Nucleon Observables in the Chiral Quark-Soliton Model Using

a Simple Non-local Approximation

I. Adjiuli¹*University of Oxford, Department of Physics,**Theoretical Physics, 1 Keble Road, Oxford, U.K.*

and

I.J.R. Aitchison²*Physics Department FM-15, University of Washington,**Seattle, WA 98195, U.S.A.*

&

TH Division, CERN, CH-1211, Geneva 23, Switzerland

and

J.A. Zuk

Max-Planck-Institut für Kernphysik, Heidelberg, Germany

&

Department of Physics, University of Manitoba, Winnipeg, Canada³

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¹Present address: British Telecom Laboratories, Ipswich, U.K.²Permanent address: University of Oxford, Dept. of Physics, Theoretical Physics, 1 Keble Road, Oxford, U.K.³Permanent address

Abstract

The 'two-point approximation' for strongly non-local effective actions is applied to the calculation of observables (after semi-classical quantization) in the chiral quark-soliton model of the nucleon. In particular, valence- and sea-quark contributions to the moment of inertia I and to the isovector magnetic moment μ_V are calculated in this approximation, and compared with results obtained from numerical solution of the full model. It is suggested that the approximation offers a simple and practical approach to such problems: It is a reliable guide concerning such questions as the stability of the classical soliton, and gives qualitatively correct answers for I and μ_V (the former being accurate to about 10%).

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1 Introduction

Non-local effective actions arise frequently in physics, typically as a result of integrating out certain degrees of freedom. A common strategy is to expand such an action in powers of derivatives of the remaining fields, thus obtaining an effective Lagrangian. But there are important cases where a derivative expansion fails, and it would be useful to have a reasonably simple (but still non-local) approximation to the full action.

In a previous publication [1], which we shall refer to as (I), we studied one such non-local approximation, called the *two-point approximation* [2–6]. (The reason for the name is that the action resembles a generalized two-point function, as seen from Eq. (2.3) below.) The non-local action which we sought to approximate arose in the particular context of the SU(2) chiral soliton model of the nucleon, namely

$$\Gamma = -iN_c \text{Tr} \ln [\not{\partial} - g(\phi_0 - \not{r}_5 \not{\vec{r}} \cdot \vec{\phi})], \quad (1.1)$$

where N_c is the number of colours, $\vec{\phi}$ is the pion field, and we assume the constraint

$$\phi_0^2 + \vec{\phi}^2 = f_\pi^2, \quad (1.2)$$

($f_\pi \simeq 93$ MeV). This is precisely a case in which the derivative expansion of Eq. (1.1) is known to fail [7], but numerical calculations using the full non-local Γ are available [8–11], so that the two-point approximation can be tested.

In (I), we applied this approximation to the sea-quark contribution to the static soliton energy (i.e., the Casimir energy), which is simply $\Gamma_B / \int dt$, where Γ_B is the Euclidean-space version of Γ ; and we then included this contribution in a self-consistent minimization of the total energy, regarded as a functional of the meson fields. The resulting solutions for the quark and meson fields led to values for quantities such as the total static energy which differed by no more than about 20% from those obtained in a numerical calculation using the full action [8], but which were much simpler and faster to compute — especially when a further (and excellent) ‘pole’ approximation to the two-point function was used.

This measure of success achieved by the two-point approximation encourages us to explore its applicability further. In the present paper, we shall complete the investigation begun in (I) by applying the

approximation also at the next stage of the chiral soliton model, viz. that in which physical observables are obtained after quantization of the classical soliton.

Following the standard approach of collective quantization (or cranking [6, 12]), the field of the static chiral soliton

$$f_\pi U_5(\vec{r}) \equiv \phi_0(\vec{r}) - i\gamma_5 \vec{r} \cdot \vec{\phi}(\vec{r}) \quad (1.3)$$

is replaced by the rotating field

$$\bar{U}_5(\vec{r}, t) = R(t) U_5(\vec{r}) R^\dagger(t), \quad (1.4)$$

where $R(t)$ is a time-dependent SU(2) matrix, whose elements are collective coordinates. It is convenient to introduce the collective angular velocity matrix Ω (with component variables Ω_a , $a = 1, 2, 3$) via

$$\Omega \equiv i\vec{R}^\dagger \dot{\vec{R}} \equiv \frac{1}{2} \Omega_a \tau_a. \quad (1.5)$$

When Eq. (1.4) is substituted into Eq. (1.1) and expanded up to second order in Ω , the additional term in the effective action represents a rotational kinetic energy, with a coefficient that can be interpreted as the moment of inertia due to the sea quarks (or vacuum polarization) $\mathcal{I}_{vac}$. In the present paper, we shall show how the approximation can be applied quite naturally to $\mathcal{I}_{vac}$ (Section 2) and also to the sea-quark contribution to the isovector magnetic moment $\mu_{V,vac}$ (Section 3). We denote the two-point approximations for these quantities by $\mathcal{I}_{vac}^{[2]}$ and $\mu_{V,vac}^{[2]}$.¹ We find that these two quantities are related by

$$\mu_{V,vac}^{[2]} = \frac{1}{3} \frac{\mathcal{I}_{vac}^{[2]}}{\mathcal{I}_{vac}}. \quad (1.6)$$

This relation was obtained previously [12] within the Skyrme model [13], but we stress that our present effective action is non-local. The possible generality of Eq. (1.6) beyond the Skyrme model was commented upon in Ref. 12.

In Section 4, we show how simple, easily computable expressions can be derived for $\mathcal{I}_{vac}^{[2]}$ and $\mu_{V,vac}^{[2]}$ by using the ‘pole approximation’ of (I). In the case of the static soliton energy, we found in (I) that the two-point approximation in general overestimated the sea-quark (or vacuum) contribution, and underestimated that of the valence level; thus the total energy was more accurately approximated than the two separate contributions. It is natural to investigate whether the same is also true for $\mathcal{I}$ and μ_V . In

¹The formula for $\mu_{V,vac}^{[2]}$ was given, without proof, in Ref. 6.

Section 5, we calculate the valence quark contributions to these quantities in the cranking approach, to leading order in the angular velocity, Ω . Numerical evaluations of the sea and valence quark contributions to $\mathcal{I}$ and μ_V , in two-point approximation, are presented in Section 6, and a comparison made with the full numerical calculations where available [10, 11]. We find that for $\mathcal{I}$ the same ‘compensation’ seems to operate as in the case of the static energy — the two-point approximation overestimating the vacuum contribution but underestimating the valence part; the total being within about 10% of the exact numerical value. For μ_V , however, this compensation does not occur: The approximation overestimates both contributions, so that the approximate μ_V is roughly twice the numerically exact one.

In summary, we suggest that the present calculations and those of (I) establish the utility of the two-point approximation for this type of problem.

2 The two-point moment of inertia $\mathcal{I}_{vac}^{[2]}$

As explained in (I), Γ of Eq. (1.1) may be replaced by

$$\tilde{\Gamma} = -\frac{1}{2}iN_c T\tau \ln \left[\frac{P^2 - m^2 - V}{P^2 - m^2} \right], \quad (2.1)$$

where $P_\mu = i\partial_\mu$, the contribution from the uniform vacuum has been subtracted,

$$V = ig\gamma \cdot \partial(\phi_0 - i\gamma_5 \vec{\tau} \cdot \vec{\phi}), \quad (2.2)$$

and $m = f_\pi g$. In the *two-point approximation*, one expands $\tilde{\Gamma}$ in powers of V , retaining only the second order term. This leads to the expression (c.f. Eq. (2.2) of (I))

$$\tilde{\Gamma}^{[2]} = \frac{iN_c}{4} \text{tr} \int \frac{d^4 q}{(2\pi)^4} F(q^2) \langle q|V|0 \rangle \langle 0|V|q \rangle \quad (2.3)$$

for the effective action in this approximation, where

$$F(q^2) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m^2)((p - q)^2 - m^2)} \quad (2.4)$$

is the two-point function (i.e., bubble graph), and ‘tr’ refers to the Dirac and SU(2) traces. These formulae are all fully covariant, and are valid whether or not the fields ϕ_0 and $\vec{\phi}$ depend on t . The calculation of the static energy in (I) proceeded by evaluating $\tilde{\Gamma}^{[2]}$ for the static hedgehog configuration, i.e., we took

$V = V_0 \equiv im\gamma \cdot \partial U_5$, with

$$\phi_0(\vec{r}) = f_\pi \cos \theta(r), \quad \vec{\phi}(\vec{r}) = f_\pi \hat{r} \sin \theta(r). \quad (2.5)$$

In the present case, we simply use the rotating hedgehog form $V = \bar{V} \equiv im\gamma \cdot \partial[RU_5 R^\dagger]$, which can easily be written as

$$\bar{V} = m\beta R(-i\vec{\alpha} \cdot \vec{\nabla} U_5)R^\dagger + m\beta R[\Omega, U_5]R^\dagger. \quad (2.6)$$

We now require the term proportional to Ω^2 when $\bar{V}$ is inserted for V in Eq. (2.3).

For static fields, q^0 is clearly constrained to be zero but, for time-dependent fields, it is not. However, we can expand $F(q^2)$ as a Taylor series in q^0 :

$$F(q^2) = F(\vec{q}^2) - (q^0)^2 \frac{\partial F}{\partial \vec{q}^2}(\vec{q}^2) + \dots \quad (2.7)$$

The Ω^2 term associated with the first term in Eq. (2.7) is clearly due to the second term in Eq. (2.6), which gives the contribution

$$\Gamma_{\text{rot},1}^{[2]} = \frac{iN_c m^2}{4} \int \frac{d^3 q}{(2\pi)^3} F(\vec{q}^2) \int dt \text{tr} \left([\Omega, \tilde{U}_5]^2 \right), \quad (2.8)$$

where $f_\pi \tilde{U}_5(\vec{q}) = \tilde{\phi}_0(\vec{q}) - i\gamma_5 \vec{\tau} \cdot \vec{\phi}_i(\vec{q})$, the $\vec{\phi}_a$ being the spatial Fourier transforms of the $\vec{\phi}_a$, $a = 0, 1, 2, 3$, and we have used the fact that $\tilde{\phi}_i(-\vec{q}) = -\tilde{\phi}_i(\vec{q})$. Since Ω commutes with $\tilde{\phi}_0$, Eq. (2.8) reduces to

$$\Gamma_{\text{rot},1}^{[2]} = iN_c g^2 \int \frac{d^3 q}{(2\pi)^3} F(\vec{q}^2) \phi_i^2(|\vec{q}|) \int dt \text{tr}_2 \left([\Omega, \tilde{\tau}_2 \hat{q}]^2 \right), \quad (2.9)$$

where ‘tr₂’ means the SU(2) trace only, and we have used the relation (c.f. Eq. (2.6) of (I))

$$\tilde{\phi}_i(\vec{q}) = -i\vec{q}_i \phi_i(|\vec{q}|), \quad (2.10)$$

for $i = 1, 2, 3$, where ϕ_i is the real-valued function

$$\phi_i(|\vec{q}|) = 4\pi f_\pi \int_0^\infty dr r^2 j_1(\vec{q}|\vec{r}) \sin \theta(r). \quad (2.11)$$

