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Quantum Hamiltonian Reduction and Non-critical W -Strings*

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Abstract

We study the quantum hamiltonian reduction of affine Lie (super)algebra by using the Wakimoto realization of the algebra and the homotopy transformation for the BRST operator. We apply this method to an affine Lie superalgebra $sl(n+1|n)^{(1)}$ and obtain the free field realization of the non-critical W_n strings.

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1 Introduction

The hamiltonian reduction method provides a systematic description of the connection between an affine Lie algebra and the integrable system of non-linear differential equations (the generalized KdV hierarchy) [1]. The second Poisson bracket structure associated with the hierarchy is nothing but the classical W -algebra, the Virasoro algebra extended by higher spin fields[2]. The W -algebra characterizes the chiral algebra of the rational conformal field theory and plays an important role in the classification of two-dimensional critical phenomena. Recently, string theories with W -algebra symmetry (non-critical W strings) have been attracted much attentions to construct a non-critical string theory defined beyond the $c = 1$ barrier [3]. Topological properties of the non-critical W -string are particularly important as we learned the case of $c \leq 1$ non-critical string theory [4]. In order to study this property, it is convenient to reformulate the non-critical W -string in view of topological conformal field theory. In fact, it has been argued in ref. [5] that the non-critical W_3 string theory has the twisted $N = 2$ super W_3 -symmetry. In the case of the super W -algebra, affine Lie superalgebras play an essential role. In this article we study the quantum hamiltonian reduction of affine Lie (super) algebras and its application to the non-critical W -string theory.

We begin with reviewing the hamiltonian reduction of an affine Lie algebra $\hat{\mathfrak{g}}$ of a simple Lie algebra $\mathfrak{g}$ with rank r . Let Δ be the set of roots $\mathfrak{g}$, Δ_+ the set of positive roots of $\mathfrak{g}$ and $\alpha_1, \dots, \alpha_r$ the simple roots of $\mathfrak{g}$. The affine Lie algebra $\hat{\mathfrak{g}}$ at level k is generated by the currents $J_\alpha(z)$ ($\alpha \in \Delta$) and $H^i(z)$ ($i = 1, \dots, r$) satisfying the operator product expansions (OPEs):

$$\begin{aligned} J_\alpha(z)J_\beta(w) &= \begin{cases} \frac{N_{\alpha,\beta}J_{\alpha+\beta}(w)}{z-w} + \dots, & \text{for } \alpha + \beta \neq 0 \\ \frac{\frac{2k}{\alpha^2}}{(z-w)^2} + \frac{\frac{2\alpha \cdot H(w)}{\alpha^2}}{z-w} + \dots, & \text{for } \alpha + \beta = 0 \end{cases} \\ H^i(z)J_\alpha(w) &= \frac{\alpha^i J_\alpha(w)}{z-w} + \dots, \\ H^i(z)H^j(w) &= \frac{k\delta_{i,j}}{(z-w)^2} + \dots. \end{aligned} \quad (1)$$

The structure constant $N_{\alpha,\beta}$ is zero if $\alpha + \beta \notin \Delta$. $\hat{\mathfrak{g}}$ admits a triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_-$, where the algebra $\hat{\mathfrak{n}}_+$ ($\hat{\mathfrak{n}}_-$) is generated by the currents which correspond to the positive (negative) roots of $\mathfrak{g}$, $\hat{\mathfrak{h}}$ by the Cartan currents $H^i(z)$. The OPE relations

(1) are induced by the infinitesimal gauge transformation: $\delta_\Lambda J(z) = [\Lambda(z), J(z)] + k\partial\Lambda(z)$ for the currents $J(z) \in \mathfrak{g}$ and the gauge parameter $\Lambda(z) \in \mathfrak{g}$. Classically these OPE relations may be regarded as the Poisson bracket structure.

Let us consider the constraints for the currents associated with the principal $sl(2)$ embedding into $\mathfrak{g}$:

$$\chi(J_\alpha(z)) = \begin{cases} 1 & \text{for } \alpha = \alpha_i, \quad i = 1, \dots, r \\ 0 & \text{for } \alpha \in \Delta_+: \text{ non-simple roots.} \end{cases} \quad (2)$$

Denote $\mathcal{M}$ the phase space with the above constraint. There still remains the residual gauge symmetry which leaves $\mathcal{M}$ invariant. $\hat{\mathfrak{n}}_-$ is the algebra which corresponds to this symmetry. We may consider the reduced phase space $\mathcal{R} = \mathcal{M}/\hat{N}_-$, where $\hat{N}_-$ is the gauge group of $\hat{\mathfrak{n}}_-$. $\mathcal{R}$ carries also a Poisson bracket structure. If we choose the special gauge (so-called the Drinfeld-Sokolov gauge), the Poisson bracket structure is the Gel'fand-Dikii algebra, which is nothing but the classical W -algebra.

To quantize this theory, we shall use the BRST formalism. Namely we introduce fermionic ghost fields $(\tilde{b}_\alpha(z), \tilde{c}_\alpha(z))$ ($\alpha \in \Delta_+$) with OPEs $\tilde{b}_\alpha(z)\tilde{c}_{\alpha'}(w) = \delta_{\alpha,\alpha'}/(z-w) + \dots$, associated with the subalgebra $\hat{\mathfrak{n}}_-$. The basic ingredient of this formalism is the BRST charge: $Q_{BRST} = \oint \frac{dz}{2\pi i} J_{BRST}(z)$, where $J_{BRST}(z)$ is the BRST current

$$J_{BRST}(z) = \sum_{\alpha \in \Delta_+} \tilde{c}_\alpha(z)(J_\alpha(z) - \chi(J_\alpha(z))) - \frac{1}{2} \sum_{\alpha, \beta \in \Delta_+} N_{\alpha, \beta}(\tilde{c}_\alpha(\tilde{c}_\beta \tilde{b}_{\alpha+\beta}))(z). \quad (3)$$

The normal ordered product $(AB)(z)$ for operators $A(z)$ and $B(z)$ is defined by $(AB)(z) = \oint_z \frac{dw}{2\pi i} \frac{A(w)B(z)}{w-z}$. The BRST operator Q_{BRST} satisfies the nilpotency condition $Q_{BRST}^2 = 0$, due to the OPE $J_{BRST}(z)J_{BRST}(w) = \text{regular}$. The physical Hilbert space may be characterized by investigating the Q_{BRST} -cohomology.

