QUANTISATION OF A PARTICLE MOVING ON A GROUP MANIFOLD

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ABSTRACT

The Hilbert space of a free massless particle moving on a group manifold is studied in details using canonical quantisation. While the simplest model is invariant under a global symmetry, $G \times G$, there is a very natural way to “factorise” the theory so that only one copy of the global symmetry is preserved. In the case of $G = SU(2)$, a simple deformation of the quantised theory is proposed to give a realisation of the quantum group, $U_q(SL(2))$. The symplectic structures of the corresponding classical theory is derived. This can be used, in principle, to obtain a Lagrangian formulation for the $U_q(SL(2))$ symmetry.
Introduction

The algebraic structures of quantum groups $U_t(G)$ have been widely studied for some time\cite{1}. However, its geometrical meaning as a symmetry in quantum field theory is still not well understood. It is therefore an intriguing task to obtain a Lagrangian formulation of the quantum group symmetry. One interesting possibility is to understand the quantum group structures in the chirally factorised two-dimensional Wess-Zumino-Witten models\cite{2,3}. However, it is very difficult to obtain an explicit formulation of the quantum group structures in these chiral models because their defining fields, so-called chiral vertex operators, are non-local operators. It turns out that we can study a simpler model which describes a particle moving on the manifold of a Lie group $G$. The idea is to study whether one can implement the quantum group structures, viewed as some deformation of $G$, by introducing nontrivial interaction to the particles. In other words, one should study a Lagrangian formulation of interacting particles with $U_t(G)$ structure where the deformation parameter $t$ can be related to the coupling constants. The aim of this paper is to investigate this possibility.

One of the common features for particles and strings moving on a group manifold is the quadratic Poisson structures among their defining fields, $U \in G$. In terms of the tensor notation, $U_1 = U \otimes 1$ and $U_2 = 1 \otimes U$, they can be written as

$$\{U_1, U_2\} = U_1 U_2 r_{12}. \quad (1)$$

In general, the matrix $r_{12}$ may depend on dynamical variables. Faddeev et al\cite{3,4} suggested that these Poisson brackets can be quantised by an exchange relation,

$$U_1 U_2 = U_2 U_1 B_{12}, \quad (2)$$

where the classical limit of the braiding matrix $B$ is $r_{12}$ in (1). For a particle moving on $G = SU(2)$, this quantisation was first discussed in [4], but the Hilbert space structure of the quantised theory was not fully discussed.

In this letter, we give a systematic derivation of the canonical structure of a free particle moving on a compact simple Lie group, $G$. In order to relate to the zero modes of the WZW model associated with $G$, we have used the same approach in [2] to quantise the particle case. It may not seem natural from the usual quantisation’s point of view, but we will justify this particular approach and show that there are several advantages. In particular, it leads to a natural way of factorising the left-invariant and the right-invariant theory. Also, it suggests a simple deformation of the non-interacting model to give a quantum realisation of $U_t(G)$. This deformation is explained in details for $U_t(SL(2))$ using two pairs of harmonic oscillators. We also derive the symplectic two-form which, in principle, should allow us to write down the Lagrangian whose symmetry is governed by $U_t(SL(2))$.

Massless Free Particle Moving on a Group Manifold

Let $g(\tau)$ denote a particle at time $\tau$ taking values on a compact simple Lie group $G$ of rank $r$. Denote $t^\alpha \equiv \{H^i, E^\alpha\}$, for $i = 1, 2, ..r$ and all the roots $\alpha$, to be a basis of the
generators for the corresponding algebra $G$. These generators are normalised such that
\[ T^a T^b = \delta^{ab}. \]
Assuming that there is no interaction among the particles, we can write down the Lagrangian for the free massless particle moving on $G$ as
\[ L = -\frac{1}{2} Tr(g^{-1} \dot{g} g^{-1} \dot{g}). \]

