A Wilson Renormalization Group Approach to Light-Front Tamm-Dancoff Scalar Field Theory

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Abstract

A program to utilize the Tamm-Dancoff approximation, on the light-front, to solve relativistic quantum field theories, is presented. We present a well defined renormalization program for the Tamm-Dancoff approximation. This renormalization program utilizes a Minkowski space version of Wilson’s renormalization group. We studied light-front $\phi^4$ field theory in 3+1 dimensions, within a two-particle truncation of Fock space. We further simplified our calculations by considering only one marginal operator and one irrelevant operator. The renormalization procedure required no more marginal or relevant operators. We derived an effective, renormalized, Hamiltonian. These techniques may be germane to the effort to find an effective, low energy, light-front Hamiltonian for quantum chromodynamics.  
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1 Introduction

As a part of a Hamiltonian approach to relativistic quantum field theory, we report here on a method which is partly analytical and partly numerical. It is a method for generating an effective Hamiltonian that has all the physics of the theory, below a cut-off energy scale, and within a certain Tamm-Dancoff truncation of Fock space [?, ?]. That is, a given truncation of Fock space will likely describe some classes of states more accurately than other classes. This paper describes a careful analysis of the renormalization problems of a simplified Tamm-Dancoff truncation of the scalar field. This simplification, used in order to bring out the essential features of the method, consists in considering the approximation of one marginal operator and one irrelevant operator. We do not investigate the validity of the Tamm-Dancoff truncation in this paper, concentrating only on reproducing the effects of high energy states within the truncation. We use a renormalization group procedure, used previously [?], which eliminates high energy states, generating an effective Hamiltonian for the low energy states. That is, the high energy sector of the theory is removed in a manner that preserves the physics of the low energy sector. The low energy physics, that is within a given Tamm-Dancoff truncation of Fock space, is preserved as the high energy sector is removed, as we show below. What sets the boundary between the high energy scales and low energy scales are the masses of the bound states one wishes to study. The masses of interest should lie in the low energy scale region.

Elimination of the high energy sector also eliminates divergences. Hence, regularization of the Hamiltonian is achieved. Renormalization is achieved because the real physical effects of the high energy sector upon the low energy sector are represented by the systematic inclusion of additional interactions.

Scalar field theory is used in order to provide as simple a framework as possible for displaying the machinery of the Tamm-Dancoff truncated, light-front field theory with the Minkowski space version of Wilson’s renormalization group [?]. For now we simply wish to develop further a realization of the Wilson conception of renormalization in field theory within the context of the light-front Tamm-Dancoff formalism [?].

Either an explicit or implicit Tamm-Dancoff truncation of Fock space is essential, because the complete vector space of a quantum field theory cannot be put on a computer. This anticipates the eventual need for a numerical solution of a theory. In contradistinction to Perry’s study [?], a Tamm-Dancoff truncation is adopted at the very beginning of the analysis. One result is that the $\beta$-function for the truncated scalar theory is not the same as that of the nontruncated theory. This does not represent a problem with our approach over and beyond those presented by the fact of a Tamm-Dancoff truncation of Fock space.
The parameters will run in a manner that is appropriate for the given truncation of the theory. We do not comment on the implications of this fact for the validity of the Tamm-Dancoff truncation except to say that the Tamm-Dancoff approximation is a variational approach to field theory. The larger the Fock space that is utilized, the more exactly will the variational solutions correspond to the exact solutions for the observables of the theory. These observables, of course, include the physical particle masses, which are invariant masses of the energy-momentum eigenstates, and particle form factors, derived from the particle wavefunctions. Scattering amplitudes for scattering between these particles can also be calculated. Also, as more of the Fock space is included, the parameters will run more analogously to the way the parameters run in the nontruncated theory.

The use of the light-front formalism is desirable, if not essential, because of the restriction to positive longitudinal momenta. This allows all vacuum structure to be forced into the zero-mode structure of a theory. If the vacuum structure can be thus isolated, it may be possible to replace this structure with interactions. This procedure is straightforward only in light-front coordinates. With the light-front formalism, the problem of solving for excited state wavefunctions, the particles, is separated from the problem of solving for the ground state wavefunction, the physical vacuum, within the Tamm-Dancoff approximation. This is simply because the physical vacuum structure has been greatly simplified. The vacuum is 'trivial'. That is, the physical vacuum is also the no-constituent state. This is not the case in equal-time Tamm-Dancoff. With this simplified vacuum, however, must come a complication of the operators of the theory, such as the Hamiltonian. In addition, the complicated operator structure of the formalism may result in a simplified excited state structure, thus providing ultimate justification of the Tamm-Dancoff truncation.

The starting point of this approach is a fixed-point Hamiltonian of a Wilson renormalization group transformation. There may be more than one fixed-point for a given canonical theory, and any number of these fixed-points may or may not be experimentally relevant. We have chosen here to work with a Gaussian, massless fixed-point Hamiltonian, a Hamiltonian consisting only of a kinetic term. Generation of the canonical Hamiltonian is an essential but secondary aspect of the method. The canonical Hamiltonian contains all the relevant and marginal operators consistent with the symmetries of the theory. We assume that the couplings for these operators are those which should run independently with the cut-off. Generation of the effective Hamiltonian is an analytical and numerical problem. Solution of the effective Hamiltonian may then be a non-perturbative, tractable, numerical problem.

Another way is now open for the non-perturbative solution of relativistic quantum field theories. The specific realizations of Wilson renormalization groups must be perturbative about the fixed-point Hamiltonian at present, and this may at times be a limitation. How-
ever, the effective Hamiltonian, in the truncated Fock space, may be open to an exhaustive numerical solution on present-day computers. The non-perturbative solution of quantum chromodynamics may be approachable by this method.

Perry has outlined the Wilson renormalization group approach to light-front field theory [?]. This report adds a Tamm-Dancoff truncation to obtain a simple, testable example of the approach and suggests the plausibility of the whole approach. An understanding of the Wilson renormalization group in the detail presented in Wilson and Kogut [?] or in Wilson [?] is desirable but not essential. Familiarity with Perry's study, which inaugurated this approach [?], is also desirable.

The next section describes the calculation of a Bloch transformation and its effects upon an observable, the two-particle to two-particle scattering amplitude within a two-particle truncation of Fock space. Section 3 discusses these calculations and presents the resulting effective Hamiltonian. Section 4 presents an outline for broader calculations, mentioning some important remaining issues which are a part of the generation of effective Hamiltonians for any field theory within light-front Tamm-Dancoff, and concludes with a summary.