There are at least two approaches to study this BRST-cohomology. One approach[6] is based on the Wakimoto realization of an affine Lie algebra[7], in which the currents are represented by free bosons with background charges and bosonic ghosts. Using this free field realization, we may study the cohomology on the Fock spaces. The other approach is to consider the cohomology on the space of the universal enveloping algebra $U(\hat{\mathfrak{g}})$ and the Clifford algebra generated by the fermionic ghost fields [8]. In this approach we decompose

the BRST current into two part: $J_{BRST}(z) = J_{BRST}^0(z) + J_{BRST}^1(z)$, where

$$\begin{aligned} J_{BRST}^0(z) &= \sum_{\alpha \in \Delta_+} \tilde{c}_\alpha(z)J_\alpha(z) - \frac{1}{2} \sum_{\alpha, \beta \in \Delta_+} N_{\alpha, \beta}(\tilde{c}_\alpha(\tilde{c}_\beta \tilde{b}_{\alpha+\beta}))(z), \\ J_{BRST}^1(z) &= - \sum_{\alpha \in \Delta_+} \tilde{c}_\alpha(z)\chi(J_\alpha(z)). \end{aligned} \quad (4)$$

Define the BRST charge Q^i by the contour integration of $J_{BRST}^i(z)$. Since the BRST charges obey the relations $(Q^0)^2 = (Q^1)^2 = \{Q^1, Q^0\} = 0$, we may consider the double complex generated by Q^0 and Q^1 . The operator Q^0 is the canonical coboundary operator of the nilpotent subalgebra $\hat{\mathfrak{n}}_+$. The operator Q^1 defines a gradation in the BRST complex associated with the principal embedding $sl(2)$ into $\hat{\mathfrak{g}}$. The latter approach is independent of the Wakimoto realization of an affine Lie algebra and very useful method to obtain the free field realization of the quantum W -currents in an explicit form. In the former approach, the W -algebra structure is not obvious but the cohomology structure is rather simple compare to the latter. We call the first approach as Bershadsky-Ooguri's approach and the latter as Feigin-Frenkel's approach.

In previous papers [10, 11], we have discussed an application of Feigin-Frenkel's approach to the non-critical W -strings and have studied their $N = 2$ super- W algebra structure. In the present paper, we will discuss the quantum hamiltonian reduction of affine Lie superalgebras $sl(n+1|n)^{(1)}$, which is a hidden symmetry of the non-critical W_n string theory, based on Bershadsky-Ooguri's approach.

This paper is organized as follows: In section 2, we will study the quantum hamiltonian reduction of affine Lie algebra by using the free field realization. We will show that the homotopy transformation simplifies the BRST-cohomology. In section 3, we will apply the homotopy transformation method to affine Lie superalgebras $sl(n+1|n)$ and discuss its relation to the non-critical W_n strings.

2 Free field realization of affine Lie algebra and the quantum hamiltonian reduction

In this section we apply the Wakimoto realization of affine Lie algebras to the quantum hamiltonian reduction. The currents of an affine Lie algebra $\hat{\mathfrak{g}}$ may be realized by using

free bosons $\varphi(z) = (\varphi_1(z), \dots, \varphi_r(z))$ and the bosonic ghosts $(\beta_\alpha(z), \gamma_\alpha(z))$ ($\alpha \in \Delta_+$) satisfying the OPEs: $\varphi_i(z)\varphi_j(w) = -\delta_{ij}\log(z-w) + \dots$, $\beta_\alpha(z)\gamma_{\alpha'}(w) = \frac{\delta_{\alpha\alpha'}}{z-w} + \dots$. In this realization the currents for a positive root $\alpha \in \Delta_+$ takes the form

$$J_\alpha(z) = \beta_\alpha + \sum_{\alpha'_1, \dots, \alpha'_n, \alpha + \alpha'_1 + \dots + \alpha'_n \in \Delta_+} C_{\alpha'_1, \dots, \alpha'_n}^\alpha \gamma_{\alpha'_1} \cdots \gamma_{\alpha'_n} \beta_{\alpha + \alpha'_1 + \dots + \alpha'_n}. \quad (5)$$

The coefficients $C_{\alpha'_1, \dots, \alpha'_n}^\alpha$ depend on the parametrization of the $\hat{N}_+$, the Lie group of the nilpotent subalgebra $\hat{n}_+$ of $\hat{\mathfrak{g}}$. We then apply the realization (5) into the BRST current (3) and study Q_{BRST} -cohomology on the direct product of Fock spaces $F = (\otimes_{\alpha \in \Delta_+} F_{\beta_\alpha, \gamma_\alpha}) \otimes (\otimes_{\alpha \in \Delta_+} F_{\tilde{b}_\alpha, \tilde{c}_\alpha}) \otimes F_\varphi$, where $F_{\beta, \gamma}$ denotes a Fock module spanned by (β, γ) fields etc. The cohomology for the BRST charge Q_{BRST} is cumbersome to manipulate in general due to the coefficients $C_{\alpha'_1, \dots, \alpha'_n}^\alpha$. We wish to simplify the BRST current J_{BRST} by a homotopy transformation R :

$$J_{BRST}(z) = e^R J_0(z) e^{-R}, \quad (6)$$

where

$$J_0(z) = \sum_{\alpha \in \Delta_+} \tilde{c}_\alpha(z) \{ \beta_\alpha(z) - \chi(J_\alpha(z)) \}. \quad (7)$$

Here we define a homotopy transformation $(A)^R(z)$ of an operator $A(z)$ by

$$e^R A(z) e^{-R} = \sum_{i=0}^{\infty} \frac{1}{i!} (A)_i^R(z), \quad (8)$$

where $(A)_i^R(z) = (\mathcal{R} (A)_{i-1}^R)_{-1}(z)$, $(A)_0^R(z) = A(z)$ and the generator R is

$$R = \int \frac{dz}{2\pi i} \mathcal{R}(z), \quad (9)$$

for a operator $\mathcal{R}(z)$ with conformal weight one. $(AB)_{-1}(z)$ means the simple pole term in the OPE between two operators $A(z)$ and $B(w)$.