This Lagrangian is invariant under a global symmetry $G \times G$, i.e. $g(\tau) \rightarrow g_1 \cdot g(\tau) \cdot g_2$, $\forall g_1 \in G, g_2 \in G$. Consequently, the equations of motion give the conservation of the two currents that generate the group transformations,
\[ \partial_\tau (g^{-1} \dot{g}) = 0, \quad \partial_\tau (\dot{g} g^{-1}) = 0. \]

A general solution of these equations can be written as $g(\tau) = u_0 e^{iP \tau} v_0$ where $u_0$ and $v_0$ are the constant elements in $G$ given by the initial conditions of (4). However, this parameterisation is ambiguous because we get the same solution after redefining the variables by $u_0 \rightarrow u_0 f_1$, $v_0 \rightarrow f_2^{-1} u_0$, and $e^{iP \tau} \rightarrow f_1^{-1} e^{iP \tau} f_2$ with constant group elements $f_1, f_2 \in G$. In order to minimise this ambiguity, we will restrict $e^{iP \tau}$ to be in the maximal torus $T$ and write the solution as
\[ g(\tau) = \tilde{u} e^{iQ \cdot H} e^{iP \cdot \tau} \tilde{v}, \quad \tilde{u} \in G/H, \quad \tilde{v} \in H \backslash G. \]

This way, the ambiguity in parameterising the solution in (5) is fixed up to the Weyl group $W$ and the maximal torus $T$. In this sense, (5) gives a map between the classical phase space, $G \times G$, and the configuration space, $G$.

According to Zuckerman, there is a closed symplectic two-form given by the Lagrangian $L$,
\[ \omega = \frac{1}{2} Tr \left\{ \delta g \delta \left( \frac{\partial L}{\partial \dot{g}} \right) \right\}. \]

This symplectic form determines the Poisson structures of the theory according to,
\[ \omega = \sum_{i,j} \omega^{ij} \delta A_i \delta A_j, \quad \{f_1, f_2\} = \sum_{i,j} \omega^{-1}_{ij} \frac{\delta f_1}{\delta A_i} \frac{\delta f_2}{\delta A_j}, \]

where $\{A_i\}$ denote a basis of the coordinates on the phase space. Since it is the space of all the solutions of the equations of motion, we substitute the solution in (5) into (6). The symplectic form is now invertible and it gives the following Poisson brackets,
\[ \{q^i, p^j\} = \delta^{ij}, \quad \{\tilde{u}_1, \tilde{v}_2\} = 0, \quad \{\tilde{u}, p^i\} = 0, \quad \{\tilde{u}, p^j\} = 0, \]
\[ \{\tilde{u}_1, \tilde{u}_2\} = \tilde{u}_1 \tilde{u}_2 r_{12} - \tilde{u}_2 \{\tilde{u}_1, i q \cdot H_2\} - \tilde{u}_1 \{i q \cdot H_1, \tilde{u}_2\}, \]
\[ \{\tilde{v}_1, \tilde{v}_2\} = -r_{12} \tilde{v}_1 \tilde{v}_2 - \{\tilde{v}_1, i q \cdot H_2\} \tilde{v}_2 - \{i q \cdot H_1, \tilde{v}_2\} \tilde{v}_1. \]

The last two brackets in (8) depend on the Poisson brackets of $\tilde{u}, \tilde{v}$ with $q \cdot H$. Their explicit forms depend on the parameterisation of the coset. However, we do not need to
specify them for the purpose of this paper. In fact, we can simplify the Poisson brackets in (8) in terms of the new variables, \( u \equiv \tilde{u} e^{i\nu} \) and \( v \equiv e^{i\nu} \tilde{v} \),

\[
\{u_1, u_2\} = u_1 u_2 r_{12}, \\
\{v_1, v_2\} = -r_{12} v_1 v_2, \quad \text{with} \quad r_{12} = \sum_{\alpha \in \Phi} \frac{i}{p \cdot \alpha} E_{\alpha} \otimes E_{-\alpha}.
\] (9)

We have denoted \( \Phi \) to be the space of roots for the Lie algebra \( G \). One can verify that these Poisson brackets satisfy the Jacobi identities. It is also straight-forward to check that \( u \) and \( v \) transform as the left-covariant and the right-covariant group elements under the currents \( L \) and \( R \).