2 Calculations

2.1 A Wilson Renormalization Group Approach

The Wilson renormalization group of Perry [?] is used in this report. The starting point is an unstable, ultraviolet fixed-point Hamiltonian. The basic ingredients of a Wilson renormalization group transformation are:

1. A Bloch transformation is the Minkowski space counterpart of a Kadanoff transformation in Euclidean space [?, ?, ?, ?, ?] which involves elimination of energy scales, progressing from ultraviolet to infrared. That is, the cut-off is lowered. Bloch transformations must typically be realized perturbatively, because presently this is the only technique generally available. This is reasonable for small enough values of the couplings.

2. Rescaling of the remaining energy scales and rescaling of the field variables must be carried out. This must be done so that a fixed-point Hamiltonian, which is an invariant of the complete transformation, may exit.

An explanation of the interrelationship of the canonical approach to field theory and the Wilson fixed-point approach is given in Wilson and Kogut [?].

Although there may be a number of different kinds of topological properties of the allowed space of Hamiltonians which will enable the procedure of cut-off lowering and rescaling to
generate renormalized Hamiltonians, the following is a possible scenario; there are at least two fixed-point Hamiltonians for the given Wilson renormalization group transformation. The infrared one must be stable. All trajectories in the neighborhood of the fixed-point must flow into the fixed-point. The ultraviolet one must be unstable. This means not all of the trajectories in the neighborhood of the fixed-point flow into the fixed-point. This scenario is likely true for the scalar field. The infrared fixed-point Hamiltonian will consist of a kinetic term and a mass term.

Essentially, the cut-off must eliminate eigenstates of $H^*$, the ultraviolet fixed-point Hamiltonian. Since one is eliminating energy levels of $H^*$, in order to follow the trajectory of Hamiltonians, one wants to rescale energy, in general. At the end of the calculation, in deriving the effective Hamiltonian, the rescalings are undone, and the original scale is restored. $H^*$ will be chosen to be a Gaussian, no interactions, massless Hamiltonian. Fig. 1 is a restatement of Fig. 12.7 of Wilson and Kogut [5], presented here for purposes of continuity, which is an expression of the above topological scenario. $H_D$ is a surface of cut-off canonical Hamiltonians, with cut-off $\Lambda_0$. As $\Lambda_0$ is made to go toward infinity, the parameters of the Hamiltonian on $H_D$ are imagined to vary with this cut-off such that the canonical surface will intersect the critical surface $C$ of a fixed-point Hamiltonian, here called $P_{\infty}$, at infinite $\Lambda_0$. The important points are that Hamiltonians, in this space, residing on the critical surface, are driven into $P_{\infty}$ by a large number of renormalization group transformations. All other Hamiltonians in the space are driven into another fixed-point, called $P_0$, by a large number of transformations.

A Bloch transformation lowers the cutoff by a finite amount, and the final cutoff, $\Lambda_f$, is given by

$$\Lambda_f = \Lambda_0 \cdot \frac{\Lambda_1}{\Lambda_0} \cdot \frac{\Lambda'_1}{\Lambda_0} \cdot \frac{\Lambda''_1}{\Lambda_0} \cdots,$$

where each $\Lambda_1$, in each factor, is associated with a given application of a Bloch transformation. That is, after a complete renormalization group transformation, the resultant Hamiltonian has the same cutoff as the original Hamiltonian, which is $\Lambda_0$, because of the rescalings, but from the point of view of the world described by the starting Hamiltonian, each successive application of the renormalization group transformation eliminates lower and lower energy scales. If $\Lambda_0$ is infinite, $\Lambda_f$ will be infinite, and if $\Lambda_0$ is finite, $\Lambda_f$ will be zero after an infinite number of transformations. So the $\Lambda_f$ associated with $P_{\infty}$ is infinite and the $\Lambda_f$ associated with $P_0$ is zero. Hence, the former fixed-point is called ‘ultraviolet’, and the latter is called an ‘infrared’ fixed-point.

Starting with a Hamiltonian on the canonical surface and applying renormalization group transformations, a trajectory will emanate, eventually going into the infrared fixed-point, if $\Lambda_0$ is finite. As one makes $\Lambda_0$ larger and larger, the trajectory will hug the critical surface,
more and more, approaching the ultraviolet fixed-point, before eventually diverging from it, going onward toward $P_0$. The consequence of this is that the Hamiltonians $Q_\alpha$ at the cutoff 1 GeV, for example, approach a limiting Hamiltonian, $Q_\infty$. $Q_\infty$ describes all the physics of the uncut-off field theory, but does so only below the cutoff of 1 GeV. This is the renormalized Hamiltonian at the cutoff $\Lambda_f = 1$ GeV. Simply cutting the canonical Hamiltonian off at 1 GeV is a zeroth-order approximation to the renormalized Hamiltonian. That is $Q_1$. $Q_2$ is a better approximation. $Q_3$ is even better, and so forth.

In our work below, we choose a Hamiltonian that is at a point on a trajectory that is near the ultraviolet fixed-point, $P_\infty$. The Hamiltonian will consist of the ultraviolet fixed-point Hamiltonian plus a perturbing interaction. For the Gaussian, interactionless, massless ultraviolet fixed-point used below, one can usually begin with a perturbation consisting of all the relevant and marginal operators of the canonical Hamiltonian of the theory. One then determines all the additional relevant and marginal operators generated by applications of the renormalization group transformation, within the given Fock space truncation. Irrelevant operators will also be generated. The new perturbation will then consist of all these relevant and marginal operators and usually of only a subset of the irrelevant operators. The terms ‘relevant’, ‘marginal’, and ‘irrelevant’ are further explained below.