Turning now to the second term in Eq. (2.7), we note that we can write, in Eq. (2.3),

$$\begin{aligned} \int \frac{d^4 q}{(2\pi)^4} (q^0)^2 \int dt dt' e^{iq^0(t-t')} (\dots) &= \int \frac{dq^0}{2\pi} \int dt dt' \left[\frac{\partial}{\partial t} \frac{\partial}{\partial t'} e^{iq^0(t-t')} \right] (\dots) \\ &= \int dt dt' \delta(t-t') \frac{\partial}{\partial t} \frac{\partial}{\partial t'} (\dots). \end{aligned} \quad (2.12)$$

so that the expansion of $F(\vec{q}^2)$ in powers of $(q^0)^2$ effectively generates an expansion in time derivatives, and thus new terms involving Ω . The corresponding term of order Ω^2 arises now from the first term in Eq. (2.6), which contributes

$$\begin{aligned}\Gamma_{\nu\alpha,1}^{[2]} &= -\frac{iN_c m^2}{4} \int \frac{d^3 q}{(2\pi)^3} \frac{\partial F}{\partial \vec{q}^2} \int dt \operatorname{tr} \left([\Omega, \vec{\tau} \vec{q} \vec{U}_5(\vec{q})] [\Omega, \vec{\tau} \vec{q} \vec{U}_5(-\vec{q})] \right) \\ &= \frac{iN_c m^2}{4} \int \frac{d^3 q}{(2\pi)^3} q^2 \frac{\partial F}{\partial \vec{q}^2} \int dt \operatorname{tr} \left([\Omega, \vec{U}_5(\vec{q})]^2 \right),\end{aligned}\quad (2.13)$$

as in Eq. (2.8). Thus the complete $O(\Omega^2)$ contribution is

$$\Gamma_{\nu\alpha}^{[2]} = iN_c g^2 \int \frac{d^3 q}{(2\pi)^3} \left[F(\vec{q}^2) + q^2 \frac{\partial F(\vec{q}^2)}{\partial \vec{q}^2} \right] \phi_i^2(\vec{q}) \int dt \operatorname{tr} \left([\Omega, \vec{\tau} \vec{q}]^2 \right). \quad (2.14)$$

Using

$$\int d\vec{q} \hat{q}_i \hat{q}_j = \frac{4\pi}{3} \delta_{ij} \quad (2.15)$$

and rotating F into Euclidean space, $F \rightarrow iF_E$, we obtain finally

$$\Gamma_{E,\nu\alpha}^{[2]} = \frac{1}{2} \mathcal{I}_{\nu\alpha}^{[2]} \int dt \Omega^\alpha(t) \Omega^\alpha(t), \quad (2.16)$$

where

$$\mathcal{I}_{\nu\alpha}^{[2]} = \frac{4N_c g^2}{3\pi^2} \int_0^\infty d|\vec{q}| \vec{q}^2 \left[F_E(\vec{q}^2) + q^2 \frac{\partial F_E(\vec{q}^2)}{\partial \vec{q}^2} \right] \phi_i^2(\vec{q}) \quad (2.17)$$

is the moment of inertia due to the sea quarks, in the two-point approximation.

We note that a derivative expansion for $\mathcal{I}_{\nu\alpha}^{[2]}$ is obtained by expanding F_E in powers of $\vec{q}^2$. Retaining only the lowest order term, and using the normalization condition (Eq. (2.12) of (I))

$$4N_c g^2 F_E(0) = 1, \quad (2.18)$$

yields the leading term in the derivative expansion of the moment of inertia, namely

$$\mathcal{I}_{\nu\alpha,0}^{[2]} = \frac{8\pi f_\pi^2}{3} \int_0^\infty dr r^2 \sin^2 \theta(r). \quad (2.19)$$

Such an expansion is expected to be valid for ‘large’ solitons ($mR \gg 1$). Eq. (2.19) agrees with the leading term in the derivative expansion of the exact expression for $\mathcal{I}_{\nu\alpha}$, as given in Ref. 6, confirming that the two-point approximation for $\mathcal{I}_{\nu\alpha}$ is exact for large solitons. We note that $\mathcal{I}_{\nu\alpha,0}^{[2]}$ is precisely the contribution associated (after cranking) with the meson-field kinetic energy as calculated by Cohen and Broniowski [14]. These authors, however, started with a local effective Lagrangian, namely a (linear)

sigma model, in which the mesons are ‘elementary fields’. Eq. (2.19) then arises from the meson-field kinetic energy in their Lagrangian. Just this same contribution, from the same source, is found in the Skyrme model [13]. However, in the present (Nambu–Jona-Lasinio) model, this kinetic energy does not appear in the original Lagrangian, but is generated as the first term in a derivative expansion of Eq. (2.1) — condition (2.18) guaranteeing the correct normalization of the kinetic energy, and hence of $\mathcal{I}_{\nu\alpha,0}^{[2]}$. From the present viewpoint, $\mathcal{I}_{\nu\alpha,0}^{[2]}$ is the leading vacuum polarization contribution to $\mathcal{I}_{\nu\alpha}$, all other vacuum polarization effects being neglected in Refs. 13 and 14. This neglect will be justified if the derivative expansion for $\mathcal{I}_{\nu\alpha}$ is valid — a point we shall discuss in Section 6.1.

We may also consider the region of small soliton size, $mR \ll 1$. We recall that, in the case of the Casimir energy E_{vac} , the dominant contribution for small R comes from the two-point approximation $E_{\text{vac}}^{[2]}$ [15, 3]. Since it also dominates at large R , it was possible to suggest [5, 6] the use of $E_{\text{vac}}^{[2]}$ as an ‘interpolation formula’ for all R , and it was this idea that we examined in detail in (I). It is natural to ask whether the same kind of argument can be made for $\mathcal{I}_{\nu\alpha}^{[2]}$. The essential qualitative point is that the expansion of Eq. (2.1) is conducted in powers of $V(p^2 - m^2)^{-1}$, and since $V \sim p$, we expect this quantity to be small both for large p (small R) and small p (large R). Thus corrections to the $O(V^2)$ term will be small in both limits, and hence $\mathcal{I}_{\nu\alpha}^{[2]}$ should be accurate for both large and small R .

Before proceeding with the discussion of the likely accuracy of $\mathcal{I}_{\nu\alpha}^{[2]}$, we should perhaps emphasise, to avoid confusion, that in the physical application considered here (and probably in most others) the soliton size is not a free parameter, but is dynamically determined by minimizing the total static energy. In fully self-consistent calculations, the complete profile $\theta(r)$ is dynamically determined; in a cruder approach, one can work with a profile of fixed shape, parametrized by some size parameter R . (One can refer to Eq. (6.1) below for some examples.) In the latter case, one minimizes the energy as a function of R , which is then fixed. In all cases, one finds that the size is typically of the order m^{-1} (the inverse of the natural mass scale), and indeed one suspects that only unusual dynamics could produce a result that is otherwise. Consequently, one is generally neither in the ‘small’ nor ‘large’ regime, in which case one must hope that the interpolation provided by the two-point approximation is reasonably good at $mR \sim 1$. The answer to this can only be determined by numerical calculations, which we present in Section 6.1. Nevertheless, it is clearly important to try to establish analytically whether there are, in fact, regimes

of mR for which $\mathcal{I}_{vac}^{[2]}$ is expected to be accurate, independently of the particular value of R (typically of order m^{-1}) relevant to the physical application. A secondary question would then be to determine (numerically, in general) how large or small R has to be before such regimes are attained, and to place the physically determined R within this context.

We already know that the derivative expansion is valid for sufficiently large $R \gg m^{-1}$. Let us continue now with the discussion of the accuracy of $\mathcal{I}_{vac}^{[2]}$ for small R . There is, in fact, a complication which becomes evident from inspection of Eq. (2.17). This formula involves the two-point function F_E , defined in Eq. (2.4), which is a divergent quantity. As explained in (I), the chiral theory is understood as an effective (rather than a renormalized) one, with a physical cut-off that is adjusted via Eq. (2.18) to yield the correct kinetic energy of the meson field. Different forms of regularization scheme and cut-off may be considered, and several were discussed explicitly in (I). (See also Ref. 3.) In particular, any standard covariant scheme (e.g. Pauli-Villars, proper-time cut-off or momentum-space cut-off) applied to $F_E(\vec{q})$ implies that F_E behaves as $|\vec{q}|^{-2}$ for large $\vec{q}^2$ (effectively small R). However, this leading behaviour cancels in the combination $F_E + \vec{q}^2 \partial F_E / \partial \vec{q}^2$ required in Eq. (2.17), and hence — for such regulators — the possibility emerges that the leading behaviour of $\mathcal{I}_{vac}^{[2]}$ for small R will be two powers of R higher than would have been anticipated from F_E alone. The actual behaviour, which is analysed carefully and comprehensively in Appendix A, depends on how rapid the long-distance fall-off of the profile is. Indeed, if $\theta(r)$ decays faster than $r^{-7/2}$ as $r \rightarrow \infty$, then one readily sees from Eq. (A.12) that the small- R behaviour expected from F_E alone (assuming $F_E \sim |\vec{q}|^{-2}$ for large $\vec{q}^2$) is proportional to R^5 ; whereas, due to the cancellation just mentioned, the true behaviour is as R^7 . We show an explicit example of this in Appendix B.

This conclusion would imply that, for such rapidly decaying profiles, $\mathcal{I}_{vac}^{[2]}$ does not necessarily dominate at small R , since other terms, not present in $\mathcal{I}_{vac}^{[2]}$, would enter at this order in R . We should note, however, that in the case of the physically motivated (but mathematically unconventional) cut-off proposed by Diakonov and Petrov [16], in which the quark mass is taken to be momentum-dependent, the above cancellation will not occur (because here $F_E \sim |\vec{q}|^{-8}$ for large $\vec{q}^2$), and so the leading small- R behaviour of $\mathcal{I}_{vac}^{[2]}$ will indeed be the dominant contribution, though the relevant exponent of R will be completely different. (See Appendix A.) Furthermore, numerical calculations show that the region in R

where the R^7 behaviour of $\mathcal{I}_{vac}^{[2]}$ is actually observable is $mR \lesssim 0.01$, which is so small that $\mathcal{I}_{vac}^{[2]}$ itself is negligible, and differences between various powers of R are quite insignificant. For all practical purposes, it is sufficient that $\mathcal{I}_{vac}^{[2]}$ goes rapidly to zero with R .

When the large-distance decay of the profile $\theta(r)$ is sufficiently slow, the detailed criterion being given in Appendix A — and this always includes the physically relevant case where the fall-off is as r^{-2} — the dependence on the soliton radius R is qualitatively different and, in fact, more favourable. From Eq. (A.9), we see that the aforementioned cancellation now plays no role in determining the small- R behaviour of $\mathcal{I}_{vac}^{[2]}$. Here, $\mathcal{I}_{vac}^{[2]} \sim R \rightarrow \text{const.} \times (mR)^{2\alpha}$, where the exponent α is given by $\theta(r) \sim_{r \rightarrow \infty} \mathcal{A}(r/R)^{-\alpha}$, regardless of the regulator. Consequently, in this case, the two-point approximation $\mathcal{I}_{vac}^{[2]}$ does become exact in the small- R limit even for $F_E \sim |\vec{q}|^{-1}$. Combining these small- R considerations with the dominance of $\mathcal{I}_{vac}^{[2]}$ for large R , we expect — as in the case of $\mathcal{E}_{vac}^{[2]}$ — that $\mathcal{I}_{vac}^{[2]}$ should provide a reasonable approximation for all R . This will be tested in the numerical work reported in Section 6.