\[
L \equiv \partial \gamma \cdot g^{-1}, \quad \{L_1, u_2\} = C_{12} u_2, \\
R \equiv g^{-1} \cdot \partial \gamma, \quad \{R_1, v_2\} = v_2 C_{12}, \quad \text{with} \quad C_{12} \equiv \sum_a t^a \otimes t_a.
\] (10)

Now, we can proceed the quantisation of these Poisson brackets. The first four brackets in (8) can be replaced with the following Dirac commutators,

\[
[q^i, p^j] = i\hbar \delta^{ij}, \quad [\tilde{u}_1, \tilde{v}_2] = 0, \quad [\tilde{u}, p^i] = 0, \quad [\tilde{v}, p^i] = 0.
\] (11)

As mentioned in the introduction, we quantise the quadratic brackets in (9) by the exchange relations,

\[
u_1 u_2 = u_2 u_1 B_{12}, \quad v_1 v_2 = B_{12}^{-1} v_2 v_1.
\] (12)

Quantum consistency requires that the braiding matrix \( B \) has to satisfy the following conditions,

- **Classical limit.**
  \[
  B = 1 \otimes 1 + i\hbar r_{12} + O(\hbar^2),
  \] (13a)

- **Antisymmetry and Unitarity.**
  \[
  B^{-1} = B^\dagger = \mathbb{P} B \mathbb{P},
  \] (13b)

- **Jacobi identities,** \[
  B_{23}(p_1) B_{13}(p_2) B_{12}(p_3) = B_{12}(p_1) B_{13}(p_2) B_{23}(p_3),
  \]

- **Locality condition,** \[
  [B, e^{i\nu(H_1 + H_2)}] = 0,
  \] (13d)

where the subscript of \( p \) denotes a shift, e.g. \( \tilde{p}_2 = \tilde{p} + \hbar \tilde{H}_2 \). This is because \( u \) and \( v \) defined in (9) do not commute with the momentum \( p \):

\[
[\tilde{p}, u_2] = \hbar u_2 \tilde{H}_2, \quad [\tilde{p}, v_2] = \hbar \tilde{H}_2 v_2.
\] (14)

The locality condition in (13d) is required to ensure that \( g(\tau) \) is local in the sense that their equal-time commutator vanishes, i.e. \([g_1(\tau), g_2(\tau)] = 0\).

The explicit solution for the braiding matrix in (13) can be obtained in a chosen representation for a given group. In this paper, we give the solution in the fundamental representation of \( G = SU(N) \). It has the form of exponentiating the classical r-matrix in (9):

\[
B = \exp \left( - \sum_{\alpha \in \Phi} \theta_\alpha E_\alpha \otimes E_{-\alpha} \right).
\] (15)
The coefficients $\theta_\alpha$’s can be determined from the Jacobi identities in (13c),

$$\sin \theta_\alpha = h/p \cdot \alpha. \quad (16)$$

Although the braiding matrix $B$ depends on the momentum $p$, the eigenvalues of $PB$ do not. They are 1 and -1 with multiplicity $\frac{1}{2}N(N + 1)$ and $\frac{1}{2}N(N - 1)$ respectively. For $N = 2$, this braiding matrix has already appeared in [3].

In order to justify that (11), (12) and (15) determine all the quantum relations, we show that the current algebras can be derived from them. First, the quantum corrections to the $SU(N)$ currents can be obtained from normal-ordering the classical currents in (10), e.g. putting $p$ to the right-hand side of $q$. We can write these currents in terms of $u$ and $v$ respectively:

$$L = iu(p \cdot H)u^{-1} + 2i\hbar H^2, \quad R = iv^{-1}(p \cdot H)v + 2i\hbar H^2. \quad (17)$$

Then, using the braiding matrix in (15), we find that the transformation laws of the covariant vertex operators are given by

$$[L_1, u_2] = i\hbar C_{12} u_2, \quad [R_1, v_2] = i\hbar v_2 C_{12}. \quad (18)$$

