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Measuring Hurst Exponents with the First Return Method

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Abstract

The First Return method has proven to be an efficient method for determining the Hurst exponent, H , of self-affine surfaces. In this note we discuss its foundations and some corrections to scaling which must be taken into account for adequate estimation of H . Using this method, we analyse a set of artificially generated surfaces with known Hurst exponent and compare the result with Fourier analysis. We also discuss the case when the surface to be analyzed has a curved bias with a maximum or minimum. In this case, the relation between the scaling exponents associated with the first-return histograms and the Hurst exponent H is different from the unbiased case.

1. INTRODUCTION

In this letter, we discuss the first-return method to measure the roughness of self-affine surfaces. As it is becoming increasingly clear that self-affine surfaces are abundant in nature, the need for accurate methods to study them is mounting [1, 2, 3]. Anyone who have tried for example to measure the roughness of such a surface, knows that it is a very delicate affaire. Even in those fortunate cases when the scaling region extends over several decades, there may be a considerable discrepancy between the results obtained with different methods of analysis. As an example, the reader should consult Table 1 of reference [4].

The first return method [4, 5] is a relatively new method that in our experience seems quite good. Since there has been no detailed discussion of it, nor any published systematic comparison of this method with other more established methods, we undertake to rectify this here. We estimate the Hurst exponent from a set of curves with predefined H using the first return method and the power spectrum method. The first return method analyses the surface in a way very different from the latter method, and it may therefore be seen as a good complement to it. Discrepancy between the two methods gives a good idea of the error in the determination of roughness.

One particularly nasty case, which is unfortunately too often encountered in practice, is that the surface to be measured has some bias. What we mean by “bias” may be illustrated as follows: Suppose we want to measure the roughness of a two-dimensional surface. We then generate an ensemble of one-dimensional cuts through this surface, all parallel. If the average profile generated from all of these one-dimensional cuts shows a curvature, the surface is biased. A bias may still exist, but in a direction that does not show up with the direction of the parallel lines chosen. This bias may alter the apparent Hurst exponents both in the first return method, and in the Fourier method. We discuss in this letter the former case, and show how we may disentangle the roughness from the drift in the first return method.

2. THE FIRST RETURN METHOD

A self-affine surface is characterized by anisotropic scaling properties. Making a cut normal to such a surface, we may characterize the profile along the cut by a function

$y = y(x)$. Statistically, a self-affine surface is invariant under the transformation:

$$\begin{cases} x \rightarrow \lambda x, \\ y \rightarrow \lambda^H y, \end{cases} \quad (1)$$

where H is the Hurst exponent — or the roughness exponent. It is this anisotropic scale invariance that is the defining property of self-affine surfaces.

A simple statistical characterization of a cut through a self-affine surface is the probability density $p(y, x)$ that the surface has a height y at position x given that it has a height $y = 0$ at $x = 0$. We consider briefly how $p(y, x)$ can be defined experimentally. A set of N self-affine curves $Y_j(x)$ defined at K points x_i and having J allowed values are generated by the same underlying process P and can thus represent the statistical nature of this process. For each x_i the frequency of occurrence of a certain value y can be measured and is proportional to the probability of finding a height y at position x_i . The normalization requirement on $p(y, x)$ along with the scaling property Eq. (1) allow us to derive

$$\lambda^H p(\lambda^H y, \lambda x) = p(y, x). \quad (2)$$

If the average height $\langle Y_j(x_1) \rangle \neq \langle Y_j(x_2) \rangle$ for $x_1 \neq x_2$, the surface has a drift. In this case, the scaling property (2) will only be valid after the drift has been subtracted.

Moments of the distribution (2) behave as

$$\langle y^k \rangle = (1/L) \int_0^L dx \int_{-\infty}^{+\infty} dy y^k p(y, x) \sim L^{kH}. \quad (3)$$

It is important to note that the above expression is valid only for positive moments, $k > 0$. In computer simulations, the “favorite” method to measure the Hurst exponent is by determining the width, $w = \sqrt{\langle y^2 \rangle - \langle y \rangle^2}$ as a function of L :

$$w \sim L^H. \quad (4)$$

This method, however, is not very practical in experiments where typically it is difficult to change the system size L . Let us note that the moment $\langle (y - y_0)^k \rangle$ where y_0 is a constant behave as

$$\langle (y - y_0)^k \rangle = \langle y^k \rangle + k y_0 \langle y^{k-1} \rangle + \dots \sim L^{kH} + a L^{(k-1)H} + \dots \quad (5)$$

Thus, $\langle (y - y_0)^k \rangle \sim L^{kH}$ to leading order.