We now give some examples for the homotopy transformation. A simplest and non-trivial example is $\mathfrak{g} = sl(3)$. Let us define currents for positive roots:

$$\begin{aligned} J_{\alpha_1}(z) &= \beta_{\alpha_1} + \gamma_{\alpha_2} \beta_{\alpha_1 + \alpha_2}, \\ J_{\alpha_2}(z) &= \beta_{\alpha_2}, \\ J_{\alpha_1 + \alpha_2}(z) &= \beta_{\alpha_1 + \alpha_2}. \end{aligned} \quad (10)$$

The BRST current for the constraints $\chi(J_{\alpha_1}) = 1$, $\chi(J_{\alpha_2}) = 1$ and $\chi(J_{\alpha_1 + \alpha_2}) = 0$ is

$$J_{BRST}(z) = \tilde{c}_{\alpha_1}(\beta_{\alpha_1} + \gamma_{\alpha_2} \beta_{\alpha_1 + \alpha_2} - 1) + \tilde{c}_{\alpha_2}(\beta_{\alpha_2} - 1) + \tilde{c}_{\alpha_1 + \alpha_2} \beta_{\alpha_1 + \alpha_2} + \tilde{c}_{\alpha_1} \tilde{c}_{\alpha_2} \tilde{b}_{\alpha_1 + \alpha_2} \quad (11)$$

Then the homotopy operator is found to be

$$\mathcal{R}(z) = -\tilde{c}_{\alpha_1} \gamma_{\alpha_2} \tilde{b}_{\alpha_1 + \alpha_2}. \quad (12)$$

The Wakimoto realization is not unique for a given $\hat{\mathfrak{g}}$. In fact, for $sl(3)$, there exists another type of the Wakimoto realization:

$$\begin{aligned} J_{\alpha_1}(z) &= \beta_{\alpha_1} + \frac{1}{2} \gamma_{\alpha_2} \beta_{\alpha_1 + \alpha_2}, \\ J_{\alpha_2}(z) &= \beta_{\alpha_2} - \frac{1}{2} \gamma_{\alpha_1} \beta_{\alpha_1 + \alpha_2}, \\ J_{\alpha_1 + \alpha_2}(z) &= \beta_{\alpha_1 + \alpha_2}. \end{aligned} \quad (13)$$

Then the homotopy operator is shown to be

$$R(z) = -\frac{1}{2} \tilde{c}_{\alpha_1} \gamma_{\alpha_2} \tilde{b}_{\alpha_1 + \alpha_2} - \frac{1}{2} \gamma_{\alpha_1} \tilde{c}_{\alpha_2} \tilde{b}_{\alpha_1 + \alpha_2}. \quad (14)$$

Although the homotopy operator depends on the explicit realization of currents, the transformed BRST currents $J_0(z)$ is independent of the choice of Wakimoto representations.

Next we study $sl(n+1)$ cases. Let us consider the Wakimoto realization of the current for a positive root $\alpha_i + \dots + \alpha_j$ ($i \leq j$):

$$J_{\alpha_i + \dots + \alpha_j}(z) = \beta_{\alpha_i + \dots + \alpha_j} + \sum_{k=j+1}^n \gamma_{\alpha_{j+1} + \dots + \alpha_k} \beta_{\alpha_i + \dots + \alpha_k}. \quad (15)$$

Then the BRST current associated with the constraints (2) is

$$J_{BRST}(z) = \sum_{\alpha \in \Delta_+} \tilde{c}_\alpha (J_\alpha - \chi(J_\alpha)) + \sum_{\alpha, \beta \in \Delta_+, \alpha < \beta} \tilde{c}_\alpha \tilde{c}_\beta \tilde{b}_{\alpha + \beta}. \quad (16)$$

Here an ordering $\alpha < \beta$ for $\alpha, \beta \in \Delta_+$ means $\alpha = \alpha_i + \dots + \alpha_j$ and $\beta = \alpha_{j+1} + \dots + \alpha_k$ for some $i \leq j < k$. We can show eq. (6) holds for

$$\mathcal{R}(z) = \sum_{k=1}^n \frac{(-1)^k}{k} \sum_{\alpha'_1 < \alpha'_2 < \dots < \alpha'_{k+1}} \tilde{c}_{\alpha'_1} \gamma_{\alpha'_2} \cdots \gamma_{\alpha'_{k+1}} \tilde{b}_{\alpha'_1 + \dots + \alpha'_{k+1}}. \quad (17)$$

The i -th term $(J)_i^R(z)$ in the l.h.s. of eq. (6) can be determined by recursively:

$$\begin{aligned} (J)_i^R(z) &= \sum_{N=i+1}^{n+1} C_N^{(i)} \sum_{\alpha'_1 < \dots < \alpha'_N} \hat{c}_{\alpha'_1} \gamma_{\alpha'_2} \dots \gamma_{\alpha'_N} \beta_{\alpha'_1 + \dots + \alpha'_N} \\ &+ \sum_{N=i+1}^{n+1} \sum_{j=2}^N C_N^{(i)} \sum_{\alpha'_1 < \dots < \alpha'_N} \hat{c}_{\alpha'_1} \gamma_{\alpha'_2} \dots \gamma_{\alpha'_{j-1}} \hat{c}_{\alpha'_j} \gamma_{\alpha'_{j+1}} \dots \gamma_{\alpha'_N} \hat{b}_{\alpha'_1 + \dots + \alpha'_N}, \end{aligned} \quad (18)$$

where the coefficients $C_N^{(i)}$ satisfy the recursion relation:

$$C_N^{(i+1)} = \sum_{N'=i+1}^{N-1} C_{N'}^{(i)} \frac{(-1)^{N-N'+1}}{N-N'} \quad (19)$$

and $C_N^{(1)} = (-1)^N/N$. The solution is given by

$$C_N^{(i)} = (-1)^i \sum_{(p_1, \dots, p_i) \in P(N-1, i)} \frac{1}{p_1 \dots p_i}, \quad (20)$$

and $P(N, m) = \{(p_1, \dots, p_m); \sum_{i=1}^m p_i = N, p_i > 0\}$ is the set of m -partitions of N . The homotopy condition (6) implies a set of identities:

$$\sum_{m=1}^{k-1} \frac{1}{m!} C_k^{(m)} = \sum_{m=1}^{k-1} \frac{(-1)^m}{m!} \sum_{(p_1, \dots, p_m) \in P(k-1, m)} \frac{1}{p_1 \dots p_m} = 0, \quad (21)$$

for $k \geq 3$. To prove (21) we use a formula

$$\epsilon^{-\sum_{n=1}^{\infty} \frac{t^n}{n}} = 1 - t. \quad (22)$$

Comparing the coefficients for t^{k-1} for $k \geq 3$ we get (21).