Following from (17) and (18), the symmetry algebras can be written in terms of the tensor-Casimir $C_{12}$ defined in (10) as follows.

$$[L_1, L_2] = i\hbar [C_{12}, L_2], \quad [R_1, R_2] = i\hbar [C_{12}, R_2]. \quad (19)$$

Furthermore, the quadratic Casimir operators for $L$ and $R$ turn out to be equal and they depend only on the momentum $p$,

$$C \equiv -Tr(L^2) \equiv -Tr(R^2) = p^2 + \hbar^2 Tr(H^2 H^2). \quad (20)$$

This implies that the Hilbert space can be decomposed into a direct sum of the diagonal products of the irreducible representations of the left and the right symmetry algebras,

$$\mathcal{H} = \bigoplus_\lambda \mathcal{V}_\lambda \otimes \overline{\mathcal{V}}_\lambda. \quad (21)$$

In order to illustrate the factorisation of the Hilbert space and to study the intertwining operators on this space, we now give an example of the above quantisation for $G = SU(2)$ in details. In particular, we will construct the left-covariant vertex $u$ explicitly in terms of harmonic oscillators and determine the Hilbert space which it acts on. To do so, it is very convenient to use the Euler parameterisation for the solution in (5):

$$u = e^{i\frac{1}{2}A\sigma_3} e^{i\frac{1}{2}D\sigma_2} e^{i\frac{1}{2}q\sigma_3}, \quad v = e^{i\frac{1}{2}q\sigma_3} e^{i\frac{1}{2}T\sigma_2} e^{i\frac{1}{2}A\sigma_3}. \quad (22)$$
The symplectic form in (6) can now be easily inverted and the quantisation of the Poisson brackets gives three pairs of harmonic oscillators,

\[ [q, p] = i\hbar, \quad [A, D] = i\hbar, \quad [\overline{A}, \overline{D}] = i\hbar, \]

(23)

where we have defined \( D \equiv p \cos D' \) and \( \overline{D} \equiv p \cos \overline{D}' \). These harmonic oscillators correspond to the six degrees of freedom in the phase space.

The left-\( SU(2) \) and the right-\( SU(2) \) currents in (17) can now be expressed in terms of the harmonic oscillators as

\[ L^3 = D, \quad L^\pm = e^{\pm iA} \sqrt{(p + D \pm \frac{h}{2})(p - D \pm \frac{h}{2})}, \]
\[ R^3 = \overline{D}, \quad R^\pm = e^{\pm i\overline{A}} \sqrt{(p + \overline{D} \pm \frac{h}{2})(p - \overline{D} \pm \frac{h}{2})}, \]

(24)

and the quadratic Casimir operators depend only on the momentum,

\[ C = -Tr(L^2) = -Tr(R^2) = (p + \frac{\hbar}{2})(p - \frac{\hbar}{2}). \]

(25)

Therefore, the irreducible representations of the symmetry algebras are labelled by the eigenvalues of the momentum, \( p = (\ell + \frac{1}{2}) \), with the spin \( \ell = 0, \frac{1}{2}, 1, \ldots \infty \). Let us denote the \((2\ell + 1)\)-dimensional irreducible representation of the left-\( SU(2) \) by \( \mathcal{V}_\ell \) and the one for the right-\( SU(2) \) by \( \overline{\mathcal{V}}_\ell \). Then, the Hilbert space can be decomposed into

\[ \mathcal{H} = \bigoplus_{\ell=0, \frac{1}{2}, 1, \ldots} \mathcal{V}_\ell \otimes \overline{\mathcal{V}}_\ell, \]

(26)

One can obtain the states in the Hilbert space as follows. Define the following orthonormal basis \( \{ |\ell; m, n\rangle \} \) with

\[ |\ell; m, n\rangle \equiv e^{i(\lambda + \frac{n}{2})m}e^{i m A}\epsilon^{inA}\epsilon^{in\overline{A}}|0\rangle. \]

(27)