Let us now show that if we make a cut through the self-affine profile $y = y(x)$, by a straight line parallel to the x axis, we find that the set of cross points between the line and the self-affine curve form a fractal set with dimension $1 - H$ [2]. For more rigorous proofs in the case when $p(y, x)$ is gaussian, see references [6, 7].

Given the probability density $p(y, x)$ we can express the probability of attaining the height y_0 at any position x as $\int_0^L dx p(y_0, x)$. But this must be proportional to the number of crossings between the line $y = y_0$ and an arbitrary self-affine curve generated by the process P . We use the scaling property (2) and expand $p(y, x)$ around its maximum at $y = 0$

$$\int_0^L dx p(y_0, x) = L^{1-H} \int_0^1 dx \left[p(0, x) + \frac{y_0}{L^H} \frac{\partial p}{\partial y}(0, x) + \frac{1}{2} \left(\frac{y_0}{L^H} \right)^2 \frac{\partial^2 p}{\partial y^2}(0, x) + \dots \right]. \quad (6)$$

The first order derivative vanishes and the number of cross points is therefore

$$M(L) = AL^{1-H} + Cy_0^2 L^{1-3H} + Dy_0^3 L^{1-4H} + \dots \quad (7)$$

Thus to leading order $M \sim L^{1-H}$, and consequently, the fractal dimension is $1 - H$. Furthermore, the scaling exponent of the leading order correction is $1 - 3H$.

There is an equivalent way of counting the number of cross points between the line $y = y_0$ and the self-affine curve $y(x)$. The negative moments, *e.g.* for $k = -2$, can be defined through

$$\lim_{\epsilon \rightarrow 0} \left\langle \frac{1}{(y - y_0)^2 + \epsilon^2} \right\rangle = \lim_{\epsilon \rightarrow 0} \int_0^L dx \int_{-\infty}^{+\infty} dy p(y, x) \frac{1}{(y - y_0)^2 + \epsilon^2}. \quad (8)$$

For any small positive number ϵ the integral is well behaved. When ϵ approaches the limit, Eq. (8) will be more and more dominated by the regions close to the crossings, $y(x) = y_0$. The contribution to the integral from each such region is essentially $1/\epsilon$, and the integral is thus simply the number of crossings per length L multiplied by $1/\epsilon$. By performing the integral on the right side of Eq. (8) we can establish directly the equivalence of this counting procedure and the expression $\int_0^L dx p(y_0, x)$. We also note that the negative moments behave differently than the positive ones, all negative moments “count” the number of crossings, $y(x) = y_0$, they only differ in the strength of the singularity in ϵ . By this line of reasoning, it is easy to see that the negative moments “count” the number of crossings $y(x) = y_0$. Only the strength of the divergence in ϵ distinguishes the various moments: The first negative moment behaves as $\log(\epsilon)$ for small ϵ , while the negative k th moment where $k < -1$, behaves as ϵ^{1-k} .

In the first return method, we measure the histogram of distances Δ between crossings of $y(x)$ and y_0 for all possible y_0 . This histogram behaves as

$$N(\Delta) \sim \Delta^{-(2-H)} . \quad (9)$$

To show this, let us assume that this histogram has the scaling form

$$N(\Delta, L) \sim \Delta^{-\alpha} L^\beta , \quad (10)$$

where α and β are unknown exponents. Note that in Eq. (10), the explicit L dependence of the histogram is included. We now calculate

$$\langle \sum_i \Delta_i^0 \rangle = \int_\delta^L d\Delta \Delta^0 N(\Delta, L) \sim L^\beta \int_\delta^L d\Delta \Delta^{0-\alpha} \sim L^\beta \delta^{1-\alpha} . \quad (11)$$

Here Δ_i is the return distance indexed from smallest to largest by i . δ reflects the lower cut-off in the self-affine cut, $y(x)$. In performing the integral we assumed that $1 < \alpha < 2$, which must be checked *posteriori*, so that it is governed by the lower integration limit. The left side of Eq. (11) is the total number of possible return intervals which must be proportional to the total length L in the x -direction. Thus,

$$\langle \sum_i \Delta_i^0 \rangle \sim L . \quad (12)$$

Comparing Eq. (11) and Eq. (12), we identify

$$\beta = 1 . \quad (13)$$

We now calculate the sum

$$\langle \sum_i \Delta_i^1 \rangle = \int_\delta^L d\Delta \Delta^1 N(\Delta, L) \sim L^\beta \int_\delta^L d\Delta \Delta^{1-\alpha} \sim L^{\beta-\alpha+2} . \quad (14)$$