For arbitrary affine Lie algebra $\hat{\mathfrak{g}}$, we take the free field realization associated with the canonical parametrization of $\hat{N}_+$ [12]:

$$\begin{aligned} J_{\alpha}(z) &= \beta_{\alpha} + \frac{1}{2} \sum_{\alpha' \in \Delta_+} N_{\alpha', \alpha} \gamma_{\alpha'} \beta_{\alpha + \alpha'} \\ &+ \sum_{n \geq 2} \frac{\hat{B}_n}{n!} \sum_{\alpha'_1, \dots, \alpha'_n \in \Delta_+} N_{\alpha'_1, \alpha'_2 + \dots + \alpha'_n + \alpha} \dots N_{\alpha'_n, \alpha} \gamma_{\alpha'_1} \dots \gamma_{\alpha'_n} \beta_{\alpha'_1 + \dots + \alpha'_n + \alpha}, \end{aligned} \quad (23)$$

where $\hat{B}_n$ is defined by $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{\hat{B}_n}{n!} x^n$. For this realization, we conjecture that the homotopy operator $\mathcal{R}(z)$ is given by

$$\begin{aligned} \mathcal{R}(z) &= \frac{1}{2} \sum_{\alpha, \alpha' \in \Delta_+} N_{\alpha', \alpha} \hat{c}_{\alpha} \gamma_{\alpha'} \hat{b}_{\alpha + \alpha'} \\ &+ \sum_{n \geq 2} \frac{(-1)^{n-1}}{n} \frac{\hat{B}_n}{n!} \sum_{\alpha, \alpha'_1, \dots, \alpha'_n \in \Delta_+} N_{\alpha'_1, \alpha'_2 + \dots + \alpha'_n + \alpha} \dots N_{\alpha'_n, \alpha} \hat{c}_{\alpha} \gamma_{\alpha'_1} \dots \gamma_{\alpha'_n} \hat{b}_{\alpha'_1 + \dots + \alpha'_n + \alpha}. \end{aligned} \quad (24)$$

Although we have checked this formula for some non-trivial examples $\mathfrak{g} = A_n$ ($n \leq 4$), B_2 , D_4 and G_2 by an explicit calculation, we have not yet prove the existence of the homotopy transformation R for arbitrary $\hat{\mathfrak{g}}$. The proof of this statement is left for future study.

We shall use this ansatz for further consideration. Since the BRST operator $Q_0 = \int \frac{dz}{2\pi i} J_0(z)$ is bilinear in free fields, the Q_0 cohomology is easily worked out[13]. We find that the Kugo-Ojima quartet $(\beta_{\alpha} - \chi(J_{\alpha}), \gamma_{\alpha}, \hat{b}_{\alpha}, \hat{c}_{\alpha})$ ($\alpha \in \Delta_+$) from $Q_0(\hat{b}_{\alpha}) = \beta_{\alpha} - \chi(J_{\alpha})$ and $Q_0(\gamma_{\alpha}) = \hat{c}_{\alpha}$. Then the Q_0 cohomology is

$$H_{Q_0}(F) \cong F_{\varphi}. \quad (25)$$

The W -algebra should be realized on the Fock space F_{φ} . A typical operator is the energy-momentum (EM) tensor which generates the Virasoro algebra. Let us consider the total EM tensor for this constrained affine Lie algebra and the ghost system:

$$T_{total}(z) = T_{\hat{\mathfrak{g}}}(z) + \mu \cdot \partial H(z) + T_{ghost}(z), \quad (26)$$

where

$$\begin{aligned} T_{\hat{\mathfrak{g}}}(z) &= \sum_{\alpha \in \Delta_+} (\beta_{\alpha} \partial \gamma_{\alpha}) - \frac{1}{2} (\partial \varphi)^2 - \frac{i\rho \cdot \partial^2 \varphi}{\alpha_+}, \\ H^i(z) &= - \sum_{\alpha \in \Delta_+} \alpha^i (\gamma_{\alpha} \beta_{\alpha})(z) + i\alpha_+ \partial \varphi^i(z). \end{aligned} \quad (27)$$

$T_{\hat{\mathfrak{g}}}(z)$ is the Sugawara form of the EM tensor of the algebra $\hat{\mathfrak{g}}$, $H^i(z)$ the Cartan currents in the free field realization. The constant α_+ is defined by $\sqrt{k + h^{\vee}}$, where $h^{\vee}$ is the dual Coxeter number of $\hat{\mathfrak{g}}$. ρ is half the sum of positive roots. T_{ghost} is the EM tensor of the ghost part:

$$T_{ghost}(z) = \sum_{\alpha \in \Delta_+} (1 - \mu \cdot \alpha) (\hat{c}_{\alpha} \partial \hat{b}_{\alpha}) - \mu \cdot \alpha (\partial \hat{c}_{\alpha} \hat{b}_{\alpha}). \quad (28)$$

The improved term $\mu \cdot \partial H(z)$ in T_{total} changes the conformal weight of a current J_{α} from 1 to $1 - \mu \cdot \alpha$. Hence the improved vector μ must satisfy the conditions $\mu \cdot \alpha_i = 1$ ($i = 1, \dots, r$) to impose the constraint at the quantum level. μ is shown to be equal to the sum of the fundamental coweight vectors. It is easy to show that the T_{total} is equal to the EM tensor $T_{W\mathfrak{g}}$ of the $W\mathfrak{g}$ algebra up to Q_0 -exact term:

$$T_{total}(z) = T_{W\mathfrak{g}}(z) + \{Q_0(z), \Lambda_0(z)\} \quad (29)$$

where

$$T_{W\mathbf{g}}(z) = -\frac{1}{2}(\partial\varphi)^2(z) - i\left(\frac{\rho}{\alpha_+} - \alpha_+\mu\right) \cdot \partial^2\varphi(z) \quad (30)$$

and

$$\Lambda_0(z) = \sum_{\alpha \in \Delta_+} (1 - \mu \cdot \alpha)(\partial\gamma_\alpha)\tilde{b}_\alpha - (\mu \cdot \alpha)\gamma_\alpha\partial\tilde{b}_\alpha. \quad (31)$$

Since $T_{total}(z)$ and $T_{W\mathbf{g}}(z)$ are homotopy invariant, we get

$$T_{total}(z) = T_{W\mathbf{g}}(z) + \{Q_{BRST}, \Lambda(z)\}, \quad \Lambda(z) = e^R \Lambda_0(z) e^{-R}, \quad (32)$$

by applying the homotopy transformation to eq. (29).