3 The two-point isovector magnetic moment $\mu_{V, \text{vac}}^{[2]}$

We need to calculate (to first order in B) the change in the static energy of the quantized soliton when it is placed in a constant external magnetic field $\vec{B}$. We introduce the external electromagnetic 4-vector potential as usual by $\partial_\mu \rightarrow \partial_\mu + iQA_\mu$, where Q is the $SU(2)$ charge operator

$$Q = \frac{1}{2N_c} + \frac{1}{2}\tau_3. \quad (3.1)$$

The ϵ -tensor-free part [17] of the resulting effective action for the rotating field (1.4) is then

$$\begin{aligned} \Gamma_A &= -\frac{iN_c}{2} T\tau \ln(\gamma \cdot \partial + iQ\gamma \cdot A - im\vec{U} \cdot \vec{\partial})(\gamma \cdot \partial + iQ\gamma \cdot A + im\vec{U} \cdot \vec{\partial}) \\ &= -\frac{iN_c}{2} T\tau \ln(p^2 - m^2 - \vec{V} \cdot 2QA \cdot P + m\gamma \cdot A(Q, \vec{U}_5)), \end{aligned} \quad (3.2)$$

where $\vec{V} = im\gamma \cdot \partial \vec{U}_5$ as before, and we are in the gauge $\partial \cdot A = 0$. We need only the term of first order in A , which is

$$\delta\Gamma_A = \frac{iN_c}{2} T\tau \left\{ \frac{1}{p^2 - m^2 - \vec{V}} (2QA \cdot P - m\gamma \cdot A(Q, \vec{U}_5)) \right\}. \quad (3.3)$$

Consider now expanding the operator $(p^2 - m^2 - \vec{V})^{-1}$ in powers of $\vec{V}$, in the spirit of the two-point

approximation already introduced for $\tilde{\Gamma}$ in the previous section. The contribution to Eq. (3.3) of the term of zeroth order in $\tilde{V}$ vanishes, while the term of order $\tilde{V}$ contributes

$$- \frac{iN_c m}{2} \text{Tr} \left\{ \frac{1}{p^2 - m^2} \tilde{V} \frac{1}{p^2 - m^2} \gamma \cdot A(\vec{Q}, \vec{U}_5) \right\}. \quad (3.4)$$

To leading (i.e., zeroth) order in the angular velocity Ω , we may consider the SU(2) matrix R (which appears in $\tilde{U}_5$ in Eq. (1.4)) to be constant, and Eq. (3.4) then reduces to

$$\delta\Gamma_{A,1}^{[2]} = - \frac{iN_c m}{2} \text{Tr} \left\{ \frac{1}{p^2 - m^2} V_0 \frac{1}{p^2 - m^2} \gamma \cdot A(\vec{Q}, U_5) \right\}, \quad (3.5)$$

where V_0 is the static expression $i m \gamma \cdot \partial U_5$, and $\vec{Q} = R^\dagger \vec{Q} R$. We note that Eq. (3.5) has two propagators $(p^2 - m^2)^{-1}$ and two 'insertions' involving U_5 , and is therefore of two-point form (c.f. Eq. (1.5) of (1)). However, another term of two-point form will arise from the $O(\tilde{V}^2)$ term in the expansion of $(p^2 - m^2 - \tilde{V})^{-1}$, namely (to zeroth order in Ω)

$$\delta\Gamma_{A,2}^{[2]} = iN_c \text{Tr} \left\{ \frac{1}{p^2 - m^2} V_0 \frac{1}{p^2 - m^2} V_0 \frac{1}{p^2 - m^2} \vec{Q} \cdot \vec{A} \cdot P \right\}. \quad (3.6)$$

The complete two-point approximation to $\tilde{\Gamma}_A$ is then the sum of Eqs. (3.5) and (3.6).

The evaluation of $\delta\Gamma_{A,1}^{[2]}$ is rather similar to that of $\tilde{\Gamma}^{[2]}$. As in Eq. (2.3), we find

$$\delta\Gamma_{A,1}^{[2]} = - \frac{iN_c m}{2} \text{tr} \int \frac{d^4 q}{(2\pi)^4} F(q^2) \langle 0 | V_0(0) \langle 0 | \gamma \cdot A(\vec{Q}, U_5) | q \rangle. \quad (3.7)$$

Since we are only interested in potentials representing constant magnetic fields, we choose $A^0 = 0$ and $\vec{A} = -\frac{1}{2} \vec{r} \times \vec{B}$. Using properties of the Dirac and SU(2) traces, as well as Eq. (2.10), we find

$$\delta\Gamma_{A,1}^{[2]} = N_c g^2 \int \frac{d^3 q}{(2\pi)^3} F(q^2) \bar{q}^a \phi_a(|\vec{q}|) \left[\frac{\partial}{\partial q^j} (\bar{q}^b \phi_b(|\vec{q}|)) \right] \varepsilon^{ijk} q^j B^k \int dt \text{tr}_2(\tau_a(\vec{Q}, \tau_b)), \quad (3.8)$$

which, using Eq. (2.15), reduces to

$$\delta\Gamma_{A,1}^{[2]} = \frac{N_c g^2}{8\pi^2} \int_0^\infty d|\vec{q}| \bar{q}^2 F(q^2) \phi_a^2(|\vec{q}|) \varepsilon^{ijk} B^k \int dt \text{tr}_2(\tau_a(\vec{Q}, \tau_j)). \quad (3.9)$$

The evaluation of $\delta\Gamma_{A,2}^{[2]}$ proceeds slightly differently, since there are three propagator factors $(p^2 - m^2)^{-1}$. Writing the functional trace in momentum space, we have

$$\begin{aligned} \delta\Gamma_{A,2}^{[2]} &= iN_c \text{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{1}{p^2 - m^2} \frac{1}{p^2 - m^2} \frac{1}{(p + q' - q)^2 - m^2} \\ &\quad \cdot \langle q | V_0(0) | 0 \rangle V_0(q) (q' - q | \vec{Q} \cdot \vec{A} \cdot p | 0). \end{aligned} \quad (3.10)$$

The matrix elements of the static V_0 constrain q^0 and q'^0 to be zero and so, for our particular A -field, we then have

$$(q' - q | \vec{Q} \cdot \vec{A} \cdot p | 0) = \frac{i}{2} \varepsilon^{ijk} p^j B^k \frac{\partial}{\partial q'^j} (2\pi)^3 \delta^3(\vec{q} - \vec{q}') \int dt \vec{Q}. \quad (3.11)$$

We also easily obtain

$$\int \frac{d^4 p}{(2\pi)^4} \frac{p^\mu}{(p^2 - m^2)^2 ((p - q)^2 - m^2)} = -q^\mu \frac{\partial F(q^2)}{\partial q^2}. \quad (3.12)$$

Manipulations similar to those leading to Eq. (3.8) now yield

$$\delta\Gamma_{A,2}^{[2]} = -2N_c g^2 \int \frac{d^3 q}{(2\pi)^3} \bar{q}^2 \frac{\partial F}{\partial \vec{q}^2} \bar{q}^a \phi_a(|\vec{q}|) \left[\frac{\partial}{\partial q^j} (\bar{q}^b \phi_b(|\vec{q}|)) \right] \varepsilon^{ijk} q^j B^k \int dt \text{tr}_2 \tau_a \tau_j \vec{Q}, \quad (3.13)$$

which reduces to

$$\delta\Gamma_{A,2}^{[2]} = - \frac{N_c g^2}{3\pi^2} \int_0^\infty d|\vec{q}| \bar{q}^2 \frac{\partial F}{\partial \vec{q}^2} \phi_a^2(|\vec{q}|) \varepsilon^{ijk} B^k \int dt \text{tr}_2 \tau_i \tau_j \vec{Q}. \quad (3.14)$$

Using the results

$$\varepsilon^{ijk} \text{tr}_2 \tau_i(\vec{Q}, \tau_j) = -2\varepsilon^{ijk} \text{tr}_2 \tau_i \tau_j \vec{Q} = -2i \text{tr}_2 \tau_a R^\dagger \tau_b R, \quad (3.15)$$

we see that Eqs. (3.9) and (3.14) combine to give, finally,

$$\delta\Gamma_A^{[2]} = \frac{1}{3} \frac{\mathcal{I}^{[2]}}{\mathcal{I}_{\text{free}}} \int dt \text{tr}_2(\tau_a R^\dagger \tau_b R) B^k, \quad (3.16)$$

where we have also used Eq. (2.17).

We can interpret Eq. (2.17) as a quantum-mechanical effective action, the elements of R being the dynamical variables as in Ref. 12. Matrix elements of the associated energy perturbation δH can then be evaluated by making use of the relation [12]

$$\langle N' | \text{tr}_2 \tau_a R^\dagger \tau_b R | N \rangle = -\frac{2}{3} \langle N' | \sigma_a \tau_b | N \rangle, \quad (3.17)$$

where N and N' are any nucleon states. We take $\vec{B}$ to be along the z -axis, and define the nucleon magnetic moment by

$$\mu_N \equiv -\frac{1}{B} \langle N | \delta H | N \rangle. \quad (3.18)$$

The isovector magnetic moment is then $\mu_V = \mu_p - \mu_n$. In this way we obtain the relation (1.6).

4 The pole approximation

We showed in (I) that the suitably regularized and cut-off two-point function was extremely well approximated by the pole form

$$F_B(\vec{q}^2) \simeq \frac{\alpha}{\vec{q}^2/m^2 + \beta^2}, \quad (4.1)$$

where from Eq. (2.18) we require $4N_c g^2 \alpha = \beta^2$. The form (4.1) actually falls within the ‘corridor’ of variation in the exact F_B represented by changes in the regularization scheme and is thus, in this sense, not really a further approximation at all.

Inserting Eq. (4.1) into Eq. (2.17), we find

$$T_{vac}^{(2)} = \frac{16\beta^4 f_\pi^2}{3} \int_0^\infty d\vec{q} \left| \frac{\vec{q}^2}{(\vec{q}^2/m^2 + \beta^2)^2} \left[\int_0^\infty d\tau \tau^2 j_1(|\vec{q}\tau| \sin \theta(\tau)) \right]^2 \right|, \quad (4.2)$$

where we have supplied the full form of $\phi_s(|\vec{q}|)$ from Eq. (2.11). In terms of the dimensionless variables

$t = |\vec{q}|/m$ and $x = m\tau$, Eq. (4.2) can be written as

$$T_{vac}^{(2)} = \frac{16\beta^4 f_\pi^2}{3} \int_0^\infty dx \int_0^\infty dt x^2 x'^2 \sin \theta(x) \sin \theta(x') \int_0^\infty dt \frac{t^2}{(t^2 + \beta^2)^2} j_1(tx) j_1(tx'). \quad (4.3)$$

The t -integral in Eq. (4.3) is very closely related to one considered in Appendix A of (I), namely

$$\int_0^\infty dt \frac{t^2}{(t^2 + \beta^2)^2} j_1(tx) j_1(tx') = \frac{\pi}{2xx'} \left(-\frac{\partial}{\partial \beta^2} G_1(x, x') \right), \quad (4.4)$$

where $G_1(x, x')$ is the coordinate-space Green's function introduced in (I), being the solution of

$$(-\partial_x^2 + 2/x^2 + \beta^2) G_1(x, x') = \delta(x - x') \quad (4.5)$$

subject to the boundary condition

$$G_1(x, x') \underset{x \rightarrow 0}{\sim} O(x^2). \quad (4.6)$$

The explicit form of $G_1(x, x')$ is given in Eq. (4.16) of (I). Performing the indicated differentiation, and returning to the variable τ , we obtain

$$T_{vac}^{(2)} = \frac{8\pi f_\pi^2}{3} \int_0^\infty d\tau \int_0^\infty d\tau' K(\tau, \tau') \sin \theta(\tau) \sin \theta(\tau'), \quad (4.7)$$

where

$$K(\tau, \tau') = \frac{1}{4\beta m} \left\{ e^{-\beta m(\tau+\tau')} [(1 + \beta m(\tau + \tau') + \beta^2 m^2 \tau \tau') (1 + \beta m(\tau + \tau')) + \beta m(\tau + \tau') + 2] - e^{-\beta m|\tau - \tau'|} [(1 + \beta m|\tau - \tau'| - \beta^2 m^2 \tau \tau') (1 + \beta m|\tau - \tau'|) + \beta m|\tau - \tau'| + 2] \right\}. \quad (4.8)$$