With the assumption that $1 < \alpha < 2$, this integral is governed by the *upper* limit, L . If we look closer at the left side of Eq. (14), we see that the sum is nothing but the length L times the average return interval. Since the total number of crossing points at a height is L^{1-H} , the average return interval is given by

$$\langle \sum_i \Delta_i^1 \rangle / L = L / L^{1-H} = L^H . \quad (15)$$

Comparing Eq. (14) and Eq. (15), we get

$$\beta - \alpha + 1 = H . \quad (16)$$

The two exponent relations Eq. (13) and Eq. (16) give for α ,

$$\alpha = 2 - H . \quad (17)$$

Since $0 < H < 1$, we see that $1 < \alpha < 2$, which is consistent with our initial assumption. We have now arrived at the correct scaling form of the return histogram

$$N(\Delta, L)/L \sim \Delta^{-(2-H)} . \quad (18)$$

This discussion has been based on the *leading* term in Eq. (7), scaling as L^{1-H} . We may include the second-leading term in the mass $M(L)$ due to non-zero y_0 . We assume a correction term of the form

$$N_{corr}(\Delta, L) \sim \Delta^{-\alpha'} L^{\beta'} . \quad (19)$$

Calculating $\langle \sum_i \Delta_i^0 \rangle \sim L$ including the correction term in (11), gives $\beta' = \beta = 1$. Including the correction term in Eq. (14) gives

$$\langle \sum_i \Delta_i^1 \rangle = L^{\beta-\alpha+2} + bL^{\beta'-\alpha'+2} . \quad (20)$$

Equation (15) now becomes

$$\langle \sum_i \Delta_i^1 \rangle / L = L / (L^{1-H} + cL^{1-3H}) = L^H - cL^{-H} + dL^{-3H} + \dots . \quad (21)$$

Thus, comparing Eq. (20) and Eq. (21), gives that $\alpha' = 2 + H$, and we have that

$$N_{corr}(\Delta, L)/L \sim \Delta^{-(2+H)} . \quad (22)$$

This correction term does *not* diminish with increasing L and will thus be present for small Δ for all system sizes. However, another correction term due to the finite size L may be as important for estimating the Hurst exponent. For a curve of length L , the *effective* length that may be observed is $L - \Delta$ rather than L . This takes into account that an interval of length Δ cannot start at an x between $L - \Delta$ and L . Large Δ s are therefore underrepresented in the histogram. The correction is obtained through multiplication by the factor $1/(1 - \Delta/L)$.

In order to illustrate how the first return method performs in practice, we have artificially generated a set of 300 self-affine curves of length $L = 1024$ with Hurst exponent $H = 0.7$. We have used the Mandelbrot-van Ness algorithm [2] with a memory constant equal to 1000. For these curves we know the baseline and no systematic drift is introduced. We therefore choose to analyze the curves directly. In an experimental system it will often be difficult to align the different samples correctly. To avoid any systematic effect of not aligning samples along the unknown baseline we would prefer to subtract the drift from each curve, *i.e.* perform the transformation

$$y(x_i) \rightarrow y(x_i) - \frac{y(x_n) - y(x_1)}{x_n - x_1} (x_i - x_1). \quad (23)$$

This is a useful way to identify — or rather approach — the direction x in surfaces where one *a priori* does not know the two principal directions x and y defined in Eq. (1).

The averaged return histogram is plotted in Fig. 1 along with a best fit. We have here fitted the function $\log N = C_1 + (H - 2)\log \Delta + \log(1 + C_2\Delta^{-2H})$ to the data. The value of the Hurst exponent is $H = 0.67 \pm 0.05$. If we subtract the drift from each curve, we would estimate the exponent to be about 5% higher.

In figure 2 we show the averaged power spectrum of the same set of self-affine curves. Here we have subtracted the drift as in Eq. (23). A straight line is fitted giving a slope of -2.39 . Theoretically[2] we expect $-(1 + 2H) = -2.40$. We estimate the Hurst exponent in this case to be 0.69 ± 0.02 .

Judging from the estimated exponents and their uncertainties we are inclined to give a higher score to the power spectrum method, although the difference is not large. The slight difference is perhaps more a consequence of the fitting procedure. For the return method we fit three parameters of a nonlinear function. Uncertainties in estimated parameters are expected to be above those for the two parameter model we use for the power spectrum. We view the two methods as complementary and in combination they may give a rather precise determination of Hurst exponents of self-affine surfaces.

3. SCALING OF BIASED SELF-AFFINE SURFACES

Let us now turn to the case of surfaces with bias. We assume that the surface may be written as

$$y(x) = x^k + r(x) \quad (24)$$

where $r(x)$ is a self-affine curve with zero drift — *i.e.* the kind of curves discussed so far in this letter. We will in the following show how a first-return analysis of the biased curve $y(x)$ leads to a change in the exponent α in the first-return histogram Eq. (18). We assume that the point $x = 0$ is included in the interval of interest.