2.2 Bloch transformation

To illustrate the techniques of the renormalization group approach to field theory, a Gaussian, massless, ultraviolet fixed-point Hamiltonian will be assumed [?, ?]. The renormalization group transformation of Perry [?] will be applied here. For this transformation, a dimensionless parameter, coupling, multiplying a given operator in the Hamiltonian, is classified as relevant, marginal, or irrelevant according to the following criteria:

For the linearized renormalization group transformation, which, for this case, is simply the rescaling part of the complete transformation, the operator will satisfy the following eigenvalue equation,

$$ L \cdot O = \rho O , $$

where $L$ is the linearized transformation. We can label $O$ with a subscript that displays the number of field operators in $O$ and a superscript that displays the number of powers of transverse momenta. For example (see Appendix for notational definitions),

$$ O_5^0 = \int \frac{dq}{q^4} |q> <q| , $$

where Eqns. (36) and (41) in the appendix enable one to see that the projection operator in Eqn. (2) is made from a product of two field operators with the vacuum projector.
Applying $L$ to any operator $O^m_n$ one finds,

$$L \cdot O^m_n = \left( \frac{\Lambda_1}{\Lambda_0} \right)^{(m-n-2)} O^m_n, \tag{3}$$

where $\Lambda_0$ is an initial cutoff on the invariant mass of allowed states, and $\Lambda_1$ is the cutoff after a Bloch transformation. Relevant couplings multiply operators for which $\rho > 1$. Since $\Lambda_0 > \Lambda_1$, the meaning is now clear. Marginal couplings multiply operators for which $\rho = 1$. Irrelevant couplings correspond to $\rho < 1$. The operators are classified likewise.

When $H^*$ is chosen to consist of only a kinetic term, $L$ is easily constructed. Let $T$ be the full renormalization group transformation. Let $H_l$ be the Hamiltonian resulting from $l$ operations of the renormalization group transformation on the ultraviolet fixed-point plus its perturbation. That is,

$$H_l = H^* + \delta H_l$$
$$H^* + \delta H_{l+1} = T[H^* + \delta H_l] = H^* + L \cdot \delta H_l + O(\delta H_l^2) \tag{4}$$

These equations define the linear operator $L$. The form of the Bloch transformation is given, to second-order, below. Then it follows from this form, and from Eqn. (4) above that the linearized $T$, which is $L$, is just the rescaling operation. The rescalings are, in turn, designed to leave $H^*$ invariant. Eqn. (3) is the resulting scaling relation.

See Eqns. 3.3, 3.6, and 3.7 of Perry [?] for the general scalar Hamiltonian allowed, by power counting, in our space of Hamiltonians. The functions $\phi^{(m,n)}_j$ below are the generalized interactions associated with the $\phi^j$ interaction that connects the $m$-particle sector with the $n$-particle sector. We truncate Fock space to allow only the one and two-particle sectors, where the general Hamiltonian is,

$$H = \int dq \frac{u^{(1,1)}_2(q)}{q^+} |q><q| + \int dq_1 dq_2 \left( \frac{u^{(2,2)}_2(q_1)}{q_1^+} + \frac{u^{(2,2)}_2(q_2)}{q_2^+} \right) |q_1, q_2><q_1, q_2| + \frac{1}{4} \int dq_1 dq_2 dq_3 \frac{u^{(2,2)}_4(q_1, q_2, -q_3, q_1 - q_2 + q_3)}{q_1^+ + q_2^+ - q_3^+} |q_1, q_2><q_3, q_1 + q_2 - q_3| \tag{5}$$

The perturbation of the fixed-point will, at first, be assumed to be a marginal operator in the $\phi^3$ interaction. That is, initially $u_4 = \lambda$, a constant (see Appendix). There are no marginal or relevant operators in the $\phi^6$ interaction, for example. There are relevant and marginal operators in the $\phi^2$ interaction. There are irrelevant operators in all interactions. As a result,

$$H_\infty = h + v \tag{6}$$
at an initial cut-off of $\epsilon_0$. This cut-off corresponds to an invariant mass of $\Lambda_0$ where,

$$\epsilon_0 = \frac{\vec{P}^2 + \Lambda_0^2}{P^+}.$$  

The ultraviolet fixed-point Hamiltonian, $h = H^*$, is

$$h = \int d\vec{q}_1 d\vec{q}_2 \theta\left(\frac{\vec{P}^2 + \Lambda_0^2}{P^+} - \frac{\vec{q}_1^2}{q_1^2} - \frac{\vec{q}_2^2}{q_2^2}\right) |q_1, q_2 > < q_1, q_2|, \quad (7)$$

where $\vec{P} = \vec{q}_1 + \vec{q}_2$ and $P^+ = q_1^+ + q_2^+$. The $\theta$-function displays the cut-off. With this fixed-point, the bosons of the basis set are massless. The perturbation, $v$, is given by,

$$v = \frac{\lambda}{4} \int d\vec{q}_1 d\vec{q}_2 d\vec{q}_3 \left|q_1, q_2 > < q_3, q_1 + q_2 - q_3\right| \frac{\vec{P}^2 + \Lambda_0^2}{P^+} \left(\frac{\vec{q}_1^2}{q_1^2} - \frac{\vec{q}_2^2}{q_2^2}\right) \left(\vec{P}^2 + \Lambda_0^2 \frac{\vec{q}_3^2}{q_3^2}\right) \left(\vec{P}^2 + \Lambda_0^2 \frac{\vec{q}_3^2}{q_3^2}\right) \theta\left(\frac{\vec{P}^2 + \Lambda_0^2}{P^+} - \frac{\vec{q}_1^2}{q_1^2} - \frac{\vec{q}_2^2}{q_2^2}\right) \theta\left(\frac{\vec{P}^2 + \Lambda_0^2}{P^+} - \frac{\vec{q}_3^2}{q_3^2}\right). \quad (8)$$

The Bloch transformation lowers the cut-off by eliminating more eigenstates of $H^*$. The resulting Hamiltonian, $H_{c_i}$, has the same low-lying eigenvalues as those of $H_{\alpha}$. The matrix elements of $H_{c_i}$, to second order in $v$, are [?],

$$< a | H_{c_i} | b > = < a | h + v | b > + \frac{1}{2} \sum_i \left(\frac{< a | v | i > < i | v | b >}{\epsilon_a - \epsilon_i} + \frac{< a | v | i > < i | v | b >}{\epsilon_b - \epsilon_i}\right), \quad (9)$$

where $|a >, |b >$ are eigenstates of $h$ with eigenvalues $\epsilon_a$ and $\epsilon_b$ respectively. These eigenvalues are below the new cut-off, $\epsilon_1$. The states $|i >$ are eigenstates of $h$ to be eliminated, with eigenvalues $\epsilon_i$. $\epsilon_1 \leq \epsilon_i \leq \epsilon_0$ ($\Lambda_1 \leq M_i \leq \Lambda_0$). Also,

$$\epsilon_a = \frac{\vec{q}_1^2 + \vec{q}_2^2}{q_1^2 + q_2^2} = \frac{\vec{P}^2 + M_a^2}{P^+}, \quad \epsilon_i = \frac{\vec{q}_1^2 + \vec{q}_2^2}{q_1^2 + q_2^2} = \frac{\vec{P}^2 + M_i^2}{P^+}, \quad \epsilon_b = \frac{\vec{q}_1^2 + \vec{q}_2^2}{q_1^2 + q_2^2} = \frac{\vec{P}^2 + M_b^2}{P^+}. \quad (10)$$