Eq. (4.7) provides a very convenient and compact closed-form expression for $T_{vac}^{(2)}$, which is essentially a non-local version of Eq. (2.19); from Eq. (4.8), the range of the non-locality is of order $(\beta m)^{-1}$. Since

$$K(\tau, \tau') \underset{\beta m \rightarrow \infty}{\sim} \delta(\tau - \tau') + \delta(\tau + \tau'), \quad (4.9)$$

we see that for large soliton size, $R \gg (\beta m)^{-1}$, Eq. (4.7) reduces to Eq. (2.19) as expected, while in general $T_{vac}^{(2)} < T_{vac,0}^{(2)}$. We observe that Eq. (2.19) scales with soliton size R (homogeneously) as R^3 . The detailed form of the behaviour for small soliton size R , in the pole-approximation, is summarised in Eq. (A.18) of Appendix A. For the self-consistent soliton solution, the leading small- R behaviour is easily seen to be R^4 (Eq. (A.20)). Equations (4.7) and (4.8) comprise the form that we shall use in numerical calculations, which we shall describe in Section 6.

5 Valence quark contributions

To calculate the contribution of the valence quarks to nucleon observables, we need to consider the Dirac equation for the valence state in the field of the rotating (i.e., cranked) hedgehog. The necessary calculations are similar to those of Cohen and Broniowski [14], but for convenience we outline the main steps here, not least because our notational conventions (which are those of Ref. 1 as far as possible) differ from those of Ref. 14. The single-particle Hamiltonian for the static problem is given by

$$H = \vec{\alpha} \cdot \vec{p} + m\beta(\cos \theta(\tau) - i\gamma_5 \vec{\tau} \cdot \vec{\tau} \sin \theta(\tau)) \quad (5.1)$$

and the valence wave-function ψ_{val} , which satisfies $H\psi_{val} = E_{val}\psi_{val}$, is written as

$$\psi_{val}(\vec{r}) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} F(r)|\xi\rangle \\ iG(r)\vec{\sigma} \cdot \vec{r}|\xi\rangle \end{pmatrix} \quad (5.2)$$

where the spin-isospin state $|\xi\rangle$ satisfies

$$(\vec{\sigma} + \vec{\tau})|\xi\rangle = 0, \quad \langle \xi|\xi\rangle = 1, \quad (5.3)$$

while F and G are normalized according to

$$\int_0^\infty dr r^2 [F^2(r) + G^2(r)] = 1. \quad (5.4)$$

The time-dependent Dirac equation for the cranked problem is

$$(\vec{\alpha} \cdot \vec{p} + m\beta\vec{U}_s)\Psi_{val}(\vec{r}, t) = i\frac{\partial}{\partial t}\Psi_{val}(\vec{r}, t). \quad (5.5)$$

A sufficiently general form [14] of the SU(2) rotation matrix $R(t)$ appearing in $\vec{U}_s$ is

$$R(t) = R_0 e^{i\vec{\tau} \cdot \vec{\omega} t/2}, \quad (5.6)$$

where R_0 is a time-independent SU(2) matrix and the angular velocity $\vec{\omega}$ is constant. For the purpose of extracting an expression for the moment of inertia, we may further restrict this form to the special case $R_0 = 1$. In any case, Eq. (5.5) can be re-expressed, in terms of the wave-function in the rotating frame

$$\Psi'_{val} = R^{-1}\Psi_{val} \text{ as}$$

$$H'\Psi'_{val}(\vec{r}, t) = i\frac{\partial}{\partial t}\Psi'_{val}(\vec{r}, t) \quad (5.7)$$

with

$$H' = H - iR^\dagger \dot{R} = H + \vec{\omega} \cdot \vec{I}, \quad (5.8)$$

where $I^a = \tau^a/2$ denote the isospin generators and the last equality in Eq. (5.8) follows from the form (5.6) for general R_0 .

Noting that H' is time-independent, we seek stationary solutions of Eq. (5.7), viz.

$$H'\psi'_{val}(\vec{r}) = E'_{val}\psi'_{val}(\vec{r}), \quad (5.9)$$

calculated perturbatively in ω . Thus, let us write

$$\begin{aligned} \psi'_{val} &= \psi'_0 + \delta\psi', \\ E'_{val} &= E_{val} + \delta E_{val}, \end{aligned} \quad (5.10)$$

where E_{val} and ψ'_0 are given by $H\psi'_0 = E_{val}\psi'_0$. Then, to first order in ω , the perturbed eigenvalue equation reads

$$(\vec{\omega} \cdot \vec{I})\psi'_0 + H\delta\psi' = E_{val}\delta\psi' + \delta E_{val}\psi'_0. \quad (5.11)$$

This, combined with the identity $\langle\psi'_0|\vec{\omega} \cdot \vec{I}|\psi'_0\rangle = 0$, leads to $\delta E_{val} = 0$, so that we are left with the equation

$$(H - E_{val})\delta\psi' = -(\vec{\omega} \cdot \vec{I})\psi'_0. \quad (5.12)$$

The general form of $\delta\psi'$ is [14]

$$\delta\psi'(\vec{r}) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} [A(\vec{r})\vec{\omega} \cdot \vec{\sigma} + B(\vec{r})(\frac{1}{2}\vec{\omega} \cdot \vec{\sigma} - \vec{\omega} \cdot \vec{r} \times \vec{\sigma})][\xi] \\ [iC(\vec{r})\vec{\omega} \cdot \vec{\tau} - D(\vec{r})(\vec{\omega} \times \vec{r}) \cdot \vec{\sigma}][\xi] \end{pmatrix} \quad (5.13)$$

where the spin-isospin state $|\xi\rangle$ is that given in Eq. (5.3). When this form is inserted into Eq. (5.12) one

obtains a set of coupled ordinary differential equations for the radial wave-functions A, B, C and D . It

is convenient to write these in dimensionless form, as in Eqs. (3.9) and (3.10) of (I). We define $z = mr$,

$u(z) = m^{-3/2}F(\vec{r})$, $v(z) = m^{-3/2}G(\vec{r})$ and similarly $(a, b, c, d) = m^{-1/2}(A, B, C, D)$. Then, we find

$$\begin{aligned} \frac{da}{dz} &= \frac{1}{3}\sin\theta\left(a + \frac{4}{3}b\right) - \frac{1}{3}(\cos\theta + \epsilon_{val})(c - 2d) - \frac{1}{2}v, \\ \frac{db}{dz} &= -\frac{3}{2}b + \sin\theta\left(2a - \frac{1}{3}b\right) + (\cos\theta + \epsilon_{val})(c + d), \\ \frac{dc}{dz} &= \frac{2}{z}d + \sin\theta c + (\cos\theta - \epsilon_{val})\left(a - \frac{2}{3}b\right) + \frac{1}{2}u, \\ \frac{dd}{dz} &= \frac{1}{z}(c - d) - \sin\theta d + (\cos\theta - \epsilon_{val})\left(a + \frac{1}{3}b\right) - \frac{1}{2}u, \end{aligned} \quad (5.14)$$

where $\epsilon_{val} = E_{val}/m$. These equations must be solved subject to the boundary conditions that the functions are regular at $z = 0$ and tend to zero exponentially at large z ; more details are given in Appendix C.

The valence-quark contributions to the angular momentum and isospin are given by

$$J_{val}^k = N_c \int d^3r \Psi_{val}^\dagger \left[\frac{1}{2}\sigma^k - i(\vec{r} \times \vec{\nabla})^k \right] \Psi_{val}, \quad (5.15)$$

$$I_{val}^a = N_c \int d^3r \Psi_{val}^\dagger \left(\frac{1}{2}\tau^a \right) \Psi_{val}. \quad (5.16)$$

The moment of inertia due to the valence quarks can be defined by the equation $J_{val}^k = I_{val}\omega^k$, valid to lowest order in the angular velocity. Having set $R_0 = 1$, it also follows that [14] $I_{val}^a = -I_{val}\omega^a$. Thus, it is convenient to evaluate the expression

$$I_{val}\omega^2 = -\vec{\omega} \cdot \vec{I} = -\frac{N_c}{2} \int d^3r \Psi_{val}^\dagger \vec{\omega} \cdot \vec{\tau} \Psi_{val}. \quad (5.17)$$

The integrand in Eq. (5.17) can be easily expressed in terms of the primed quantities whence, to leading order in ω , we find

$$I_{val}\omega^2 = -N_c \int d^3r (\delta\psi')^\dagger \vec{\omega} \cdot \vec{\tau} \psi'_0. \quad (5.18)$$

On inserting expressions (5.2) and (5.13) for ψ'_0 and $\delta\psi'$, respectively, into Eq. (5.18), this leads to the

result

$$I_{\text{val}} = \frac{N_c}{m} \int_0^\infty dx x^2 \left[au + \frac{1}{3}(c - 2d)v \right]. \quad (5.19)$$

For the valence-quark contribution to the isovector magnetic moment of the nucleon (defined by

$\mu_V = \mu_p - \mu_n$), we must evaluate the expression

$$\mu_{N,\text{val}} = N_c \langle N \uparrow | \frac{1}{2} \int d^3r \vec{r} \times \vec{V}_3 | N \uparrow \rangle, \quad (5.20)$$

where here

$$V_3^\mu = \frac{1}{2} \bar{\Psi}_{\text{val}} \gamma^\mu \tau_3 \bar{\Psi}_{\text{val}}. \quad (5.21)$$

We need only compute this to zeroth order in ω , since

$$V_3^k = \frac{1}{2} (\psi_0^\dagger)^\dagger R_0^\dagger \gamma^0 \tau_3 R_0 \psi_0' + O(\omega^2), \quad (5.22)$$

keeping R_0 general now. Writing

$$R_0^\dagger \tau_3 R_0 = \sum_i \frac{1}{2} \tau_i \text{tr}(\tau_i R_0^\dagger \tau_3 R_0), \quad (5.23)$$

and recalling Eq. (3.17), we obtain

$$\mu_{V,\text{val}} = \frac{2N_c}{9m} \int_0^\infty dx x^3 uv. \quad (5.24)$$

6 Numerical results

6.1 Sea-quark contributions

Our numerical evaluations will make use of the pole approximation, both for $I_{\text{vac}}^{[2]}$ as given by Eq. (4.7), and for the self-consistent profile $\theta^{[2]}(\tau)$ (in the two-point approximation) as described in Section 4 of (1). We shall use relation (1.6) for $\mu_{V,\text{vac}}^{[2]}$. We begin, however, by discussing the leading term in the derivative expansion of I_{vac} , namely $I_{\text{vac},0}^{[2]}$ as given by Eq. (2.19).