The highest weight state of spin \( \ell \) is \( |\ell; \ell, \ell\rangle \) since it is annihilated by the raising operators \( L^+ \) and \( R^+ \). The rest of the states in the representation can be obtained by applying the lowering operator \( L^- \) and \( R^- \) to the highest weight state:

\[ \mathcal{V}_\ell \otimes \overline{\mathcal{V}}_\ell \equiv \{ L^-_m R^-_n |\ell; \ell, \ell\rangle, \quad m, n = -\ell, -\ell + 1, \ldots \}. \]

(28)

These current generators act on the states according to

\[ L^\pm |\ell; m, n\rangle = \sqrt{(\ell \mp m)(\ell \pm m + 1)}|\ell; m \pm 1, n\rangle, \]
\[ R^\pm |\ell; m, n\rangle = \sqrt{(\ell \mp n)(\ell \pm n + 1)}|\ell; m, n \pm 1\rangle. \]

(29)
This realisation makes it easy for us to study the action of the group element \( g(\tau) \) and the left-covariant vertex operator \( u \) on the Hilbert space. For example, in the spin \( \frac{1}{2} \) representation, \( g(\tau) \) defined in (5) is a unitary \( 2 \times 2 \) matrix (normal-ordered),

\[
g_{ab}(\tau) = a g_{ab}^+ - b g_{ab}^-, \quad \text{for} \quad a, b = \pm 1,
\]

where \( g^\pm \) takes a state of spin \( \ell \) into a new state of spin \( \ell \pm \frac{1}{2} \). To be more precise, we have

\[
g_{ab}^\pm(\tau)|\ell; m, n\rangle = \frac{1}{2} \epsilon^{\pm i(\ell + \frac{1}{2})h \tau} \sqrt{1 + \frac{n b + 1}{\ell + \frac{1}{2}} \sqrt{1 + \frac{a m}{\ell + \frac{1}{2}(1 \pm 1)}}} \left| \ell \pm \frac{1}{2}; m + \frac{a}{2}, n + \frac{b}{2} \right\rangle.
\]

On the other hand, applying the left-covariant vertex \( u \) on a state \( |\ell; m, n\rangle \) in \( \mathcal{H} \), we find that it gives the Wigner coefficients\(^6\) of \( SU(2) \) according to

\[
u_{ab}|\ell; m, n\rangle = n_{ab}(-1)^{\frac{1}{2}(1+a) - \frac{1}{2} m - \ell} \sqrt{2 \ell + 1} \left( \begin{array}{c} \ell \ \ell + \frac{3}{2} \ -m - \frac{a}{2} \\ m \ \frac{3}{2} \ -m + \frac{a}{2} \end{array} \right) \left| \ell + \frac{b}{2}; m + \frac{a}{2}, n \right\rangle \not\in \mathcal{H},
\]

where \( n_{ab} = 1 \) except for \( n_{-+} = -1 \). Unlike \( g(\tau) \), \( u \) takes a state in \( \mathcal{H} \) outside of \( \mathcal{H} \). This is because the spin of the representation is changed while the quantum number \( n \) does not. If we omit \( n \) by projecting it down to 0, then \( u \) becomes a well-defined operator in the left-projected space,

\[
P_L(\mathcal{H}) \equiv \bigoplus_{\ell \geq 0} V^L_\ell, \quad \text{where} \quad V^L_\ell \equiv \{ L^{\ell-m}|\ell; 0\rangle, \quad m = -\ell, -\ell + 1, \ldots \\}.
\]

This projected space can be identified as the Hilbert space for the left-invariant theory defined by \( u \). All the states in different representations are connected to one another by \( u \) in the following way:

\[
\begin{align*}
V^L_0 & \overset{u_{a+}}{\longrightarrow} V^L_1 \quad \overset{u_{a+}}{\longrightarrow} V^L_2 \quad \overset{u_{a+}}{\longrightarrow} \cdots \\
\overset{u_{a-}}{\longleftarrow} & \quad \overset{u_{a-}}{\longleftarrow} & \quad \overset{u_{a-}}{\longleftarrow}
\end{align*}
\]

Notice that the left-index of \( u \) labels the new representation and the right-index labels the weight of the new states.