Now the first second-order term from Eqn. (9) is,

$$\frac{1}{2} \sum_i \left(\frac{< a | v | i > < i | v | b >}{\epsilon_a - \epsilon_i}\right) = < a | v' | b > \quad (11)$$

where $v'$ is $v$ with $u_4$ replaced by $\delta u_4$, and

$$\delta u_4 = -\frac{\lambda^2}{64 \pi^2} \int_0^1 dx \int_{\Lambda_0^2 x (1-x)} \frac{dr^2}{r^2} \sum_{n=0}^\infty \frac{x(1-x)}{z(1-z)} \frac{s^2}{r^2} n. \quad (12)$$
\( \Lambda_1 \) is the invariant mass cut-off associated with \( \epsilon_1 \), and
\[
\frac{\tilde{s}^2}{z(1-z)} < \Lambda_1^2 \leq \frac{\tilde{r}^2}{x(1-x)} \leq \Lambda_0^2 ;
\]
(13)
where,
\[
\epsilon_a = \frac{\tilde{p}^2}{P^+} + \frac{\tilde{s}^2}{P^+ z(1-z)} ,
\]
(14)
and,
\[
\epsilon_i = \frac{\tilde{p}^2}{P^+} + \frac{\tilde{r}^2}{P^+ x(1-x)} ,
\]
(15)
and,
\[
q_{1a} = (z P^+, z \tilde{P} + \tilde{s}) ,
\]
\[
q_{2a} = ((1-z) P^+, (1-z) \tilde{P} - \tilde{s}) .
\]
(16)
\((z, \tilde{s})\) are Jacobi coordinates for \(|a>, (x, \tilde{r})\) are Jacobi coordinates for \(|i>, \) and we now introduce \((y, \tilde{t})\) as Jacobi coordinates for \(|b>\).

\( \lambda \) multiplies a marginal operator contribution to \( u_4. \) \( \delta u_4 \) consists of a marginal contribution, \( n = 0 \) in Eqn. (12), and an infinite number of irrelevant operator contributions, \( n \geq 1 \) in (12). Considering the term with \( \epsilon_i \) in the denominator as well, the other second-order term in Eqn. (9), and ignoring all the irrelevant operator contributions,
\[
\delta u_4 = -\frac{\lambda^2}{32 \pi^2} \int_0^1 \int_{\Lambda^2_x(1-x)}^{\Lambda^2_y(1-x)} \frac{dy^2 dx}{\tilde{s}^2} + O(\tilde{s}^2) + O(\tilde{r}^2)
\]
(17)
Keeping the leading irrelevant operator contributions to \( u_4, n = 1 \) in Eqn. (12), which are generated by a Bloch transformation, at second order,
\[
\delta u_4 = -\frac{\lambda^2}{16 \pi^2} ln(\frac{\Lambda_0}{\Lambda_1}) - \frac{\lambda^2}{64 \pi^2} (\frac{1}{\Lambda_1^2} - \frac{1}{\Lambda_0^2}) \frac{\tilde{s}^2}{z(1-z)} + O(\tilde{s}^4)
\]
\[
-\frac{\lambda^2}{64 \pi^2} (\frac{1}{\Lambda_1^2} - \frac{1}{\Lambda_0^2}) \frac{\tilde{t}^2}{y(1-y)} + O(\tilde{t}^4) + O(\tilde{s}^2 \tilde{t}^2) .
\]
(18)
Let \( u_4, \) in \( u, \) consist only of contributions from the leading irrelevant operator. This contribution is
\[
\frac{\alpha}{\Lambda_0^2} (\frac{\tilde{s}^2}{z(1-z)} + \frac{\tilde{r}^2}{x(1-x)}) ,
\]
where \( \alpha \) is the irrelevant coupling strength, and \((z', \tilde{s}')\)
is made from \( q_1 \) and \( q_2 \) in \( v \), and \( (y', t') \) is made from \( (q_3, q_1 + q_2 - q_3) \) in \( v \). The second order Bloch transformation gives, to the leading irrelevant operator,

\[
\delta u_4 = -\frac{3\alpha^2}{64\pi^2} \left( \frac{1}{1 - \frac{\Lambda_1^2}{\Lambda_0^2}} \right) \frac{1}{\pi^2} \left( \frac{s'^2}{2} \right) \left( \frac{\mu^2}{2} \right) \left( \frac{y'^2}{2} \right) \left( \frac{1 - y'^2}{1 - y'^2} \right) \left( \frac{1 - \mu'^2}{1 - \mu'^2} \right) + \mathcal{O}(g'^2) + \mathcal{O}(w'^2) + \mathcal{O}(s'^4 + t'^4) \right).
\]

These results correspond to the diagrams in Figs. 3a and 3d, respectively, keeping the marginal and leading irrelevant operators, where the incoming and outgoing lines are the states \( |a> \), \( |b> \), and the vertices are the interaction \( v \), and the internal lines are the states \( |i> \).

Fig. 2a displays the vertex associated with the marginal coupling \( \lambda \). Fig. 2b shows the leading irrelevant vertex, where the leading irrelevant operator with strength \( \alpha \) is associated with the vertex. Figs. 3b and 3c must also be included in a second order analysis. The four second-order diagrams, 3a, 3b, 3c, and 3d, and the two first order diagrams, 2a and 2b, illustrate how the Bloch transformation can be summarized as a change of coupling strengths. That is, these parameters run.