As in the case of the static soliton energy, it is important to examine the reliability, or otherwise, of the derivative expansion for I_{vac} . Diakonov *et al.* [6] argued that I is expected to be sensitive to the outer part of the nucleon, where the profile function $\theta(\tau)$ is more slowly varying, and hence that a derivative

expansion of I_{vac} should be valid. Indeed, these authors calculated $I_{\text{vac},0}^{[2]}$ using a particular profile of fixed shape with one variational parameter, and found that it agreed well with the empirical value of I (as determined from the Δ - N mass difference); they also asserted that the valence-quark contribution could be neglected. We shall see, however, that the numerical agreement in Ref. 6 is fortuitous — the true physical situation as regards the sea and valence contributions being different from that suggested in Ref. 6. We also note, concerning $I_{\text{vac},0}^{[2]}$ that this term is identical with the entire meson-field contribution as calculated by Cohen & Broniowski [14] and that, as pointed out after Eq. (2.19), the accuracy of their calculation (in this respect) depends upon the validity of the derivative expansion for I_{vac} .

Turning then to the numerical comparisons, we show in Table 1, for a representative quark mass of $m = 372$ MeV, the values of $I_{\text{vac}}^{[2]}$ and of $I_{\text{vac}}^{[2]}$ (in fm) for five different one-parameter profiles of fixed shape, defined as follows:

$$\begin{aligned} \text{Stereo } n : \quad \theta(\tau) &= 2 \arctan(R/\tau)^n, \quad n = 2, 3 \\ \text{Exp} : \quad \theta(\tau) &= \pi \exp(-\tau/R), \\ \text{Linear} : \quad \theta(\tau) &= \begin{cases} \pi(1 - \tau/R) & \text{for } \tau \leq R \\ 0 & \text{for } \tau > R, \end{cases} \\ \text{Tanh} : \quad \theta(\tau) &= \pi[1 - \tanh(\tau/R)]. \end{aligned} \quad (5.1)$$

In each separate case, the parameter R was determined by minimizing the total static energy as calculated in two-point approximation. The ‘Stereo 2’ profile is the one used by Diakonov *et al.* [6], and it can be seen from the Table that it gives the largest value of $I_{\text{vac},0}^{[2]}$. For comparison, the empirical value of I , as determined from the Δ - N mass difference, is $I \simeq 1.006$ fm, which is indeed in fair agreement with the ‘Stereo 2’ value (and it could be improved by somewhat reducing m). It is clear that $I_{\text{vac},0}^{[2]}$ depends quite sensitively on the detailed form of $\theta(\tau)$ — varying by more than a factor of two. By contrast, the variation in the static soliton energy, for the same profiles, is of the order of 10% or less. (See Table 2 of (1).) Table 1 also shows that the same sensitivity to the profile function holds for the full two-point approximation to I_{vac} . Thus, for essentially the same energy, quite different values of $I_{\text{vac}}^{[2]}$ can be obtained with these one-parameter profiles. We expect this to be a feature of the exact moment of inertia functional.

In Figure 1, we have plotted graphs of $I_{\text{vac}}^{[2]}$, normalized by the lowest-order derivative contribution

$\mathcal{I}_{vac,0}^{[2]}$ versus mR for each of the five one-parameter profiles listed in Eq. (6.1). The approximation of $\mathcal{I}_{vac}^{[2]}$ by $\mathcal{I}_{vac,0}^{[2]}$ is good when these curves attain values close to unity, but we observe that this typically does not happen to within 10% until mR is at least twice the value at which the energy functional is minimised (as given in the second column of Table 1), and hence far from the region of interest.

Of course, it is much more satisfactory to use the self-consistent $\theta(r)$ that solves the Euler-Lagrange equation, including the full sea contribution in the energy functional. Strictly within the context of the two-point approximation, this means using what we shall call $\theta^{[2]}(r)$ — the self-consistent profile calculated in (1) in two-point approximation for the sea energy. However, it is also interesting to evaluate $\mathcal{I}_{vac,0}^{[2]}$ and $\mathcal{I}_{vac}^{[2]}$ using the exact profile $\theta^{ex}(r)$, obtained by using the full sea energy [18]. (Note that quantisation is not considered in this reference.) In Table 2, we show the values of $\mathcal{I}_{vac,0}^{[2]}$ and $\mathcal{I}_{vac}^{[2]}$ calculated with $\theta^{[2]}(r)$ and with $\theta^{ex}(r)$ [18], for $m = 350, 400$ and 500 MeV, together with results with the results of an exact evaluation of $\mathcal{I}_{vac}$ as reported in Ref. 10. Another exact evaluation of $\mathcal{I}_{vac}$ is available in Fig. 1 of Ref. 11, which shows somewhat smaller values in the range $0.25 - 0.30$ fm.

We may draw a number of conclusions from these results. First, in all cases we have

$$\mathcal{I}_{vac,0}^{[2]} > \mathcal{I}_{vac}^{[2]} > \mathcal{I}_{vac}^{ex} \quad (6.2)$$

where by $\mathcal{I}_{vac}^{ex}$ we mean the exactly calculated value. The first inequality can be understood analytically by referring to Eq. (2.17) and observing that, for all values of $\vec{q}^2$,

$$0 < F_B(\vec{q}^2) + \vec{q}^2 \frac{\partial}{\partial \vec{q}^2} F_B(\vec{q}^2) < F_B(0), \quad (6.3)$$

being significantly less than the upper bound in the region $\vec{q}^2 \sim m^2$ where the term $\phi^2(|\vec{q}|)$ will be peaked in a self-consistent or variational calculation; whereas, to leading order in the derivative expansion, the expression above is replaced precisely by its maximum value. Thus, the leading order term $\mathcal{I}_{vac,0}^{[2]}$ is always an overestimate, and $\mathcal{I}_{vac}^{[2]}$ provides a better estimate, at least for the range of m -values considered. Indeed, for the approximate profile $\theta^{[2]}$, $\mathcal{I}_{vac,0}^{[2]}$ is roughly a factor of two times the exact $\mathcal{I}_{vac}$, while $\mathcal{I}_{vac}^{[2]}$ lies midway between the two. For this profile, the derivative expansion is not a good approximation. However, it is interesting to note that the use of the exact profile θ^{ex} reduces both $\mathcal{I}_{vac,0}^{[2]}$ and $\mathcal{I}_{vac}^{[2]}$, bringing the latter quite close to $\mathcal{I}_{vac}^{ex}$. It also somewhat reduces the difference between $\mathcal{I}_{vac,0}^{[2]}$ and $\mathcal{I}_{vac}^{[2]}$, suggesting that the derivative expansion for $\mathcal{I}_{vac}$ is reasonably good if the exact profile is used, as was

argued in Ref. 6¹. Nevertheless, the sensitivity of all quantities to the detailed form of the profile makes it difficult to draw a general conclusion on this point.

We can now understand the status of the calculation of $\mathcal{I}_{vac}$ presented in Ref. 6, based only on $\mathcal{I}_{vac,0}^{[2]}$ and evaluated for the ‘Stereo 2’ profile. From Tables 1 and 2, we see that for this profile, $\mathcal{I}_{vac,0}^{[2]}$ is a serious overestimate of $\mathcal{I}_{vac}$, being more than twice too large; the derivative expansion is not a reliable approximation for this profile. The large numerical value obtained for $\mathcal{I}_{vac,0}^{[2]}$ close to the empirical value of $\mathcal{I}$, led to the erroneous conclusion that $\mathcal{I}_{vac}$ could be neglected. In fact, as we shall see, $\mathcal{I}_{vac}$ and $\mathcal{I}_{vac}$ are generally of the same order of magnitude (which is why the overestimate in Ref. 6 could mimic the correct result).

In the calculation of Cohen & Broniowski [14], $\mathcal{I}$ is the sum of $\mathcal{I}_{vac,0}^{[2]}$ and $\mathcal{I}_{vac}$, with θ calculated self-consistently, but neglecting all vacuum polarisation effects except for the meson-field kinetic energy. Using $m = 500$ MeV, they found $\mathcal{I}_{vac,0}^{[2]} \simeq 0.725$ fm. It is difficult to compare their results directly with ours or with Ref. 10 since, among other things, they used a linear rather than a non-linear σ -model for the meson fields; however, our results would certainly suggest that the vacuum polarisation contribution to $\mathcal{I}$ of Cohen & Broniowski is a considerable overestimate. It is probable that this is compensated by an underestimate of the valence contribution to $\mathcal{I}$ (for which they obtained 0.445 fm), since such compensation does emerge (for $\mathcal{I}$, but not for $\mu\nu$) in our own calculations of the valence contributions, to which we now turn.

6.2 Valence contributions and totals

We obtain $\mathcal{I}_{vac}$ via Eq. (5.19) from the solutions of Eqs. (5.14) and the unperturbed wavefunctions u and v . As in the case of $\mathcal{I}_{vac}$, we shall consider both a ‘consistent’ approach to the two-point approximation, in which we insert u, v, ε_{vac} and $\theta^{[2]}$ from (1) into Eqs. (5.14), and also an alternative evaluation in which Eqs. (5.14) are solved using θ^{ex} and the corresponding ε_{vac} , u and v from Ref. 18.

Figure 2 shows a, b, c and d for $m = 400$ MeV, in the consistent two-point approximation, using the

¹ The same conclusion can be inferred from Fig. 1 of Ref. 11, despite a statement in the text to the contrary.

7 Conclusions

The present study, and that of its companion (I), has been undertaken principally in order to test the practicality and reliability of a simple non-local approximation to a particular non-local action which arises in a realistic physical problem. The example is one in which numerical solutions to the full problem have been obtained, thus enabling our approximation to be tested. We have shown that the approximation can be straightforwardly applied both at the stage of obtaining the 'classical' soliton (but including the vacuum polarization contribution to the energy) and at the subsequent stage of quantization. As regards the first stage, it is worth emphasizing again that the approximation does succeed in producing a stable soliton — something that the derivative expansion fails to do — and the gross properties of the nucleon are within about 20% of those obtained from the exact calculations. Similarly, the approximation is qualitatively successful when applied to the semi-classical quantization of the soliton, leading to simple expressions for I_{vac} and $\mu_{V,vac}$. It would be nice to understand why the valence- and sea-quark contributions seemed to compensate for I (as happens also for the total static soliton energy in (I)) but not for μ_V , since then we might have a general understanding of the likely accuracy of the approximation in calculating such quantities. Nevertheless, even in the absence of this, we would still suggest that the approximation will be a reasonable guide to the values of appropriate physical observables in similar problems.

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results of (I) (from which we have $\epsilon_{val} = 0.244$ MeV for this value of m). Though different in some details, these functions are qualitatively similar to those shown in Figure 2(a) of Ref. 14, and they are representative of our two-point results for all values of m ; those using θ^{ss} are also similar. The corresponding results for I_{val} for $m = 350, 400$ and 500 MeV are shown in Table 3, which also reproduces the I_{vac} contributions from Table 2, and lists the totals for I . The first three rows are the results obtained using the two-point approximation throughout; the second three rows are those using θ^{ss} (and ϵ_{val} , u and v) obtained in Ref. 18; and the three remaining rows are quoted from the full calculations of Ref. 10.