3 An application to non-critical W_n strings

In this section we shall study the quantum hamiltonian reduction of an affine Lie superalgebra $sl(n+1|n)^{(1)}$ and its relation to the non-critical W_n strings.

The roots Δ of a Lie superalgebra $\mathbf{g}$ may be decomposed into two classes; even (bosonic) roots and odd (fermionic) roots, which correspond to the commuting generators and anti-commuting generators, respectively. Denote the set of even (odd) roots as Δ^0 (Δ^1), the set of positive even (odd) roots as Δ_+^0 (Δ_+^1). An affine Lie superalgebra $\hat{\mathbf{g}}$ of a Lie superalgebra $\mathbf{g}$ with rank r is generated by bosonic currents $J_\alpha(z)$ ($\alpha \in \Delta^0$), $H^i(z)$ ($i = 1, \dots, r$) and fermionic currents $j_\beta(z)$ ($\beta \in \Delta^1$) satisfying OPEs:

$$\begin{aligned} J_\alpha(z)J_\beta(w) &= \begin{cases} \frac{N_{\alpha,\beta}J_{\alpha+\beta}(w)}{z-w} + \dots, & \text{for } \alpha + \beta \neq 0, \\ \frac{\frac{2k}{\alpha^2}}{(z-w)^2} + \frac{\frac{2\alpha H(w)}{\alpha^2}}{z-w} + \dots, & \text{for } \alpha + \beta = 0. \end{cases} \\ j_\alpha(z)j_\beta(w) &= \begin{cases} \frac{N_{\alpha,\beta}J_{\alpha+\beta}(w)}{z-w} + \dots, & \text{for } \alpha + \beta \neq 0. \\ \frac{k}{(z-w)^2} + \frac{\alpha H(w)}{z-w} + \dots, & \text{for } \alpha + \beta = 0. \end{cases} \\ j_\alpha(z)J_\beta(w) &= \frac{N_{\alpha,\beta}j_{\alpha+\beta}(w)}{z-w} + \dots, \\ H^i(z)J_\alpha(w) &= \frac{\alpha^i J_\alpha(w)}{z-w} + \dots, \quad H^i(z)j_\beta(w) = \frac{\beta^i j_\beta(w)}{z-w} + \dots, \\ H^i(z)H^j(w) &= \frac{k\delta_{i,j}}{(z-w)^2} + \dots. \end{aligned} \quad (33)$$

Let us consider the constraints $\chi(J_\alpha)$ and $\chi(j_\beta)$ for positive root currents associated the principal $sl(2)$ embedding. In this case, however, we need to consider the second class

constraints in general. In such a case, it is convenient to introduce auxiliary fields which change the second class constraints to the first class. We introduce fermionic ghosts $(\tilde{b}_\alpha(z), \tilde{c}_\alpha(z))$ ($\alpha \in \Delta_+^0$) and bosonic ghosts $(\tilde{\beta}_\alpha(z), \tilde{\gamma}_\alpha(z))$ ($\alpha \in \Delta_+^1$) for bosonic and fermionic constraints, respectively. The BRST current is given by

$$\begin{aligned} J_{BRST}(z) &= \sum_{\alpha \in \Delta_+^0} \tilde{c}_\alpha(J_\alpha - \chi(J_\alpha)) + \sum_{\alpha \in \Delta_+^1} \tilde{\gamma}_\alpha(j_\alpha - \chi(j_\alpha)) \\ &\quad - \frac{1}{2} \sum_{\alpha, \beta \in \Delta_+^0} N_{\alpha,\beta}(\tilde{c}_\alpha(\tilde{c}_\beta \tilde{b}_{\alpha+\beta}))(z) - \frac{1}{2} \sum_{\alpha \in \Delta_+^0, \beta \in \Delta_+^1} N_{\alpha,\beta}(\tilde{c}_\alpha(\tilde{\gamma}_\beta \tilde{\beta}_{\alpha+\beta}))(z). \end{aligned} \quad (34)$$

By studying this cohomology, we shall have the $W\mathbf{g}$ algebra associated with a Lie superalgebra $\mathbf{g}$, including $N = 1$ and $N = 2$ supersymmetric W -algebras [9].

Now we study the quantum hamiltonian reduction of affine Lie superalgebras $sl(n+1|n)^{(1)}$. Note that a Lie superalgebra admits various choice of simple root system. This means that constraints and therefore the cohomological analysis depends essentially on a simple root system of $\mathbf{g}$. In fact, for a Lie superalgebra $\mathbf{g} = sl(n+1|n)$ we have at least different free field realization of the $N = 2$ super W_n -algebra. Let e_i ($i = 1, \dots, n+1$) and δ_a ($a = 1, \dots, n$) be an orthonormal basis, which satisfies $e_i \cdot e_j = \delta_{i,j}$, $\delta_a \cdot \delta_b = -\delta_{a,b}$. If we take a purely odd simple root system

$$\alpha_{2i-1} = e_i - \delta_i, \quad \alpha_{2i} = \delta_i - e_{i+1}, \quad (35)$$

for $i = 1, \dots, n$, we have the manifestly supersymmetric $N = 2$ super- W_n algebra, which is the chiral algebra of Kazama-Suzuki's $N = 2$ CP_n coset model [14].

It is known that there is another simple root system:

$$\alpha_i = e_i - e_{i+1} (i = 1, \dots, n), \quad \alpha_{n+1} = e_{n+1} - \delta_1, \quad \alpha_{a+n+1} = \delta_a - \delta_{a+1} (a = 1, \dots, n-1). \quad (36)$$

In ref. [5], it is argued that the classical hamiltonian reduction of $sl(n+1|n)^{(1)}$ give rise to the $N = 2$ super W_n -algebra which appears in the non-critical W_n -string theory. The quantum hamiltonian reduction based on Feigin-Frenkel's approach was studied in refs. [10, 11]. In the following, we study the quantum hamiltonian reduction of $sl(n+1|n)^{(1)}$ by using the Wakimoto realization of the currents. For the simple roots system (36) the even positive roots are $e_i - e_j$ ($i < j$) and $\delta_a - \delta_b$ ($a < b$). The odd positive roots are