For Δ small enough so that the curvature induced by the drift x^k in Eq. (24) is negligible, the first return histogram Eq. (18) must be unchanged. However, at some cutoff Δ_M , which must be a function of x , the histogram is cut off. Let us therefore rewrite Eq. (18) as

$$N(\Delta, x) = \Delta^{-(2-H)} F(\Delta/\Delta_M(x)) , \quad (25)$$

where $F(t)$ is a constant for $t \rightarrow 0$ and drops faster to zero than any power law for $t \rightarrow \infty$. We will now argue that the cutoff $\Delta_M(x)$ behaves as

$$\Delta_M(x) \sim x^{-\nu(k)} . \quad (26)$$

Over a distance δx , an unbiased fractional brownian motion moves $\delta y = \delta x^H$ in height. The drift x^k , induces a change in height $kx^{k-1}\delta x$. When these two are equal, we get an estimate of the maximum distance δx_M over which we may regard the curve $y(x)$ as unbiased. Thus, $\delta x_M^H \sim x^{k-1}\delta x_M$, or $\delta x_M(x) \sim x^{-(k-1)/(1-H)}$. This maximum distance δx_M must be equal to the cutoff $\Delta_M(x)$, and as a result, we determine

$$\nu(k) = \frac{k-1}{1-H} . \quad (27)$$

The first return histogram Eq. (25) is defined locally at x . In order to determine the *global* first return histogram, we need to integrate Eq. (25) over all x ,

$$N(\Delta) = \int_0^\infty dx \Delta^{-(2-H)} F(\Delta x^{(k-1)/(1-H)}) . \quad (28)$$

After a change of variable $x \rightarrow \Delta^{(1-H)/(k-1)} x$, we find

$$N(\Delta) \sim \Delta^{-(2-H+(1-H)/(k-1))} , \quad (29)$$

so that in this case, the exponent α of Eq. (17) is changed into

$$\alpha = 2 - H + \frac{1-H}{k-1} . \quad (30)$$

This change in exponent comes about because the cutoff Δ_M diverges where the slope of the drift is zero, $dx^k/dx = 0$. Thus, this kind of drift should be encountered fairly often in

practice: All that is needed is for the drift to have an extremum somewhere. Furthermore, we see that the case $k = 2$ is the most likely value for k to be encountered; most extrema are quadratic. In this case,

$$\alpha = 3 - 2H . \quad (31)$$

Thus, failing to take the drift into account, thus using Eq. (17) rather than Eq. (31) in such cases, leads to a severe underestimate of H .

In Fig. 3, we show the first return histogram of a biased curve $y(x) = r(x) + 10^4 \cdot (x/L)^2$, where $r(x)$ is a Mandelbrot-van Ness fractional Brownian motion of the same type as studied in Figs. 1 and 2, and where $x \in [0, L]$. The slope of the fit shown in Fig. 3 is 1.78 ± 0.15 . Using Eq. (31), we find $H = 0.63 \pm 0.08$ consistent with $H = 0.7$. When we compare Fig. 1 and Fig. 3 we see clearly a reduction in the data on long length scales, as the method only probes return distances less than an upper cutoff decided by the bias amplitude.

In conclusion, we have compared the first return method with Fourier analysis, and found that they perform approximately equally well in determining roughness exponents. We have furthermore generalized the first return method to self-affine surfaces with drift, showing how the drift leads to severe corrections even to the lowest order in the measured quantities. This is important as drift is very typical in experimentally encountered situations.

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Figure Captions

- Fig. 1 The first return histogram $N(\Delta)$ averaged over 50 Mandelbrot-van Ness fractional Brownian motions of length 4096, memory constant $M = 1000$ and $H = 0.7$. A finite size correction has been applied to the data, giving the pronounced upward curvature for large Δ . The superimposed curve is a best fit having the form $\log N(\Delta) = 3.10 + (H - 2) \log \Delta + \log (1 + 2.11 \Delta^{-2H})$ with $H = 0.67$.
- Fig. 2 Averaged power spectrum of the same data as in Fig. 1. The straight line represents a best fit, and has a slope of -2.40 ± 0.01 .
- Fig. 3 The first return histogram for the biased fractional Brownian motion $y(x) = r(x) + 10^4 \cdot (x/L)^2$, averaged over the same 50 samples as in Fig. 1. The slope of the best fit is 1.78, which according to Eq. (31) gives $H = 0.63 \pm 0.08$.

Fig. 1

Fig. 2

Fig. 3