Comparing the first and seventh rows of Table 3, we see that in all cases I_{val} is underestimated in the two-point approximation — by about 20%. However, as noted in the previous section, I_{vac} is overestimated (as seen from rows two and eight), such that there is a tendency towards compensation, and the values for the total $I^{[2]}$ are within about 10% of $I^{[2]}$ (rows three and nine). This is a satisfactory result, but we cannot claim that the two-point approximation will generally be so accurate, since we have not been able to understand this compensation from a general viewpoint — and indeed it does not occur for μ_V , as we shall see shortly. The fourth row of Table 3 shows I_{val} calculated using θ^{ss} etc. from Ref. 18. We believe that this row should in principle equal the seventh row, since the basic physics (viz. the cranking approximation) is the same, although it is implemented differently in our Section 5 and in Section 4 of Ref. 10. (However, it may be noted that the exact results do not quite agree among themselves, the values for I_{val} given in Ref. 11 lying some 10% above those of Ref. 10; again, there is a kind of compensation, and the total I for Refs. 10 and 11 are very similar.) Since, as we saw previously, $I_{val}^{[2]}$ is still overestimated even using θ^{ss} (row five), the totals of row six all lie above those of row nine, though not by more than 10%.

Table 4 collects together the corresponding results for μ_V . This time we see that $\mu_{V,val}$ is overestimated when the 'two-point' wavefunctions u and v of Ref. 1 are used in Eq. (5.24) (though our value is actually very similar to that of Ref. 14). Since, as we saw earlier, $\mu_{V,vac}^{[2]}$ is also an overestimate, the resulting total $\mu_V^{[2]}$ is about twice as large as the value obtained in Ref. 10. Again, we believe that line four should in principle be the same as line seven. We see no reason to attach any significance to the fact that $\mu_V^{[2]}$ is actually much closer to the experimental value than is $\mu_V^{[2]}$.

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Appendix A

In this Appendix, we study the asymptotic behaviour of the two-point approximation to I_{vac} for hedgehog solitons of small size. For this purpose, we shall consider background hedgehog configurations described by fixed profile functions $\theta(r/R)$ parameterized by their radius R , in the limit $R \rightarrow 0$. A similar analysis for the energy functional has been carried out in Ref. 3. As was done there, we suppose that the long-distance behaviour of the profile is given by the power-law fall-off

$$\theta(z) \underset{z \rightarrow \infty}{\sim} \mathcal{A} z^{-\alpha}, \quad (\text{A.1})$$

where here $z = r/R$. From Eq.(2.17), we see that the expression for $I_{vac}^{[2]}$ may be cast into the form

$$I_{vac}^{[2]} = \frac{16\pi m^2}{3g^2} R^6 \int_0^\infty dq q^2 f(q^2/m^2) I^2(qR), \quad (\text{A.2})$$

In this Appendix, we write $q \equiv |\vec{q}|$. We have defined

$$f(q^2/m^2) \equiv \frac{1}{F_E(0)} \left[F_E(q^2) + q^2 \frac{\partial F_E(q^2)}{\partial q^2} \right], \quad (\text{A.3})$$

so that $f(0) = 1$. Also, (c.f. Eq. (4.5) of Ref. 3)

$$I(\kappa) \equiv \int_0^\infty dz z^2 \hat{f}_1(\kappa z) \sin \theta(z). \quad (\text{A.4})$$

Let us now introduce the dimensionless variables $\rho \equiv \beta m R$, $t \equiv q/\beta m$, so that $qR = \rho t$, where β is an arbitrary scale factor. Then

$$I_{vac}^{[2]} = \frac{16\pi}{3\beta^3 g^2 m} \rho^6 \int_0^\infty dt t^2 f(\beta^2 t^2) I^2(\rho t). \quad (\text{A.5})$$

Next, we introduce δ such that $0 < \delta < 1$ and take $\rho \ll 1$. After splitting the t -integral at $t = \rho^{-\delta}$ into two intervals, it follows that

$$\begin{aligned} I_{vac}^{[2]} &\underset{\rho \rightarrow 0}{\sim} \frac{16\pi}{3\beta^3 g^2 m} \rho^6 \int_0^{\rho^{-\delta}} dt t^2 f(\beta^2 t^2) I^2(\rho t) + \frac{16\pi}{3\beta^3 g^2 m} \rho^3 \int_{\rho^{-\delta}}^\infty dt t^2 f_{asymp} \left(\frac{\beta^2 t^2}{\rho^2} \right) I^2(t) \\ &\equiv (I) + (II). \end{aligned} \quad (\text{A.6})$$

We have defined $I_{asymp}(\kappa)$ such that $I(\kappa) \sim_{\kappa \rightarrow 0} I_{asymp}(\kappa)$, so that $I_{asymp}(\kappa)$ is the leading asymptotic term for small κ . Its explicit form can be extracted from the analysis in Section 4 of Ref. 3 where the same quantity arose, and is given by

$$I_{asymp}(\kappa) = \begin{cases} -\left(\frac{\Gamma(3-\alpha)}{\alpha-1} \cos \frac{\alpha\pi}{2}\right) \mathcal{A} \kappa^{\alpha-3} & \text{for } 1 < \alpha < 4 \\ \frac{\mathcal{A}}{3} \kappa \ln \frac{1}{\kappa} & \text{for } \alpha = 4 \\ \frac{\kappa}{3} \int_0^\infty dz z^3 \sin \theta(z) & \text{for } \alpha > 4. \end{cases} \quad (\text{A.7})$$

Also, $f_{asymp}(\beta^2 t^2)$ is defined to be the leading asymptotic term of $f(\beta^2 t^2)$ in the large- t^2 limit, and we assume the form

$$f_{asymp}(\beta^2 t^2) = \frac{f_\infty}{t^{2\nu}}. \quad (\text{A.8})$$

The term (I) dominates the $\rho \rightarrow 0$ behaviour for (i) $\frac{3}{2} < \alpha < \nu + \frac{3}{2}$ with $\alpha < 4$, in which case

$$I_{vac}^{[2]} \underset{\rho \rightarrow 0}{\sim} \frac{16\pi \mathcal{A}^2}{3\beta^3 g^2 m} \left(\frac{\Gamma(3-\alpha)}{\alpha-1} \cos \frac{\alpha\pi}{2} \right)^2 \left(\int_0^\infty dt t^{2(\alpha-2)} f(\beta^2 t^2) \right) \rho^{2\alpha}; \quad (\text{A.9})$$

(ii) $\alpha = 4$ with any $\nu > \frac{5}{2}$, in which case

$$I_{vac}^{[2]} \underset{\rho \rightarrow 0}{\sim} \frac{16\pi \mathcal{A}^2}{27\beta^3 g^2 m} \left(\int_0^\infty dt t^4 f(\beta^2 t^2) \right) \rho^6 \ln^2 \rho; \quad (\text{A.10})$$

and (iii) $\alpha > 4$ with any $\nu > \frac{5}{2}$, in which case

$$I_{vac}^{[2]} \underset{\rho \rightarrow 0}{\sim} \frac{16\pi}{27\beta^3 g^2 m} \left(\int_0^\infty dz z^3 \sin \theta(z) \right)^2 \left(\int_0^\infty dt t^4 f(\beta^2 t^2) \right) \rho^6. \quad (\text{A.11})$$

The term (II) dominates the $\rho \rightarrow 0$ behaviour for (i) $\alpha > \nu + \frac{3}{2}$ with $\alpha < 4$ and (ii) $\alpha > 4$ with $\nu < \frac{5}{2}$, which in both cases gives rise to

$$I_{vac}^{[2]} \underset{\rho \rightarrow 0}{\sim} \frac{16\pi f_\infty}{3\beta^3 g^2 m} \left(\int_0^\infty d\kappa \kappa^{2(1-\nu)} I^2(\kappa) \right) \rho^{3+2\nu}. \quad (\text{A.12})$$

Both the terms (I) and (II) contribute to the $\rho \rightarrow 0$ behaviour for (i) $\alpha = \nu + \frac{3}{2}$ with $\alpha < 4$, in which case

$$I_{vac}^{[2]} \underset{\rho \rightarrow 0}{\sim} \frac{32\pi \mathcal{A}^2 f_\infty}{3\beta^3 g^2 m} \left(\frac{\Gamma(3/2-\nu)}{2\nu+1} (1 + \sin \nu\pi) \right)^2 \rho^{3+2\nu} \ln \frac{1}{\rho}; \quad (\text{A.13})$$

(ii) $\alpha > 4$ with $\nu = \frac{5}{2}$, in which case

$$I_{vac}^{[2]} \underset{\rho \rightarrow 0}{\sim} \frac{16\pi f_\infty}{27\beta^3 g^2 m} \left(\int_0^\infty dz z^3 \sin \theta(z) \right)^2 \rho^6 \ln \frac{1}{\rho}; \quad (\text{A.14})$$

and (iii) $\alpha = 4$ with $\nu = \frac{5}{2}$, in which case

$$\mathcal{I}_{vac}^{[2]} \sim_{\rho \rightarrow 0} \frac{16\pi A^2 f_{\infty}}{81\beta^3 g^2 m} \rho \ln^3 \frac{1}{\rho}. \quad (\text{A.15})$$

For the pole approximation (introduced in Eq. (4.1)), the function $f(\beta^2 t^2)$ clearly assumes the form

$$f(\beta^2 t^2) = \frac{1}{(t^2 + 1)^2} \sim_{t \rightarrow \infty} \frac{1}{t^4}. \quad (\text{A.16})$$

Thus, we see that $f_{\infty} = 1$ and $\nu = 2$. (Note that we have identified the scale parameter β with the parameter of the same name entering the pole form (4.1) of the two-point function.) Specialising the foregoing general discussion to these particular values, and taking account of the integral

$$\int_0^\infty dt t^{2(\alpha-2)} f(\beta^2 t^2) = \int_0^\infty dt \frac{t^{2(\alpha-2)}}{(t^2 + 1)^2} = \frac{\pi}{2} \left(\frac{5}{2} - \alpha \right) \frac{1}{\cos \alpha \pi}, \quad (\text{A.17})$$

we obtain the following results for the pole approximation:

$$\begin{aligned} \frac{3}{2} < \alpha < \frac{7}{2} &: \mathcal{I}_{vac}^{[2]} \sim_{\rho \rightarrow 0} \frac{4\pi^2 A^2}{3\beta^3 g^2 m} \cos \alpha \pi \left(\frac{\Gamma(3-\alpha)}{\alpha-1} \cos \frac{\alpha \pi}{2} \right)^2 \rho^{2\alpha}, \\ \alpha = \frac{7}{2} &: \mathcal{I}_{vac}^{[2]} \sim_{\rho \rightarrow 0} \frac{128\pi A^2}{75\beta^3 g^2 m} \rho^7 \ln \frac{1}{\rho}, \\ \alpha > \frac{7}{2} &: \mathcal{I}_{vac}^{[2]} \sim_{\rho \rightarrow 0} \frac{16\pi}{3\beta^3 g^2 m} \left(\int_0^\infty d\kappa \kappa^{-2} \mathcal{I}^2(\kappa) \right) \rho^7, \end{aligned} \quad (\text{A.18})$$

and, in the last case, we have

$$\int_0^\infty d\kappa \kappa^{-2} \mathcal{I}^2(\kappa) = \frac{\pi}{6} \int_0^\infty dx \int_0^\pi dx' \left(x^2 + \frac{1}{2} x'^2 \right) x'^3 \sin \theta(x) \sin \theta(x'). \quad (\text{A.19})$$

For the physically relevant value of $\alpha = 2$, we have

$$\mathcal{I}_{vac}^{[2]} \sim_{\rho \rightarrow 0} \frac{4\pi^2 A^2}{3\beta^3 g^2 m} \rho^4. \quad (\text{A.20})$$

Note that $A = 2$ for the ‘Stereo 2’ profile. Finally, we point out that profiles with $\alpha \leq \frac{5}{2}$ have not been considered since $\mathcal{I}_{vac}^{[2]}$ is divergent in these cases. To see this, one should recall that, for $1 < \alpha < 4$, $\mathcal{I}(\kappa) \sim_{\kappa \rightarrow 0} f_{\infty} \kappa^{\alpha-3}$. Then it is clear that convergence of the t -integral in Eq. (A.5) at the $t \rightarrow 0$ limit requires $\alpha > \frac{3}{2}$.