Similarly, for the right-invariant theory, \( v \) acts on the right-projected Hilbert space with \( m = 0 \),

\[
P_R(\mathcal{H}) \equiv \bigoplus_{\ell \geq 0} V^R_\ell, \quad \text{where} \quad V^R_\ell \equiv \{ R^{\ell-n}|\ell; 0\rangle, \quad n = -\ell, -\ell + 1, \ldots \\}.
\]

The difference from \( u \) is that the right-index of \( v \) labels the new representation and the left-index labels the weight of the new states.
Another remark is that the exchange matrix $B(p)$ can be identified as the Racah (6j-symbols) matrix for $SU(2)$ when it acts on a physical state with eigenvalue, $p = \ell + \frac{1}{2}$ of $\ell > 1$.

$$B_{m,m',n'}(p) = \delta_{m+n,m'+n'}\left\{ \frac{1}{2}, \ell - \frac{n'}{2} \right. \left. \ell - \frac{n}{2} \right\} \quad \text{for } m,n = \pm 1. \quad (36)$$

Although $B$ is singular when $p$ is at the origin, i.e. $\ell = -\frac{1}{2}$, the braiding matrix $B(p)$ is well defined in $\mathcal{H}$ since this representation is excluded in $\mathcal{H}$.

**Quantum realisation of $U_t(SL(2))$**

In the previous section, we formulated the quantum theory of the left-$SU(2)$ invariant particle in terms of two pairs of harmonic oscillators. In this section, we propose a deformation of those results to give a realisation for $U_t(SL(2))$. We shall use $t = q^\frac{n}{2}$ to be the deformation parameter rather than $q$ as in [7]. This is because the defining relations for the quantum group actually depend on $q^\frac{n}{2}$ rather than $q$.

Let us recall the defining relations of the generators for $U_t(SL(2))$,

$$[H, X_{\pm}] = \pm X_{\pm}, \quad [X_+, X_-] = [2H], \quad \text{with} \quad [S] \equiv \frac{t^s - t^{-s}}{t - t^{-1}}. \quad (37)$$

The key observation to obtain a realisation of (37) is that we can modify (deform) the $L^\pm$ generators given in (24) by replacing the momentum operators $p$ with $[p]$, i.e.

$$S_3 \equiv D, \quad S_\pm \equiv e^{\pm iA} \sqrt{|p + D + \frac{\hbar}{2}|[p - D - \frac{\hbar}{2}|. \quad (38)$$

Using $[q,p] = [A, D] = i\hbar$, we can check that

$$[S_3, S_{\pm}] = \pm \hbar S_\pm, \quad [S_+, S_-] = [\hbar][2S_3]. \quad (39)$$

At the classical limit when $\hbar \to 0$, this algebra is not the usual Lie algebra $SL(2)$. In other words, the deformation occurs already at the classical level. For $\hbar = 1$, (39) gives a realisation of (37) and we shall set $\hbar = 1$ from now on. The quadratic Casimir operator depends on $p$ only,

$$C \equiv S_+ S_- + [S_3][S_3 + 1] = [p - \frac{1}{2}][p + \frac{1}{2}]. \quad (40)$$

Thus, the representations of this algebra are labelled by the eigenvalues of $[p - \frac{1}{2}] = [\ell]$. These representations are irreducible when $\ell$ is $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Using the following orthonormal basis,

$$\left\{ |\ell; m\rangle \equiv e^{i(\ell + \frac{1}{2})\eta} e^{imA} |0\rangle \right\}. \quad (41)$$

we obtain all the states in the spin $\ell$ representation by applying the lowering operator $S_-$ to the highest weight states, $|\ell; \ell\rangle$,

$$V_\ell = \{ S_-^m |\ell; \ell\rangle, \quad m = -\ell, -\ell + 1, \ldots, \ell \}. \quad (42)$$
While, the highest weight state \(|\ell; \ell\rangle\) is annihilated by the raising operators \(S_+\).