$e_i - \delta_a$. We introduce the n pairs of the fermionic fields $(\psi_a^\dagger, \psi_a)$ ($a = 1, \dots, n$) with weights $(3/2 + n - a, -1/2 - n + a)$. The operator product expansions are given by $\psi_a(z)\psi_b^\dagger(w) = \frac{\delta_{a,b}}{z-w} + \dots$. The constraints are

$$\begin{aligned}\chi(J_{e_i - e_j})(z) &= \delta_{i,j-1}, \quad \text{for } i < j, \\ \chi(j_{e_i - \delta_a})(z) &= \delta_{i,n+1}\psi_a^\dagger(z), \\ \chi(J_{\delta_a - \delta_b})(z) &= \delta_{a,b-1} + \psi_a\psi_{n+1+b}^\dagger(z), \quad \text{for } a < b.\end{aligned}\quad (37)$$

which are consistent with OPEs for the subalgebra $\hat{n}_+$:

$$\begin{aligned}J_{e_i - e_j}(z)J_{e_k - e_l}(w) &= \frac{-\delta_{j,k}J_{e_i - e_j}(w)}{z-w} + \dots, \quad (i < j \leq k < l), \\ J_{\delta_a - \delta_b}(z)J_{\delta_c - \delta_d}(w) &= \frac{-\delta_{b,c}J_{\delta_a - \delta_d}(w)}{z-w} + \dots, \quad (a < b \leq c < d), \\ J_{e_i - e_j}(z)j_{e_k - \delta_a}(w) &= \frac{-\delta_{j,k}J_{e_i - e_a}(w)}{z-w} + \dots, \quad (i < j), \\ J_{\delta_a - \delta_b}(z)j_{e_i - \delta_c}(w) &= \frac{\delta_{a,c}J_{e_i - e_b}(w)}{z-w} + \dots, \quad (a < b).\end{aligned}\quad (38)$$

The Wakimoto realization of an affine Lie superalgebra can be done by introducing bosonic ghosts $(\beta_\alpha(z), \gamma_\alpha(z))$ for even positive roots $\alpha \in \Delta_+^0$, fermionic ghosts $(\eta_\alpha(z), \xi_\alpha(z))$ for odd positive roots $\alpha \in \Delta_+^1$, free bosons $\varphi(z) = (\varphi_1(z), \dots, \varphi_r(z))$. For $\mathfrak{g} = sl(n+1|n)^{(1)}$, the currents for positive roots are given by

$$\begin{aligned}J_{e_i - e_j}(z) &= \beta_{e_i - e_j} + \sum_{k=j+1}^{n+1} \gamma_{e_j - e_k} \beta_{e_i - e_k} + \sum_{a=1}^n \eta_{e_j - \delta_a} \eta_{e_i - \delta_a}, \\ j_{e_i - \delta_a}(z) &= \eta_{e_i - \delta_a} + \sum_{c=a+1}^n \gamma_{\delta_a - \delta_c} \eta_{e_i - \delta_c}, \\ J_{\delta_a - \delta_b}(z) &= \beta_{\delta_a - \delta_b} + \sum_{c=b+1}^n \gamma_{\delta_b - \delta_c} \beta_{\delta_a - \delta_c}.\end{aligned}\quad (39)$$

We wish to simplify the BRST current (34) by a homotopy transformation R :

$$J_{BRST}(z) = e^R J_0(z) e^{-R}, \quad (40)$$

where

$$J_0(z) = \sum_{i < j} \hat{c}_{e_i - e_j} (\beta_{e_i - e_j} - \delta_{i,j+1}) + \sum_{a < b} \hat{c}_{\delta_a - \delta_b} (\beta_{\delta_a - \delta_b} - \delta_{a,b+1}) + \sum_{i,a} \hat{\gamma}_{e_i - \delta_a} (\eta_{e_i - \delta_a} - \delta_{i,n+1} \psi_a^\dagger). \quad (41)$$

The homotopy operator $\mathcal{R}(z)$ would take the form

$$\begin{aligned}\mathcal{R}(z) &= - \sum_{i < j < k} \hat{c}_{e_i - e_j} \gamma_{e_j - e_k} \hat{b}_{e_i - e_k} - \sum_{i < j, a} \hat{c}_{e_i - e_j} \xi_{e_j - \delta_a} \hat{\beta}_{e_i - \delta_a} - \sum_{a < b < c} \hat{c}_{\delta_a - \delta_b} \gamma_{\delta_b - \delta_c} \hat{b}_{\delta_a - \delta_c} \\ &\quad - \sum_{i, b < c} \hat{\gamma}_{e_i - \delta_b} \gamma_{\delta_b - \delta_c} \hat{\beta}_{e_i - \delta_c} + \sum_{a < b} \psi_a \hat{c}_{\delta_a - \delta_b} \hat{\beta}_{e_{n+1} - \delta_b} - \sum_{a < b} \psi_a \gamma_{\delta_a - \delta_b} \eta_{e_{n+1} - \delta_b} + \dots,\end{aligned}\quad (42)$$

The remaining terms would be determined recursively. At present we can show that these terms are fixed for $n = 1, 2$ by an explicit calculation. Results are as follows:

$$\begin{aligned}\mathcal{R}(z) &= -\hat{c}_{e_1 - e_2} \xi_{e_2 - \delta_1} \hat{\beta}_{e_1 - \delta_1}, \quad (n = 1) \\ \mathcal{R}(z) &= -\hat{c}_{e_1 - e_2} \gamma_{e_2 - e_3} \hat{b}_{e_1 - e_3} - \sum_{i < j, a} \hat{c}_{e_i - e_j} \xi_{e_j - \delta_a} \hat{\beta}_{e_i - \delta_a} \\ &\quad - \hat{\gamma}_{e_2 - \delta_1} \gamma_{\delta_1 - \delta_2} \hat{\beta}_{e_2 - \delta_2} - \hat{\gamma}_{e_1 - \delta_1} \gamma_{\delta_1 - \delta_2} \hat{\beta}_{e_1 - \delta_2} + \psi_1 \hat{c}_{\delta_1 - \delta_2} \hat{\beta}_{e_3 - \delta_2} - \psi_1 \gamma_{\delta_1 - \delta_2} \eta_{e_3 - \delta_2} \\ &\quad + \frac{1}{2} (\hat{c}_{e_1 - e_2} \gamma_{e_2 - e_3} \xi_{e_3 - \delta_1} \hat{\beta}_{e_1 - \delta_1} + \hat{c}_{e_1 - e_2} \gamma_{e_2 - e_3} \xi_{e_3 - \delta_2} \hat{\beta}_{e_1 - \delta_2} + \hat{c}_{e_1 - e_2} \xi_{e_2 - \delta_1} \gamma_{\delta_1 - \delta_2} \hat{\beta}_{e_1 - \delta_2} \\ &\quad + \hat{c}_{e_2 - e_3} \xi_{e_3 - \delta_1} \gamma_{\delta_1 - \delta_2} \hat{\beta}_{e_2 - \delta_2} + \hat{c}_{e_2 - e_3} \psi_1 \gamma_{\delta_1 - \delta_2} \hat{\beta}_{e_2 - \delta_2}) \\ &\quad - \frac{1}{3} \hat{c}_{e_1 - e_2} \gamma_{e_2 - e_3} \xi_{e_3 - \delta_1} \gamma_{\delta_1 - \delta_2} \hat{\beta}_{e_1 - \delta_2} - \frac{1}{6} \hat{c}_{e_1 - e_2} \gamma_{e_2 - e_3} \xi_{e_3 - \delta_1} \gamma_{\delta_1 - \delta_2} \hat{\beta}_{e_1 - \delta_2}, \quad (n = 2).\end{aligned}\quad (43)$$