Appendix B

An explicit demonstration of the $O(R^7)$ behaviour of $\mathcal{I}_{vac}^{[2]}$ in the limit of small soliton size R , for profiles with rapidly decaying tails, is provided by the linear profile. In this case, given the pole approximation

to the two-point function, one can derive a closed analytical expression for $\mathcal{I}_{vac}^{[2]}$ as a function of R . (An analogous result for the Casimir energy was derived in Ref. 4.) To do this, we begin by writing

$$\begin{aligned} \mathcal{I}_{vac}^{[2]} &= \frac{8\pi^2}{3} \int_0^\infty d\tau \int_0^\infty d\tau' \tau \tau' \sin \theta(\tau/R) \sin \theta(\tau'/R) \left(\tau \left(-\frac{\beta^2 m^2}{\partial_\tau^2 + 2/\tau^2 + \beta^2 m^2} \right) |\tau'\rangle \right. \\ &= -\frac{4\pi}{3\beta^3 g^2 m} \rho^3 \frac{d}{d\rho} \int_0^\infty dz \int_0^\infty dz' z z' \sin \theta(z) \sin \theta(z') \left(z \left(-\frac{1}{\partial_z^2 + 2/z^2 + \rho^2} \right) |z'\rangle \right) \\ &= \frac{4\pi}{3\beta^3 g^2 m} \left(5 - \rho \frac{d}{d\rho} \right) \int_0^\infty dz \int_0^\infty dz' z z' G_1(z, z'; \beta = 1) \sin \theta(z/\rho) \sin \theta(z'/\rho), \end{aligned} \quad (\text{B.1})$$

where $\rho = \beta m R$ and

$$\begin{aligned} G_1(z, z'; \beta) &\equiv \langle z | -\frac{1}{\partial_z^2 + 2/z^2 + \beta^2} | z' \rangle \\ &= \frac{1}{2\beta^3 z z'} \left[(1 + \beta(z + z') + \beta^2 z z') e^{-\beta(z+z')} - (1 + \beta|z - z'| - \beta^2 z z') e^{-\beta|z-z'|} \right] \end{aligned} \quad (\text{B.2})$$

as in Eq. (4.16) of (1).

The profile to be inserted here is given by

$$\theta(z) = \begin{cases} \pi(1-z) & \text{for } z \leq 1 \\ 0 & \text{for } z > 1. \end{cases} \quad (\text{B.3})$$

Then, after a calculation similar to the one leading to Eq. (6.2) in Ref. 4, we arrive at the result

$$\begin{aligned} \int_0^\infty dz \int_0^\infty dz' z z' G_1(z, z'; \beta = 1) \sin \theta(z/\rho) \sin \theta(z'/\rho) &= \frac{1}{6} \Delta \rho^2 + \frac{1}{2} \Delta (1 - \Delta) \rho^2 \\ &- \frac{1}{4} \Delta^2 [6 + (1 - \Delta)^{-1}] \rho - \frac{1}{2} \Delta (1 - \Delta) (1 + 2\Delta)^2 + \frac{1}{2} e^{-2\rho} \Delta (1 - \Delta) (1 + 2\Delta + \rho)^2, \end{aligned} \quad (\text{B.4})$$

where

$$\Delta = \left(1 + \frac{\pi^2}{\rho^2} \right)^{-1}. \quad (\text{B.5})$$

Using this expression in Eq. (B.1), we obtain

$$\begin{aligned} \mathcal{I}_{vac}^{[2]} &= \frac{4\pi}{3\beta^3 g^2 m} \left\{ \frac{1}{2} \Delta^2 \rho^3 + \frac{1}{2} \Delta (1 - \Delta) (1 + 4\Delta) \rho^2 - \frac{1}{2} [12 + (1 - \Delta)^{-1}] \Delta^2 \rho \right. \\ &\quad \left. - \Delta (1 - \Delta) (1 + 2\Delta) \left(\frac{\rho}{2} + \Delta + 8\Delta^2 \right) \right. \\ &\quad \left. + e^{-2\rho} \Delta (1 - \Delta) (1 + 2\Delta + \rho)^2 (1 + 2\Delta + \rho) + (2\Delta + \rho)^2 - 2\Delta (1 - 2\Delta) \right\}. \end{aligned} \quad (\text{B.6})$$

In the limit of large soliton size, we find that

$$\mathcal{I}_{vac}^{[2]} \sim_{\rho \rightarrow \infty} \frac{4\pi}{3\beta^3 g^2 m} \left(\frac{1}{6} - \frac{1}{4\pi^2} \right) \rho^3, \quad (\text{B.7})$$

which agrees with the expression derived from Eq. (2.19). In the limit of small soliton size, we find that

$$\begin{aligned} T_{4dc}^{[2]} &\sim_{\rho \rightarrow 0} \frac{4\pi}{3\beta^3 g^2 m} \left(\frac{1}{15} \Delta \rho^5 - \frac{1}{16} \Delta^2 \rho^3 - \frac{21}{128} \Delta^3 \rho \right) \\ &\sim_{\rho \rightarrow 0} \frac{4}{3\pi \beta^3 g^2 m} \left(\frac{1}{15} - \frac{1}{16\pi^2} - \frac{21}{128\pi^4} \right) \rho^7, \end{aligned} \quad (\text{B.8})$$

as expected from the general discussion of Appendix A.

Appendix C

In order to study the asymptotic behaviour of the unperturbed Dirac wavefunctions $u(z)$, $v(z)$ and of the cranking wavefunctions a , b , c , d in the limits of large and small z , we begin by setting the profile $\theta(z)$ to its limiting values, i.e. $\sin \theta(z) \rightarrow 0$, $\cos \theta(z) \rightarrow \nu$, where $\nu = +1$ applies to the limit $z \rightarrow \infty$ and $\nu = -1$ to the limit $z \rightarrow 0$. Then, if we introduce $f(z) = zu(z)$, $g(z) = zv(z)$, the unperturbed radial Dirac equations assume the form

$$\left(\frac{d}{dz} - \frac{1}{z} \right) f(z) = -(\nu + \lambda) g(z), \quad (\text{C.1})$$

$$\left(\frac{d}{dz} + \frac{1}{z} \right) g(z) = -(\nu - \lambda) f(z), \quad (\text{C.2})$$

where here and throughout this appendix we write $\lambda \equiv \epsilon_{\text{rel}}$. The second order equation for $f(z)$ becomes

$$\left(\frac{d^2}{dz^2} - \mu^2 \right) f(z) = 0, \quad (\text{C.3})$$

where $\mu^2 = 1 - \lambda^2$. The physical solution for $z \gg 1$ is

$$f(z) \sim_{z \rightarrow \infty} C_1 e^{-\mu z}, \quad (\text{C.4})$$

in which case

$$g(z) \sim_{z \rightarrow \infty} C_1 \sqrt{\frac{1-\lambda}{1+\lambda}} e^{-\mu z}. \quad (\text{C.5})$$

For $z \ll 1$, the solution that avoids an unphysical singularity in $g(z)$ at $z = 0$ is given by

$$f(z) = \frac{C_2}{\mu} \sinh \mu z \sim_{z \rightarrow 0} C_2 z. \quad (\text{C.6})$$

Correspondingly,

$$g(z) = \frac{C_2}{1-\lambda} \left[\cosh \mu z - \frac{\sinh \mu z}{\mu z} \right] \sim_{z \rightarrow 0} \frac{1}{2}(1+\lambda) C_2 z^2. \quad (\text{C.7})$$

The limits $z \rightarrow 0$ and $z \rightarrow \infty$ can also be studied in a unified manner for the cranking wavefunctions a , b , c , d . Again on making the substitutions $\sin \theta \rightarrow 0$, $\cos \theta \rightarrow \nu$, we observe that the cranking equations

(5.14) for $\psi \equiv (a, b, c, d)^T$ can be written as

$$\frac{d}{dx} \psi + \frac{1}{x} \begin{pmatrix} 1 & 0 \\ 0 & M^{-1} \end{pmatrix} Z \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \psi - \begin{pmatrix} 0 & (\nu + \lambda)M \\ (\nu - \lambda)M^{-1} & 0 \end{pmatrix} \psi = \phi, \quad (\text{C.8})$$

where $Z = \text{diag}(0, 3, 2, -1)$, $\phi = \frac{1}{2}(-v, 0, u, -v)^T$ and

$$M = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ 3 & 3 \end{pmatrix}. \quad (\text{C.9})$$

It is useful to note that

$$M \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} M^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{C.10})$$

By performing the transformation

$$\psi' = x \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \psi, \quad (\text{C.11})$$

we obtain

$$\frac{d}{dx} \psi' + \frac{1}{x} (Z - 1) \psi' - \begin{pmatrix} 0 & \nu + \lambda \\ \nu - \lambda & 0 \end{pmatrix} \psi' = \phi', \quad (\text{C.12})$$

where

$$\phi' = x \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \phi, \quad (\text{C.13})$$

and we note that $Z - 1 = \text{diag}(-1, 2, 1, -2)$. If we now introduce the components $\psi' = (\alpha_0, \alpha_1, \beta_0, \beta_1)^T$, then we see that α_0 and β_0 are coupled according to the inhomogeneous equations

$$\frac{d}{dx} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} - \begin{pmatrix} 0 & \nu + \lambda \\ \nu - \lambda & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = -\frac{x}{2} \begin{pmatrix} v \\ u \end{pmatrix}, \quad (\text{C.14})$$

while α_1 and β_1 are coupled according to the homogeneous equations

$$\frac{d}{dx} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} - \begin{pmatrix} 0 & \nu + \lambda \\ \nu - \lambda & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = 0. \quad (\text{C.15})$$

Let us first look at the equations for α_1, β_1 ; these can be expressed as

$$-\frac{d^2 \beta_1}{dx^2} + \frac{2}{x^2} \beta_1 + \mu^2 \beta_1 = 0, \quad (\text{C.16})$$

where $\mu^2 = 1 - \lambda^2$, and

$$\alpha_1 = \frac{1}{\nu - \lambda} \left(\frac{d}{dx} - \frac{2}{x} \right) \beta_1. \quad (\text{C.17})$$

The two linearly independent solutions of Eq. (C.16) are given by

$$\beta_1^\pm(z) = \frac{1}{2\mu} \left(1 \mp \frac{1}{\mu z} \right) e^{\pm \mu z}. \quad (\text{C.18})$$

To study the $z \rightarrow \infty$ limit, we must clearly choose $\beta_1(z) = k_1 \beta_1^-(z)$. Then,

$$\begin{aligned} \beta_1(z) &\sim_{z \rightarrow \infty} \frac{k_1}{2\mu} e^{-\mu z}, \\ \alpha_1(z) &\sim_{z \rightarrow \infty} \frac{k_1}{1+\lambda} e^{-\mu z}, \end{aligned} \quad (\text{C.19})$$

where we have set $\nu = +1$, or equivalently

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \sim_{z \rightarrow \infty} k_1' \begin{pmatrix} 1 \\ -\sqrt{\frac{1-\lambda}{1+\lambda}} \end{pmatrix} e^{-\mu z}. \quad (\text{C.20})$$

To avoid an unphysical singularity, the appropriate combination for studying the $z \rightarrow 0$ limit is $\beta_1 = k_2(\beta_1^- + \beta_1^+)/2$, i.e.

