states \( \rho_0 \) with nontrivial \( V(\rho_0) \) is of considerable interest.

A related important question is the following: Let us take two squeezing transformations \( P_1, P_2 \in \Pi \) belonging to equivalence classes \((a_1, b_1), (a_2, b_2)\) respectively, which could coincide. The product \( P_1, P_2 \) will in general be of the form \( S(X,Y) P \) with \( S(X,Y) \in U(2) \) and \( P \in \Pi \). If \( P \) belongs to the equivalence class \((a, b)\) we wish to determine this class in terms of \( P_1 \) and \( P_2 \). Using the fact \( Tr(P^T) = Tr((P_1 P_2)(P_1 P_2)^T) \) and the eq. (3.18) we arrive at the following relations

\[
2[cosh(a - b) + cosh(a + b)] = Tr((P_1, P_2)(P_1, P_2)^T)
\]

\[
2[cosh(2a - b) + cosh(2a + b)] = Tr(((P_1 P_2)(P_1 P_2)^T)^T) \tag{6.7}
\]

which can be solved to find \((a, b)\). We note here that \((a, b)\) do not depend only upon

\((a_1, b_1), (a_2, b_2)\) but also on the actual elements chosen from each class. So we do not have a notion of class multiplication among these equivalence classes.

Finally we call attention to our considerations in Section V and to the need for being able to experimentally implement or realize general passive elements of the subgroup \( U(2) \) of \( Sp(4, \mathbb{R}) \). Once this is achieved, for any given state we can bring out in an explicit or manifest fashion its squeezing nature (provided it is squeezed) by altering its variance matrix and making the least eigenvalue appear in the leading position on the diagonal. This also means that we would be able to experimentally measure the fluctuation in the quadrature variable isolating the least eigenvalue. As extensively discussed elsewhere, these considerations which exploit the richness of the geometry underlying the symplectic group do not require complete diagonalization of the variance matrix at all [11].

In the following paper we shall examine \( U(2) \)-invariant properties of two mode squeezed states which go beyond the level of second order moments of the quadrature operators.
REFERENCES

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A recent paper exploiting these ideas for squeezing problems is D. Han, Y. S. Kim, M. E. Noz and L. Yeh. Journal of Mathematical Physics 34, 5493 (1993).


FIG. 1. Equivalence classes of two-mode squeezing transformations. Caves-Schumaker (C-S) and single mode limits: squeezed thermal region.