Applying the same deformation to the momenta in (22), we obtain the the following \(2 \times 2\) matrix, \(U\),

\[
U \equiv e^{t A_{\alpha \beta}} \left( \begin{array}{cc} x & -y \\ y & x \end{array} \right) e^{t q_{\alpha \beta}}, \quad \begin{cases} x \equiv t^{-(D-p)/2} \sqrt{[p + D]/[2p]}, \\ y \equiv t^{-(D+p)/2} \sqrt{[p - D]/[2p]} \end{cases}
\]  

(43)

The phase \(t^{-(D-p)/2}\) in \(x\) and a similar one in \(y\) are introduced such that \(\det(U) = 1\). Notice that \(U\) is not unitary any more under this deformation. We will call \(U\) the “Wigner” operator for \(U_t(SL(2))\) because its matrix elements between states in \(V_\ell\) and \(V_{\ell\pm}\) given the Wigner coefficients for \(U_t(SL(2))\). Thus, we have the following intertwining structure,

\[
\begin{array}{cccc}
V_0 & U_{a+} \quad & U_{a-} \quad & V_+ \\
U_{a+} & V_+ & U_{a-} & V_+ \\
U_{a-} & U_{a+} & V_+ & V_+
\end{array}
\] 

(44)

In order to study how the quantum group acts on the Hilbert space, we investigate the transformation of \(U\) under the quantum group symmetry generated by (38). Using (39) and (43), we can write down the following commutators

\[
t^{\pm S_3} U_{ab} = U_{ab} t^{\pm S_3 t^{\pm 1}/2}, \quad \forall a, b = \pm 1; \quad [t^{-S_3} S_\pm, U] = -t^{-2S_3 \pm 1/2} S_\mp U.
\]  

(45)

The second equation in (45) can also be written as

\[
S_\pm(U_{+b}, U_{-b}) = (U_{+b}, U_{-b}) \left( t^{\pm S_3} S_\pm - t^{\pm 1} S_\mp t^{-S_3} \right).
\]  

(46)

This implies that the transformation law of \(U\) under \(U_t(SL(2))\) symmetry can be formulated in terms of the coproducts; namely, for every element \(q \in U_t(SL(2))\), \(q\) acts on \(U\) in the representation of spin \(1/2\) according to

\[
q U = U(\Pi^{1/2} \otimes id) \Delta(q).
\]  

(47)

We have denoted \(\Pi^{1/2}(q)\) to be the spin \(1/2\) representation of \(q\). These cocycles are determined by the ones for the generators of \(U_t(SL(2))\),

\[
\Delta(t^{S_3}) = t^{S_3} \otimes t^{S_3}, \\
\Delta(S_\pm) = t^{S_3} \otimes S_\pm + S_\mp \otimes t^{-S_3}.
\]  

(48)

It is also possible to study the commutation relation of the deformed Wigner operator \(U\) with itself. It turns out that it obeys the following exchange relation,

\[
R U_1 U_2 = U_2 U_1 B(p),
\]  

(49)
where \( B(p = \ell + \frac{1}{2}) \) gives the Racah \( \{6j\} \)-matrix\(^8\) for \( U_t(SL(2)) \),

\[
B(p) = \begin{pmatrix}
 t^\frac{1}{2} & 0 & 0 \\
 0 & t^{-\frac{1}{2}} \sqrt{\frac{(2p+1)(2p-1)}{2p}} & -t^{-\frac{1}{2}} - \frac{2p}{2p} \\
 0 & t^{-\frac{1}{2} + 2p} \frac{1}{2p} & t^{-\frac{1}{2}} \sqrt{\frac{(2p+1)(2p-1)}{2p}} \\
 0 & 0 & t^\frac{1}{2}
\end{pmatrix},
\]

(50)

and \( R \) is the constant R-matrix for \( U_t(SL(2)) \),

\[
R = t^{-\frac{1}{2}} \begin{pmatrix}
 t & 0 & 0 \\
 0 & 1 & 0 \\
 0 & t & 0
\end{pmatrix}.
\]

(51)

As it turns out, Eq. (49) gives the operator identity for the IRF-Vertex transformation discussed in [7].