$$\beta_1(z) = \frac{k_2}{2\mu} \left[\cosh \mu z - \frac{\sinh \mu z}{\mu z} \right] \sim_{z \rightarrow 0} \frac{k_2 \mu}{6} z^2. \quad (\text{C.21})$$

Then we also have

$$\begin{aligned} \alpha_1(z) &= -\frac{k_2}{2(\lambda+1)} \left[\sinh \mu z - \frac{3}{\mu z} \left(\cosh \mu z - \frac{\sinh \mu z}{\mu z} \right) \right] \\ &\sim_{z \rightarrow 0} -\frac{k_2}{30(\lambda+1)} (\mu z)^3, \end{aligned} \quad (\text{C.22})$$

having set $\nu = -1$. On introducing $k_2' = -k_2 \mu^3 / [30(\lambda+1)]$, we can write this result as

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \sim_{z \rightarrow 0} k_2' \begin{pmatrix} z^3 \\ \frac{5}{6} z^2 \end{pmatrix}. \quad (\text{C.23})$$

The individual equations for α_0, β_0 which follow from Eq. (C.14) can be cast in the form

$$\left(\frac{d^2}{dz^2} - \mu^2 \right) \alpha_0 = -\lambda f, \quad (\text{C.24})$$

$$\beta_0 = \frac{1}{\nu + \lambda} \left[\left(\frac{d}{dz} - \frac{1}{z} \right) \alpha_0 + \frac{1}{2} g \right]. \quad (\text{C.25})$$

Because of Eq. (C.3), the particular solution of Eq. (C.24) can be expressed as

$$\alpha_0^p(z) = -\frac{\lambda}{2\mu^2} z \frac{df(z)}{dz}. \quad (\text{C.26})$$

The general solution which respects the correct boundary conditions, whether at $z \rightarrow 0$ or $z \rightarrow \infty$, is then given by

$$\alpha_0(z) = k f(z) - \frac{\lambda}{2\mu^2} z \frac{df(z)}{dz}, \quad (\text{C.27})$$

with k an arbitrary constant. Noting that

$$\left(\frac{d}{dz} - \frac{1}{z} \right) \alpha_0^p(z) = -\frac{\lambda}{2} z f(z), \quad (\text{C.28})$$

and taking account of Eq. (C.1), we obtain

$$\beta_0(z) = \frac{1}{\nu + \lambda} \left[-\frac{\lambda}{2} z f(z) + \left(\frac{1}{2} - k(\nu + \lambda) \right) g(z) \right]. \quad (\text{C.29})$$

To study the problem at $z \gg 1$, where $\nu = +1$ applies, we must insert the forms (C.4) and (C.5). It follows immediately that

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \sim_{z \rightarrow \infty} \frac{C_1 \lambda}{2(1+\lambda)} \begin{pmatrix} -\sqrt{\frac{1-\lambda}{1+\lambda}} \\ -1 \end{pmatrix} z e^{-\mu z}, \quad (\text{C.30})$$

in which the arbitrary constant k does not appear. To study the problem at $z \ll 1$, where $\nu = -1$ applies, we must take for $f(z)$ and $g(z)$ the forms (C.6) and (C.7). Then we see that

$$\begin{aligned} \alpha_0 &\sim_{z \rightarrow 0} C_2 \left(k - \frac{\lambda}{2\mu^2} \right) z, \\ \beta_0 &\sim_{z \rightarrow 0} -\frac{C_2}{3} \left(k(1+\lambda) - \frac{2\lambda-1}{2(\lambda-1)} \right) z^2. \end{aligned} \quad (\text{C.31})$$

On introducing $k' = k - \lambda/(2\mu^2)$, we can rewrite this result as

$$\begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \sim_{z \rightarrow 0} C_2 \begin{pmatrix} k' z \\ -\frac{1}{3} [k'(1+\lambda) + \frac{1}{2}] z^2 \end{pmatrix}. \quad (\text{C.32})$$

To reconstruct the original wavefunction $\psi(z)$, we note that

$$\psi(z) = \frac{1}{z} \begin{pmatrix} 1 & 0 \\ 0 & M^{-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \frac{1}{z} \begin{pmatrix} \alpha_0 \\ -\beta_0 + \frac{2}{3} \beta_1 \end{pmatrix}. \quad (\text{C.33})$$

Thus, it follows from Eqs. (C.20) and (C.30) that the $z \gg 1$ limit is given by

$$\psi(z) \sim_{z \rightarrow \infty} \frac{C_1 \lambda}{2(1+\lambda)} \begin{pmatrix} \sqrt{\frac{1-\lambda}{1+\lambda}} \\ O(z^{-1}) \\ +1 \\ -1 \end{pmatrix} e^{-\sqrt{1-\lambda} z}. \quad (\text{C.34})$$

The $z \ll 1$ limit is furnished by Eqs. (C.32) and (C.23), and can be expressed as

$$\psi(z) \sim_{z \rightarrow 0} \begin{pmatrix} K_1 \\ K_2 z^2 \\ K_3 z \\ K_4 z \end{pmatrix}, \quad (\text{C.35})$$

with

$$\begin{aligned} K_3 &= \frac{1}{3} \left[\frac{1}{2} C_2 + (1+\lambda) K_1 - \frac{10}{1-\lambda} K_3 \right], \\ K_4 &= -\frac{1}{3} \left[\frac{1}{2} C_2 + (1+\lambda) K_1 + \frac{5}{1-\lambda} K_3 \right]. \end{aligned} \quad (\text{C.36})$$

However, closer inspection of Eq. (C.35) reveals that we were not justified in setting $\sin \theta(z)$ to zero everywhere in Eqs. (5.14). Specifically, this is the case with the equation for $b(z)$ in this set. We note that $b(z) \sim_{z \rightarrow 0} K_3 z^2$, $a(z) \sim_{z \rightarrow 0} K_1$ implies that one cannot throw away the term $a(z) \sin \theta(z)$ there, as it is of the same order of magnitude as b/z , for example. Instead, one should write $\sin \theta(z) \sim_{z \rightarrow 0} bz$.

Then one obtains, for $z \ll 1$,

$$5K_3 z - 2K_1 h z + (1-\lambda)(K_3 + K_4)z = 0, \quad (\text{C.37})$$

so that

$$K_3 + K_4 = -\frac{1}{1-\lambda}(5K_3 - 2hK_1). \quad (\text{C.38})$$

We combine this with the relation

$$K_3 - 2K_4 = \frac{1}{2} C_2 + (1+\lambda) K_1 \quad (\text{C.39})$$

to obtain the final relations for $z \ll 1$, which replace Eq. (C.36):

$$\begin{aligned} K_3 &= \frac{1}{3} \left[\frac{1}{2} C_2 + (1+\lambda) K_1 - \frac{2}{1-\lambda}(5K_3 - 2hK_1) \right], \\ K_4 &= -\frac{1}{3} \left[\frac{1}{2} C_2 + (1+\lambda) K_1 + \frac{1}{1-\lambda}(5K_3 - 2hK_1) \right]. \end{aligned} \quad (\text{C.40})$$

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Table captions

Table 1: $T_{vac,0}^{[2]}$ (Eq. (2.19)) and $T_{vac}^{[2]}$ (Eq. (4.7)) for $m = 372$ MeV as evaluated (in fm) using the five different one-parameter profiles given in Eq. (6.1).

Profile	mR	$T_{vac,0}^{[2]}$ (fm)	$T_{vac}^{[2]}$ (fm)	$T_{vac}^{[2]}/T_{vac,0}^{[2]}$
Stereo 2	1.0	0.929	0.731	0.787
Stereo 3	1.1	0.388	0.229	0.590
Exp	1.2	0.844	0.605	0.717
Tanh	1.8	0.783	0.530	0.677
Lin	2.0	0.315	0.123	0.390

Table 2: Rows 1 & 2: $T_{vac,0}^{[2]}$ and $T_{vac}^{[2]}$ evaluated (for $m = 350, 400$ & 500 MeV) using the self-consistent profile of the two-point approximation $g^{[2]}$ calculated in (1). Rows 2 & 3: The same two quantities evaluated for the exact self-consistent profile of Ref. 18. Row 4: The exact calculation of I reported in Ref. 10.

Table 3: Rows 1–3: I_{vac} (Eq. (5.19)), $T_{vac}^{[2]}$, and the total $I^{[2]} = I_{vac} + T_{vac}^{[2]}$ evaluated using the profile of (1). Rows 4–6: The same quantities calculated with the profile of Ref. 18. Rows 6–9: The exact calculations reported in Ref. 10.

Table 4: Contributions to $\mu\nu$ calculated according to the same scheme as in Table 3. Note that the relation $\mu_{\nu,vac}^{[2]} = \frac{1}{3}T_{vac}^{[2]}$ of Eq. (1.6) has been used.

Figure captions

Figure 1: The ratio $T_{vac}^{[2]}/T_{vac,0}^{[2]}$ for $m = 372$ MeV as a function of mR using the five different one-parameter profiles given in Eq. (6.1).

Figure 2: The cranking wavefunctions a, b, c, d for $m = 400$ MeV using the two-point self-consistent profile of Ref. 1.

Table 2

Profile	I (fm)	$m = 350$ MeV	$m = 400$ MeV	$m = 500$ MeV
$g^{[2]}$ (Ref. 1)	$T_{vac,0}^{[2]}$	0.765	0.846	0.844
	$T_{vac}^{[2]}$	0.558	0.632	0.648
g^{ex} (Ref. 18)	$T_{vac,0}^{[2]}$	0.609	—	0.596
	$T_{vac}^{[2]}$	0.454	—	0.458
g^{ex} (Ref. 10)	T_{vac}^{ex}	0.384	0.378	0.357

Figure 1

Table 3

Profile	r (fm)	$m = 350$ MeV	$m = 400$ MeV	$m = 500$ MeV
$\theta^{[2]}$ (Ref. 1)	r_{val}	0.928	0.662	0.423
	$r_{vac}^{[2]}$	0.558	0.632	0.648
	$r^{[2]}$	1.486	1.294	1.071
	$r^{[2]}$	1.258	—	0.514
θ^{ss} (Ref. 18)	r_{val}	0.454	—	0.458
	$r_{vac}^{[2]}$	1.712	—	0.972
	$r^{[2]}$	1.229	0.831	0.530
θ^{ss} (Ref. 10)	r_{val}	0.384	0.378	0.357
	$r_{vac}^{[2]}$	1.613	1.209	0.887
	$r^{[2]}$	1.006	—	—
Experiment	r	1.006	—	—

Table 4

Profile	μ_V (fm)	$m = 350$ MeV	$m = 400$ MeV	$m = 500$ MeV
$\theta^{[2]}$ (Ref. 1)	$\mu_{V,val}$	0.203	0.197	0.185
	$\mu_{V,vac}^{[2]}$	0.186	0.211	0.216
	$\mu_V^{[2]}$	0.389	0.408	0.401
	$\mu_V^{[2]}$	0.186	—	0.152
θ^{ss} (Ref. 18)	$\mu_{V,val}$	0.151	—	0.153
	$\mu_{V,vac}^{[2]}$	0.337	—	0.305
	$\mu_V^{[2]}$	0.144	0.133	0.119
θ^{ss} (Ref. 10)	$\mu_{V,val}$	0.088	0.082	0.070
	$\mu_{V,vac}^{[2]}$	0.232	0.215	0.189
	$\mu_V^{[2]}$	0.494	—	—
Experiment	μ_V	0.494	—	—

Figure 2