The difference equations that represent the leading effects of the Bloch transformation are, using \( 4!g = \lambda \) and \( 4!w = \alpha \),

\[
\begin{align*}
g'_{n+1} &= g_n - \frac{3g_n^2}{2\pi^2} \frac{\Lambda_0}{\Lambda_1} - \frac{3g_n w_n}{2\pi^2} \left( 1 - \frac{\Lambda_1^2}{\Lambda_0^2} \right), \\
&\quad -\frac{3w_n^2}{8\pi^2} \left( 1 - \frac{\Lambda_0^4}{\Lambda_0^4} \right), \\
w'_{n+1} &= w_n - \frac{9w_n^2}{8\pi^2} \left( 1 - \frac{\Lambda_1^2}{\Lambda_0^2} \right) - \frac{3g_n w_n}{2\pi^2} \left( \frac{\Lambda_0}{\Lambda_1} \right) \\
&\quad -\frac{3g_n^2}{8\pi^2} \left( \frac{\Lambda_0^2}{\Lambda_1^2} - 1 \right),
\end{align*}
\]

where the parameter set \( (g'_{n+1}, w'_{n+1}) \) is generated from the parameter set \( (g_n, w_n) \) by a single Bloch transformation. A parameter set \( (g_{n+1}, w_{n+1}) \) results from \( (g_n, w_n) \) by a complete renormalization group transformation. This transformation is given below.

### 2.3 T-matrix

We now calculate an observable. If the above approximation to the full Bloch transformation is reasonable, this observable should be nearly independent of the cut-off, below the new cut-off. We use the two-particle to two-particle scattering amplitude, for interacting but massless scalar bosons, to test the procedure.
The S-matrix is given in terms of the T-matrix by \[ S = \delta_{ij} - 2\pi i \delta(E_i - E_j) < \phi_i | T(E) | \phi_j > \],

where \( \phi_i, \phi_j \) are two-particle eigenstates of \( \hat{h} \) in Eqn. (7). The T-matrix is given by the perturbative series,

\[ T(E) = < \phi_i | H_1 | \phi_j > + < \phi_i | H_1 | \frac{1}{E - \hat{H}_0 + i\epsilon} H_1 | \phi_j > + \ldots \] (22)

Now, assuming \( H_1 \) consists only of the generalized \( \phi^4 \) interaction, that is,

\[ u^{(2,\xi)}_4 = 4!(g + \frac{w^2}{\Lambda_0^2} \frac{1}{(1 - z^j)}) + \frac{w^2}{\Lambda_0^2} \frac{1}{(1 - y^j)} \],

we have the following results,

\[ < \phi_i | H_1 | \phi_j > = 384 \pi^3 (g + \frac{2M^2}{\Lambda_0^2}) \delta(p_{i}^1 + p_{j}^1 - p_{i}^2 - p_{j}^2), \] (24)

\[ < \phi_i | H_1 | \frac{1}{E - \hat{H}_0 + i\epsilon} H_1 | \phi_j > = - \delta(p_{i}^1 + p_{j}^1 - p_{i}^2 - p_{j}^2), \]

\[ 288 \pi \left[ (g + \frac{2M^2}{\Lambda_0^2})^2 \ln \left( \frac{\Lambda_0^2}{M^2} - 1 \right) + 2w(g + \frac{2M^2}{\Lambda_0^2}) \right] \left( \frac{A^2}{2} - M^2 \Lambda^2 \right) + i\pi \left( g + \frac{2M^2}{\Lambda_0^2} \right)^2 \],

where \( E_i = E_j \), \( M \) is the invariant mass of the scattering states, and \( \Lambda \) is the cut-off currently in effect. The pole in the integrand is handled with the aid of the relation,

\[ \frac{1}{x + i\epsilon} = P \frac{1}{x} - i\pi \delta(x). \] (26)

The difference equations which determine the running of the parameters caused by successive Bloch transformations, without intervening rescalings, are given by,

\[ g'_{n+1} = g'_n - 3\frac{g'_n^2}{2\pi^2} \frac{A_n}{A_{n+1}} - 3\frac{g'_n w'_n}{2\pi^2} \left( \frac{A^2_n - A^2_{n+1}}{\Lambda_0^2} \right) \]

\[ - 3\frac{w'_n}{8\pi^2} \left( \frac{A^4_n - A^4_{n+1}}{\Lambda_0^4} \right), \]

\[ w'_{n+1} = w'_n - 9\frac{w'_n^2}{8\pi^2} \left( \frac{A^2_n - A^2_{n+1}}{\Lambda_0^2} \right) - 3\frac{g'_n w'_n}{\pi^2} \frac{A_n}{A_{n+1}} \]

\[ - 3\frac{g'_n^2}{8\pi^2} \left( \frac{A^2_0}{A^2_{n+1} - A^2_n} \right). \] (27)

The results are shown in Figs. 4, 5, and 6. Results are shown, in Fig. 5, for the case of a marginal contribution to \( u_4 \), and only a marginal correction to \( u_4 \) being kept. \( w \) is set to
zero and is not allowed to run. Only $g$ is allowed to run. Also, results are shown in Fig. 6 for the case in which a marginal and the leading irrelevant operator contributions to $u_4$ are kept, and both are allowed to run. It must be understood that the Eqns. (20) do not govern the running parameters in these cases, because there are no intervening rescaling operations. Rather a slightly altered set of difference equations must be used, Eqns. (27), which keep track of the absolute lower cut-off.

Fig. 4 shows the cut-off dependence of an observable when the couplings of the theory are not allowed to run with the cut-off. Plotted is the real part of a two-boson to two-boson scattering amplitude as a function of the invariant mass of the scattering states. The interaction is a $\phi^4$ interaction consisting of a marginal operator and the leading irrelevant operator. The marginal coupling is fixed at 1 and the irrelevant coupling is fixed at 0. Fig. 5 shows how this cut-off dependence is lessened when the marginal coupling is allowed to run, and the irrelevant coupling is left fixed at 0. Fig. 6 shows that the cut-off dependence is lessened even more when both couplings are allowed to run. Deviations from cut-off independence is strongest near the cut-off, as shown in Figs. 5 and 6. Good results are obtained with the retention of just the first marginal correction, for small coupling. Better results are obtained with the retention of the first irrelevant operator. The irrelevant operator’s effects are quite noticeable nearer the cut-off. Retention of all irrelevant operators should eliminate all cut-off dependence.

Since $\frac{x^2}{2(1-x)}$ is the invariant mass squared of a state, the termination of the Bloch transformation after a few orders of the interaction and after a finite number of irrelevant operators indicates that this transformation has been expanded in powers of the couplings and powers of the ratio $\frac{M^2}{N^2}$.

For the case of a relevant and a marginal operator, in a two-particle truncation, the following term must be added to the perturbing interaction $v$,

$$v_m = \int d\tilde{q}_1^i d\tilde{q}_2^i \left( \frac{m^2}{q_1^i} + \frac{m^2}{q_2^i} \right) |q_1^i, q_2^i \rangle \langle q_1^i, q_2^i| \ .$$

Hence, the sum of Figs. 3a, 7a, 7b, and 7c is given by,

$$\frac{1}{2} \sum_i ( <a|v|i><i|v|b> (\frac{1}{\epsilon_a - \epsilon_i} + \frac{1}{\epsilon_b - \epsilon_i}) ) \ ,$$

where Fig. 2c is the relevant operator vertex.