We assume that the existence of R for $n \geq 3$. Let us study the Q_0 -cohomology which acting on the Fock space $F = F_{\tilde{\mathfrak{g}}} \otimes F_{ghosts} \otimes (\otimes_{a=1}^n F_{\psi_a^\dagger, \psi_a})$ where $F_{\tilde{\mathfrak{g}}} = F_\varphi \otimes (\otimes_{\alpha \in \Delta_+^0} F_{\beta_\alpha, \gamma_\alpha}) \otimes (\otimes_{\alpha \in \Delta_+^1} F_{\eta_\alpha, \xi_\alpha})$ is the Wakimoto modules and $F_{ghosts} = (\otimes_{\alpha \in \Delta_+^0} F_{\beta_\alpha, \gamma_\alpha}) \otimes (\otimes_{\alpha \in \Delta_+^1} F_{\eta_\alpha, \xi_\alpha})$ the Fock modules for ghost fields. It is easy to see that

$$\begin{aligned}Q_0(\tilde{\gamma}_{e_i - e_j}) &= \hat{c}_{e_i - e_j}, \quad Q_0(\tilde{b}_{e_i - e_j}) = \hat{\beta}_{e_i - e_j} - \delta_{i,j+1}, \\ Q_0(\tilde{\gamma}_{\delta_a - \delta_b}) &= \hat{c}_{\delta_a - \delta_b}, \quad Q_0(\tilde{b}_{\delta_a - \delta_b}) = \hat{\beta}_{\delta_a - \delta_b} - \delta_{a,b+1}, \\ Q_0(\xi_{e_i - \delta_a} - \delta_{i,n+1} \psi_a) &= (1 + \delta_{i,n+1}) \gamma_{e_i - \delta_a}, \quad Q_0(\hat{\beta}_{e_i - \delta_a}) = \eta_{e_i - \delta_a} - \delta_{i,n+1} \psi_a^\dagger.\end{aligned}\quad (44)$$

Then $(\tilde{\gamma}_{e_i - e_j}, \hat{c}_{e_i - e_j}, \tilde{b}_{e_i - e_j}, \hat{\beta}_{e_i - e_j} - \delta_{i,j+1})$, $(\tilde{\gamma}_{\delta_a - \delta_b}, \hat{c}_{\delta_a - \delta_b}, \tilde{b}_{\delta_a - \delta_b}, \hat{\beta}_{\delta_a - \delta_b} - \delta_{a,b+1})$ and $(\xi_{e_i - \delta_a} - \delta_{i,n+1} \psi_a, \gamma_{e_i - \delta_a}, \hat{\beta}_{e_i - \delta_a}, \eta_{e_i - \delta_a} - \delta_{i,n+1} \psi_a^\dagger)$ form the Kugo-Ojima quartets. We define new fermionic fields

$$(b_a(z), c_a(z)) \equiv ((\eta_{e_{n+1} - \delta_a} + \psi_a^\dagger)/\sqrt{2}, (\xi_{e_{n+1} - \delta_a} + \psi_a)/\sqrt{2}), \quad (45)$$

which are physical operators in the BRST cohomology. Therefore we get

$$H_{Q_0}(F) \cong F_{\neq} \oplus (\oplus_{a=1}^n F_{b_a, c_a}). \quad (46)$$

The total EM tensor is given by

$$T_{total}(z) = T_{\hat{\mathbf{g}}}(z) + \mu \cdot \partial H(z) + T_{ghosts}(z) + T_{\psi, \psi^\dagger}(z), \quad (47)$$

where $\alpha_+ = \sqrt{k+1}$ and

$$T_{\hat{\mathbf{g}}}(z) = -\frac{1}{2}(\partial\varphi)^2 - \frac{i\rho \cdot \varphi}{\alpha_+} + \sum_{\alpha \in \Delta_+^0} \beta_\alpha \partial\gamma_\alpha - \sum_{\alpha \in \Delta_+^1} \eta_\alpha \partial\xi_\alpha, \quad (48)$$

is the EM tensor of $\hat{\mathbf{g}}$. ρ is defined by $\frac{1}{2}(\sum_{\alpha \in \Delta_+^0} \alpha - \sum_{\alpha \in \Delta_+^1} \alpha)$. For $sl(n+1|n)$ case, ρ becomes

$$\rho = \rho_{sl(n+1)} + \hat{\rho}_{sl(n)} - \frac{1}{2}\nu, \quad (49)$$

where $\rho_{sl(r)}$ ($\hat{\rho}_{sl(r)}$) is half the sum of positive roots of $sl(r)$ with positive (negative) metric: $\sum_{i=1}^r \frac{r+1-2i}{2} e_i$ ($\sum_{i=1}^r \frac{r+1-2i}{2} \delta_i$), ν measures the $U(1)$ charge of $\mathbf{g}$: $n \sum_{i=1}^{n+1} e_j - (n+1) \sum_{a=1}^n \delta_a$.¹ The improved vector μ is given by $\mu = \rho_{sl(n+1)} - \hat{\rho}_{sl(n)}$. The Cartan currents is expressed as

$$H^i(z) = i\alpha_+ \partial\varphi^i(z) - \sum_{\alpha \in \Delta_+^0} \alpha^i (\gamma_\alpha \beta_\alpha)(z) - \sum_{\alpha \in \Delta_+^1} \alpha^i (\xi_\alpha \eta_\alpha)(z). \quad (50)$$