In this realisation of \( U_t(SL(2)) \), we do not restrict the value of the parameter \( t \). In the limit when \( t = 1 \), we recover the results of the undeformed model in the previous section. For the interest of the chiral WZW models, we need to consider when the deformation parameter \( t \) is a root of the unity, i.e. \( t = e^{i \frac{2\pi}{k}} \) for some positive integer \( k \). In this case, we have only a finite number of integrable representations which corresponds to \( \ell = 0, \frac{1}{2}, 1, \ldots, k/2 \). Although the braiding matrix in (50) becomes singular when \( p = 0 \) or \( p = (k + 2)/2 \), i.e. \( \ell = -\frac{1}{2} \) or \( \ell = (k + 1)/2 \), these singularities are not harmful when we consider only the integrable representations.

**Symplectic Structure and the Lagrangian formulation for \( U_t(SL(2)) \)**

We have given a simple realisation of the quantum group generators and the Wigner operator \( \mathcal{U} \) in terms of two pairs of harmonic oscillators. In order to understand the geometrical meaning of this quantum theory and the origin of the deformation, one would like to have a Lagrangian formulation for its classical theory. It seems very plausible that \( \mathcal{U} \) can be taken as the defining variable (particle) which transform covariantly under \( U_t(SL(2)) \). Indeed, by taking the classical limits of the quantum relations in the previous section, we obtain a set of Poisson brackets given by the following closed two-form in terms of \( \mathcal{U} \),

\[
\omega = i\delta q\delta p + i\delta A\delta D,
\]

\[
= \frac{i}{4\gamma} \text{Tr}\{i4\gamma\delta p\sigma^3\mathcal{U}^{-1}\delta\mathcal{U} + e^{-i2\gamma\sigma^3}\mathcal{U}^{-1}\delta\mathcal{U}e^{i2\gamma\sigma^3}\mathcal{U}^{-1}\delta\mathcal{U}
\]

\[
+ \frac{1}{4}(Z^{-1}_+\delta Z_+ + \hat{\omega}(Z^{-1}_+\delta Z_+))\hat{\omega}^{-1}((\delta Z_-Z_+^{-1} + \hat{\omega}(Z_-Z_+^{-1}))\hat{\omega}^{-1})\}.
\]

(52)

where \( \hat{\omega} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is the Weyl reflection and \( Z_+ \) and \( Z_- \) are elements in the Borel subgroups \( B_+ \) and \( B_- \) defined in

\[
Z = Z_+ Z_- \equiv \mathcal{U}e^{i2\gamma\sigma^3}\mathcal{U}^{-1}.
\]

(53)
Although $Z_\pm$ are defined up to an element $h$ in the maximal torus, i.e. $Z_+ \rightarrow Z_+ h$, and $Z_- \rightarrow h^{-1} Z_-$, the symplectic form in (52) is not affected by this re-definition. $\gamma$ is related to the deformation parameter according to $t = e^{i\gamma}$.

A similar symplectic form had been discussed by Alekseev and Malkin\cite{9} in a more general context. Their approach is to deform the Kirillov symplectic form for Lie-Poisson groups. It is plausible that our result provides an explicit example of their approach. Once the symplectic form is determined, we can choose the classical Hamiltonian $H_0$ to be quadratic Casimir given in (40) and write down the Lagrangian for the interacting particle with $U_\theta(SL(2))$ symmetry as

$$L = \text{Tr} \{ \Pi(\mathcal{U}) \partial_\tau \mathcal{U} - H_0 \},$$

(54)

where $\Pi(\mathcal{U})$ is the conjugate momentum of $\mathcal{U}$ as in the symplectic form $\omega = \text{Tr}\{\delta \mathcal{U} \delta \Pi\}$. However, it is difficult to obtain a simple way of writing the Lagrangian in terms of $\mathcal{U}$ other than a perturbative expression in powers of $\ln t$. We hope to consider this further in the near future.

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