Now Fig. 7a equals,

$$2 \int \frac{d\tilde{q}_1^i d\tilde{q}_2^i}{\epsilon_a or \epsilon_b} \frac{<a|q_1^i, q_2^i><q_1^i, q_2^i|b>}{\left( \frac{m^2}{q_1^i} + \frac{m^2}{q_2^i} \right)^2} ,$$

gives zero when it acts upon $|a>$ or $|b>$. Likewise Figs. 7b and 7c give zero.
So the relevant coupling, \( \mu \), where \( m^2 = \mu \Lambda^2 \), runs only because of the rescaling, and not because of the Bloch transformation, to second order in the transformation.

Inclusion of the one-particle sector will not change any of these results. Inclusion of the one- and three-particle sectors will affect, slightly, the way the relevant coupling runs, because of the diagram in Fig. 8.

2.4 The Difference Equations

In accordance with Perry [7], the rescaling step of a renormalization group transformation is done according to the following rules: (a) Multiply each factor of transverse momentum that appears, including those in the measure, by \( \frac{\Lambda_0}{\Lambda_1} \). (b) For every field that appears in an operator, such as four fields appearing in the operator in Eqn. (8), multiply the operator by a factor of \( \frac{\Lambda_0}{\Lambda_1} \). (c) Finally, multiply the entire Hamiltonian by a factor of \( \left( \frac{\Lambda_0}{\Lambda_1} \right)^2 \).

The difference equations that are obtained with the entire renormalization group transformation, in the two-particle truncation, are the following,

\[
\begin{align*}
\mu_{n+1} & = \frac{\Lambda_0^2}{\Lambda_1^2} \mu_n, \\
g_{n+1} & = g_n - \frac{3g_n^2}{2\pi^2} \ln \frac{\Lambda_0}{\Lambda_1} - \frac{3g_n w_n}{2\pi^2} \left( 1 - \frac{\Lambda_1^2}{\Lambda_0^2} \right) \\
& \quad - \frac{3w_n^2}{8\pi^2} \left( 1 - \frac{\Lambda_1^4}{\Lambda_0^4} \right), \\
w_{n+1} & = \frac{\Lambda_0^2}{\Lambda_1^2} w_n - \frac{9w_n^2}{8\pi^2} \frac{\Lambda_0^2}{\Lambda_1^2} \left( 1 - \frac{\Lambda_1^2}{\Lambda_0^2} \right) - \frac{3g_n w_n}{\pi^2} \frac{\Lambda_1^2}{\Lambda_0^2} \ln \frac{\Lambda_0}{\Lambda_1} \\
& \quad - \frac{3g_n^2}{8\pi^2} \left( 1 - \frac{\Lambda_1^2}{\Lambda_0^2} \right). \\
\end{align*}
\]

3 The Effective Hamiltonian

In order to generate the effective, renormalized, Hamiltonian, the starting cut-off must be raised toward infinity while the couplings are sent toward their values at the fixed-point. The number of iterations of the renormalization group transformation it takes to get to the final lower cut-off is thereby increased toward infinity. With this limiting process, the running parameters must go to a non-zero limit, or the theory is trivial.

The difference equations (30), for the two-particle truncated scalar field, suggest that the parameters, except the mass, do go to zero in this limit. So the effective Hamiltonian for this model of the scalar field, using this fixed-point, is trivial. However, as Fig. 9 implies, if the couplings are small, the couplings go to zero weakly. For example, for a starting \( g \) of .1, a decay of its value by 50% occurs only after a cut-off decrease of a factor of 10^30,
or after 100 iterations of a reduction in the cut-off by a factor of 2. Figs. 9 and 10 also show that the irrelevant coupling does track with the marginal coupling after some initial transience. The irrelevant coupling depends upon the cut-off only through a functional dependence upon the marginal coupling. This tracking sets in long before the marginal coupling decays appreciably. This means that in the renormalization process for this model of the scalar field, the cut-off cannot really be taken to infinity, but it can be made very large. A calculation of a truncated model of QED is also expected to be strictly trivial. Assuming that with effectively no truncation of the Fock space our methods give the correct $\beta$-function for a theory, then our methods suggest that non-asymptotically free theories will be strictly trivial. In fact, our methods are expected to give strictly non-trivial results only for asymptotically free theories.

If we expand $w$ in a power series in $g$, to second order, in accordance with the coupling coherence of Perry and Wilson [?], and if the difference equations (30) are used to solve for the coefficients, we obtain,

$$w = -\frac{3g^2}{8\pi^2}$$

Figs. 9 and 10 show that, after some initial transience caused by the initial values of $w$ and $g$, $w$ goes to this function of $g$, for small values of $g$, after a large number of successive renormalization group transformations. This is also in accordance with Wilson’s observation of his general solution of the difference equations with relevant, marginal, and irrelevant couplings [?]. That is, after a large number of renormalization group transformations, the final irrelevant couplings will become strongly dependent upon the values of the final couplings of the independent relevant and marginal operators and only weakly dependent on the initial values of the irrelevant couplings.

The interaction term in the Hamiltonian, after a large number of iterations of the renormalization group transformation, then becomes, keeping only up to the leading irrelevant operator,

$$v = 3!g \int d\tilde{q}_1^\dagger d\tilde{q}_2^\dagger d\tilde{q}_3^\dagger \left[ q_1^2 q_2^2 q_3^2 + q_1 q_2 q_3 + q_1 + q_2 + q_3 \right].$$

$$\left( \frac{\tilde{P}^2 + \Lambda_0^2}{P^+} - \frac{\tilde{q}_1^2}{q_1^2} - \frac{\tilde{q}_2^2}{q_2^2} \right) \left( \frac{\tilde{P}^2 + \Lambda_0^2}{P^+} - \frac{\tilde{q}_3^2}{q_3^2} - \frac{\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3}{q_1^2 + q_2^2 + q_3^2} \right).$$

$$(1 - \frac{3g}{8\pi^2} \left( \frac{z'(1-z')}{{\Lambda}_0^2} + \frac{y'(1-y')}{{\Lambda}_0^2} \right)).$$

In the final Hamiltonian, all the energy-momentum variables must be expressed in the same units. To obtain the effective Hamiltonian, all the transverse momenta, which are the new variables, must be multiplied by a factor of $\frac{\Lambda_0}{\Lambda_f}$ to get the old variables. $\Lambda_f$ is the final cut-off.
The entire Hamiltonian must also be multiplied by $\frac{\Lambda^2}{\Lambda_0^2}$ to undo the rescaling of the energy eigenvalues of the initial Hamiltonian. Also, the rescaling of the fields must be undone, so that each field variable which appears must be multiplied by a factor of $\frac{\Lambda^2}{\Lambda_0^2}$. The net result is that to obtain the effective Hamiltonian replace every occurrence of $\frac{\Lambda^2}{\Lambda_0^2}$ in the Hamiltonian with a $\frac{\Lambda^2}{\Lambda_0^2}$.