The EM tensors $T_{ghosts}(z)$ and $T_{\psi, \psi^\dagger}(z)$ for ghost fields and auxiliary fields are given by

$$T_{ghosts}(z) = \sum_{\alpha \in \Delta_+^0} (1 - \mu \cdot \alpha) \tilde{c}_\alpha \partial \tilde{b}_\alpha - (\mu \cdot \alpha) \partial \tilde{c}_\alpha \tilde{b}_\alpha + \sum_{\alpha \in \Delta_+^1} (1 - \mu \cdot \alpha) \tilde{\beta}_\alpha \partial \tilde{\gamma}_\alpha - (\mu \cdot \alpha) \tilde{\gamma}_\alpha \partial \tilde{\beta}_\alpha, \quad (51)$$

$$T_{\psi, \psi^\dagger}(z) = \sum_{a=1}^n (e_{n+1} - \delta_a) \cdot \mu \partial \psi_a \psi_a - (1 - (e_{n+1} - \delta_a) \cdot \mu) \psi_a^\dagger \partial \psi_a.$$

We can show

$$T_{total}(z) = T_{W\mathbf{g}}(z) + \{Q_0, \Lambda_0(z)\}, \quad (52)$$

where

$$T_{W\mathbf{g}}(z) = -\frac{1}{2}(\partial\varphi)^2(z) - i\left(\frac{\rho}{\alpha_+} - \alpha_+ \mu\right) \cdot \partial^2 \varphi(z) + \sum_{a=1}^n \left(-\frac{3}{2} - n + a\right) b_a \partial c_a(z) + \left(-\frac{1}{2} - n + a\right) \partial b_a c_a(z), \quad (53)$$

¹Note that even subalgebra of $sl(n+1|n)$ is $sl(n+1) \oplus sl(n) \oplus \mathfrak{u}(1)$.

and

$$\begin{aligned} \Lambda_0(z) = & \sum_{\alpha \in \Delta_+^0} (1 - \mu \cdot \alpha) \tilde{b}_\alpha \partial \gamma_\alpha - (\mu \cdot \alpha) (\partial \tilde{b}_\alpha) \gamma_\alpha \\ & + \sum_{\alpha \in \Delta_+^1, \alpha \neq e_{n+1} - \delta_a} (1 - \mu \cdot \alpha) \tilde{\beta}_\alpha \partial \xi_\alpha - (\mu \cdot \alpha) \partial \tilde{\beta}_\alpha \xi_\alpha \\ & + \frac{1}{2} \sum_{a=1}^n (-1 + (e_{n+1} - \delta_a) \cdot \mu) (\eta_{e_{n+1} - \delta_a} - \psi_a^\dagger) \partial (\xi_{e_{n+1} - \delta_a} - \psi_a) \\ & + (e_{n+1} - \delta_a) \cdot \mu \partial (\eta_{e_{n+1} - \delta_a} - \psi_a^\dagger) (\xi_{e_{n+1} - \delta_a} - \psi_a). \end{aligned} \quad (54)$$

Here we have used $(e_{n+1} - \delta_a) \cdot \mu = -\frac{1}{2} - n + a$. $T_{W\mathbf{g}}(z)$ is the EM tensor of the $N=2$ super W_n -algebra. By applying a homotopy transformation, we can show that $T_{total}(z)$ is Q_{BRST} -equivalent to $T_{W\mathbf{g}}(z)$.

There exists a $U(1)$ current $U(z)$ in this $N=2$ super W_n -algebra:

$$U(z) = -i\alpha_+ \nu \cdot \partial\varphi(z) + \sum_{a=1}^n c_a b_a(z), \quad (55)$$

which is BRST-equivalent to the operator

$$\nu \cdot \left(H - \sum_{\alpha \in \Delta_+^0} \alpha \tilde{c}_\alpha \tilde{b}_\alpha - \sum_{\alpha \in \Delta_+^1} \alpha \tilde{\gamma}_\alpha \tilde{\beta}_\alpha - \sum_{a=1}^n (e_{n+1} - \delta_a) \psi_a \psi_a^\dagger \right). \quad (56)$$

We find that the twisted EM tensor $T(z) = T_{W\mathbf{g}}(z) + \frac{1}{2}\partial U(z)$ becomes

$$\begin{aligned} T(z) = & -\frac{1}{2}(\partial\varphi)^2(z) - i\left\{\left(\frac{1}{\alpha_+} - \alpha_+\right)\rho_{sl(n+1)} + \left(\frac{1}{\alpha_+} + \alpha_+\right)\hat{\rho}_{sl(n+1)}\right\} \cdot \partial^2 \varphi(z) \\ & + \sum_{a=1}^n (-2 - n + a) b_a \partial c_a(z) + (-1 - n + a) \partial b_a c_a(z). \end{aligned} \quad (57)$$

This is exactly the EM tensor of the non-critical W_n -strings. Here we have used a formula:

$$\frac{\rho}{\alpha_+} - \alpha_+ \mu - \frac{\alpha_+ \nu}{2} = \left(\frac{1}{\alpha_+} - \alpha_+\right)\rho_{sl(n+1)} + \left(\frac{1}{\alpha_+} + \alpha_+\right)\hat{\rho}_{sl(n+1)}, \quad (58)$$

where $\hat{\rho}_{sl(n+1)} = \hat{\rho}_{sl(n)} - \nu/2$. $-\nu/2$ may be regarded as the n -th fundamental weight of $sl(n+1)$, in the embedding of $sl(n)$ into $sl(n+1)$.

4 Conclusions and discussion

In this paper we have examined the quantum hamiltonian reduction of affine Lie (super)algebra from the view point of the homotopy transformation. By choosing a special

type of embedding of $sl(2)$ into a Lie superalgebra $sl(n+1|n)$, we have established the connection between the affine Lie superalgebra $sl(n+1|n)^{(1)}$ and the non-critical W_n -strings, at the level of the Fock modules. In order to study the irreducible representation, we need to take into account the resolution by screening operators, which is defined as a commutant of the chiral algebra. In the present approach, however, the W -algebra structure of this model is not clear. On the other hand, in the Feigin-Frenkel approach, the W -algebra can be computed in principle, although explicit construction of the algebra is quite complicated. It is desirable to synthesis the present method and the Feigin-Frenkel approach in order to get more simplified picture for the quantum construction of the W -algebra associated with a Lie (super)algebra.

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