Finally, the effective Hamiltonian is the following,

$$H_{eff} = \int d\tilde{q} (\tilde{q}^2 + \frac{\mu \Lambda^2}{\tilde{q}^2}) |q| < q|$$

$$+ \int d\tilde{q}_1 d\tilde{q}_2 \theta\left(\frac{\tilde{P}^2 + \Lambda^2}{\tilde{q}_1^2} - \frac{\tilde{q}_1^2}{\tilde{q}_2^2}\right) \left(\frac{\tilde{q}_2^2 + \mu \Lambda^2}{\tilde{q}_2^2} + \frac{\tilde{q}_2^2}{\tilde{q}_1^2}\right) |q_1, q_2| < q_1, q_2|$$

$$+ 3! \int d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3 |q_1, q_2, q_3| < q_3, q_1 + q_2 - q_3|$$

$$\theta\left(\frac{\tilde{P}^2 + \Lambda^2}{\tilde{q}_1^2} - \frac{\tilde{q}_1^2}{\tilde{q}_2^2}\right) \left(\frac{\tilde{P}^2 + \Lambda^2}{\tilde{q}_2^2} - \frac{\tilde{q}_2^2}{\tilde{q}_3^2}\right) - (\tilde{q}_1 + \tilde{q}_2 - \tilde{q}_3)^2$$

$$= \left(1 - \frac{3g}{8\pi^2} \frac{\tilde{s}^2}{z'(1-z')\Lambda^2_j} + \frac{\tilde{t}^2}{y'(1-y')\Lambda^2_j}\right).$$

The interaction has an extra term due to the leading irrelevant operator. In fact, there are an infinite number of these extra interactions due to an infinite number of irrelevant operators, of decreasing importance. More of these interactions can be systematically included. These extra interactions represent the effects of the high energy sector of the theory upon the low energy sector, within this truncation of Fock space.

A straightforward working out of the second-order Bloch transformations, within the two-particle truncation, for which the vertices, interactions, consist of the irrelevant operators $(\frac{\tilde{s}^2}{z(1-z)})^n$ and $(\frac{\tilde{t}^2}{y(1-y)})^m$, and products of these, where $n, m = 1, 2, ...$, do not generate marginal operators in addition to the one that we have assumed. That is, within the two-particle truncation, second-order Bloch transformations with vertices that are functions of $\frac{\tilde{s}^2}{z(1-z)}$ and $\frac{\tilde{t}^2}{y(1-y)}$ generate interactions that are, in turn, functions of these quantities. This symmetry guarantees that no new marginal operators are generated. The bare Hamiltonian has the structure we have assumed for our perturbation of the fixed-point. We do not expect this symmetry to generalize beyond the two-particle Tamm-Dancoff truncation. This symmetry does make the theory with a two-particle Tamm-Dancoff truncation much easier to study than the full theory.
4 Conclusions

To broaden any calculation of this type there are some obvious and not-so-obvious options. The order in the interaction and the number of irrelevant operators kept, both in the renormalization group transformation, could be increased. The Fock space could be increased. It may turn out that in doing this, more relevant and marginal operators will be generated. One may then implement the broader program of coupling coherence of Perry and Wilson [?, ?, ?]. This program re-institutes lost symmetries, and in the process, limits the number of independent relevant and marginal operators. Implementation of the solution scheme of Wilson [?] may need to be carried out. This method of solution of the difference equations gives all the marginal, relevant, and irrelevant couplings in terms of the independent relevant and marginal couplings, given at the low cut-off. Having obtained the effective Hamiltonian, one then may carry out a non-perturbative, numerical, or possibly analytical, solution of the resulting integral equations for the eigenvalues and wavefunctions or scattering amplitudes.

We have carried out the procedure of Perry [?], but in a Tamm-Dancoff truncation of Fock space. The couplings run differently because of the truncation. We do not get the same $\beta$-function as that of the nontruncated scalar field, for example. The Tamm-Dancoff truncation does introduce errors, as we expect. However, the couplings run in the manner appropriate for the given truncation of the Fock space.

Although the effective Hamiltonian is trivial, which agrees with other non-perturbative studies [?], for small coupling, a non-trivial effective Hamiltonian can still be found. This non-trivial effective Hamiltonian results from making the starting cut-off large but not infinite. One then declares that all higher energy scales are unimportant to the problem. This suggests that, in this manner, our procedure could be applied to other non-asymptotically free theories with small couplings, such as QED.

We have shown how a non-perturbative study of QCD may be approached. Since QCD is asymptotically free, triviality will not be an issue. After obtaining the truncated canonical Hamiltonian, one uses the renormalization group transformation to lower the cut-off until the couplings are borderline perturbative. One then implements the coupling coherence program and solves the renormalization group difference equations. This gives the Hamiltonian with added effective interactions. The rescalings are undone, and the effective, low energy Hamiltonian, in a Tamm-Dancoff truncation, is obtained.

The Tamm-Dancoff truncation, with a good renormalization program, is a variational approach to the field theoretic problem. In non-relativistic quantum mechanics the variational approach is a powerful, popular approach. All variational approaches come with the caveat that certain states are described better, by a given truncation, than other states. All states
are described equally well with no truncation. A Hilbert space or Fock space cannot be put on a computer, however. Truncation is necessary. In this regard, ways of generating the effective, non-local interactions that compensate for the fact of the truncation have not been found. Ways of finding these interactions would indeed be desirable. Those states that are described well are well described, in part, because of the interactions induced by removing high energies. For these states, the missing interactions needed to compensate for the truncation are less important. The reverse is true for those states that are not well described by a given Tamm-Dancoff truncation and renormalization. A given truncation may be horrible for all eigenstates of the Hamiltonian. One's knowledge must be used, either derived from experiment or otherwise, to choose truncations that are going to be useful. The usefulness of one's guesses for a truncation can be tested by looking for stability of the results against further increases of the Fock space.

The methods presented here promise to assist in a non-perturbative solution of asymptotically free theories in their strong coupling regimes. The solution of non-asymptotically free theories in their strong coupling regimes remains an open question, from the point of view of this approach.

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References


Appendix

We want to generate the Tamm-Dancoff truncated free scalar field Hamiltonian. Starting with,

$$H_0 = \int dx^- d^2 x \frac{1}{2} \phi(x)(-\partial^2 + m^2) \phi(x), \quad (35)$$

which is the free Hamiltonian in the Schrödinger representation (it contains no light-front “time” dependence, $x^+$) where,

$$x^+ = x_0 + x_3,$$
$$x^- = x_0 - x_3,$$
$$\phi(x) = \phi(x^-, x_1, x_2),$$

$$= \int \frac{dq^+ d^2 q}{16\pi^3 q^+} \theta(q^+) \left[ a(q) e^{-iq \cdot x} + a^\dagger(q) e^{iq \cdot x} \right] , \quad (36)$$

and $(x_0, x_1, x_2, x_3)$ are normal spacetime coordinates.

$$q^+ = q_0 + q_3,$$
$$\vec{q} = (q_1, q_2),$$
$$a(q) = a(q^+, \vec{q}). \quad (37)$$

$q^+$ is the ‘longitudinal’ momentum. $\vec{q}$ is the ‘transverse’ momentum. Here the operators $a(q)$, $a^\dagger(q)$ satisfy,

$[a(q), a^\dagger(q')] = 16\pi^3 q^+ \delta(q - q'), \quad (38)$

and,

$$d\vec{q} = \frac{dq^+ d^2 q}{16\pi^3 q^+}. \quad (39)$$

This implies,

$$H_0 = \int d\vec{q} \left( \frac{\vec{q}^2 + m^2}{q^+} \right) a^\dagger(q) a(q). \quad (40)$$

Now, a multi-particle ket is created with the application of creation operators, $a^\dagger(q)$, where,

$$|q_1, q_2, ... > = a^\dagger(q_1) a^\dagger(q_2) ... |0 >. \quad (41)$$

The rules for the overlap of these kets can be inferred from the following;

$$< k | q > = 16\pi^3 q^+ \delta(k^+ - q^+) \delta(k - \vec{q}) = 16\pi^3 q^+ \delta(k - q), \quad (42)$$

19
\begin{equation}
<k_1, k_2|q_1, q_2> = \left(16\pi^3\right)^2 q_1^+ q_2^+ \left(\delta(k_1 - q_1)\delta(k_2 - q_2) + \delta(k_1 - q_2)\delta(k_2 - q_1)\right) \tag{43}
\end{equation}

\begin{equation}
<k_1, k_2, ...|q_1, q_2, ...> = \text{etc.}
\end{equation}

The resolution of the identity is,

\begin{equation}
1 = |0><0| + \int d\tilde{q}|q><q| + \frac{1}{2!} \int d\tilde{q}d\tilde{k}|q, k><q, k| + ... \tag{44}
\end{equation}

and this implies,

\begin{equation}
H_0 = \int d\tilde{q} \left(\frac{\tilde{q}^2 + m^2}{q^+}\right) |q><q| + \int d\tilde{q}d\tilde{k} \left(\frac{\tilde{q}^2 + m^2}{q^+} + \frac{\tilde{k}^2 + m^2}{k^+}\right) |q, k><q, k| + ...
\end{equation} \tag{45}

If

\begin{equation}
H_1 = \frac{\lambda}{4!} \int d\tilde{x} d^2 x :\phi^4(x): \tag{46}
\end{equation}

then in the one-two particle truncation,

\begin{equation}
H_1 = \frac{\lambda}{4} \int d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3 |q_1, q_2><q_3, q_1 + q_2 - q_3|_{q_1^+ q_2^+ q_3^+} \tag{47}
\end{equation}
Figure Captions

Figure 1: Renormalization group trajectories emanating from a canonical surface $H_D$ of cutoff $\phi^4$ Hamiltonians. The canonical surface intersects the critical surface $C$ at $\Lambda_0 = \infty$. $Q_\infty$ is the renormalized Hamiltonian, at the cut-off $\Lambda_f = 1$.

Figure 2: Diagrams describing the boson-boson interactions of the basis set. (a) The marginal part of the $\phi^4$ interaction. (b) Leading irrelevant part of the $\phi^4$ interaction. (c) Relevant part of the $\phi^2$ interaction.

Figure 3: Diagrams contributing to the second-order Bloch transformation.

Figure 4: Shown is the real part of the transition amplitude, for two (massless)-boson to two-boson scattering, versus the invariant mass of the scattering states. The starting cutoff is 100 units, and it is lowered by factors of $\sqrt{2}$. The couplings are held fixed. The observable is seen to change with the cutoff.

Figure 5: Shown is the same observable as in Fig. 4, except that the marginal coupling, for the $\phi^4$ interaction, is allowed to run according to the second-order Bloch transformations. The cutoff dependence is dramatically reduced.

Figure 6: Shown is the same observable as in Figs. 4 and 5. The marginal and leading irrelevant couplings of the $\phi^4$ interaction are both allowed to run according to the second-order Bloch transformation. The cutoff dependence is decreased even further.

Figure 7: Second-order diagrams with a combination of relevant vertices of the $\phi^3$ interaction and marginal vertices of the $\phi^4$ interaction.

Figure 8: A second-order diagram involving the one and three-particle sectors of Fock space.
Figure 9: The irrelevant coupling, $w$, is plotted versus the marginal coupling, $g$ (solid curve). The values of each are generated by the full renormalization group difference equations, where the cutoff is lowered by a factor of two at each iteration of the transformation. The starting values, $(g_0, w_0)$, are $(.1, 0.)$. Approximately 100 iterations are shown. The dashed curve shows the function $w = -3g^2/(8\pi^2)$. After a large enough number of iterations of the renormalization group transformation, $w$ approaches this function of $g$.

Figure 10: The same quantities are plotted, as in Fig. 9, except the starting values, $(g_0, w_0)$, are $(.1, .5)$, for the solid curve. The dashed curve is the same as in Fig. 9.