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COMPUTER PROGRAMS FOR ANALYTICAL PERTURBATIVE CALCULATIONS IN HIGH ENERGY PHYSICS

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ABSTRACT

A short review of the present status of computer packages for the high order analytical perturbative calculations is presented. The mathematical algorithm and the quantum field theory methods used are briefly discussed. The most recent computer package *HEPLoops* for analytical computations in high energy physics up to four-loops is discussed.

1. Introduction

A recent significant progress in a precise high energy physics experiment requires an adequate progress in development of calculational methods and the computations of various observables from the first principles of the theory. This can consistently be done within the perturbation theory for the processes with large momentum transfer. A significant efforts have been made to develop the methods of perturbation theory to study a wide class of problems including calculation of various cross-sections and decay widths, structure functions for deep inelastic scattering processes, coefficient functions in operator product expansions, renormalization group functions and renormalization constants in various quantum field theory models etc.¹ Some of the methods of higher order analytical perturbative calculations have been reviewed recently in Ref. 2.

The application of certain theoretical methods, such as the renormalization group, dispersion relations, operator product expansions etc., results the effective reduction of the problem of evaluation of physical quantities to the problem of computation of corresponding Feynman diagrams. In each order of perturbation theory contributions in the physical observables come from a finite set of Feynman diagrams with the same number of internal momentum integrations (number of loops). In general, integral corresponding to a Feynman diagram is ultraviolet and infrared divergent. A most convenient way to handle those divergences is to use the dimensional regularization method³ and 't Hooft's minimal subtraction (MS) prescription⁴ for ultraviolet renormalization. Using the available calculational methods,⁵ in many cases one can significantly simplify the problem by setting to zero of a dimensional

parameters - usually all masses of the particles involved and some of the external momenta. Another physical quantities allow expansion in powers of small m^2/s or s/m^2 (m is the mass of the particle and $\sqrt{s}$ is the center-of-mass energy) with mass independent coefficients. For a high enough energy or a large mass calculation of a first few terms in this expansion gives satisfactory approximation (e.g., in calculation of total cross-sections and decay widths). Thus, one deals with a wide class of problems, which can effectively be reduced to the problem of evaluation of massless multiloop dimensionally regularized propagator type (with only one external momentum) Feynman diagrams. The analytical recursive type algorithms for one-, two- and three-loop massless propagator type Feynman integrals were developed in Ref. 6. In the next section, the main ideas of this algorithm will be discussed.

In practice, a calculation of physical quantities within perturbation theory is very cumbersome and tedious already beyond the one-loop level, especially in realistic quantum field theory models, like QCD. However, the existing mathematical algorithm⁶ allow convenient implementation within algebraic programming systems like REDUCE,⁷ SCHOONSCHIP,⁸ FORM,⁹ MATHEMATICA,¹⁰ etc., which makes it possible to do high order perturbative calculations up to the four-loop level. In the later sections some of the existing computer programs will be discussed.

2. The algorithm

The evaluation of a given physical quantity to a particular order of the perturbation theory includes number of different steps. As a rule, one starts with the Lagrangian of a given quantum field theory model, which fixes corresponding Feynman rules - a graphical technique, which gives a special rules to describe propagators of the particles involved and their interactions.¹¹ According to the particular Feynman rules, one can express the considered physical quantity as a perturbation series in terms of interaction constant. A set of Feynman diagrams with the same number of loops contribute at particular order of perturbation theory. The first step involves generation of the topologically distinct Feynman graphs with the given number of loops. At one- and two-loop levels this problem can be solved without computer assistance. The situation changes at higher orders and a reliable solution of the problem requires a special computer programs. Such a programs were offered in earlier works for QED.¹² At this point there is an essential problem of recognition of topologically identical diagrams, which is important from a viewpoint of full automation of the multiloop calculations. For QCD the problem is harder and still awaits its solution. In the existing four-loop QCD calculation¹³ the graph generation problem has been solved without computer involvement. At the next step an analyses of the Feynman diagrams is necessary. This involves recognition of divergent ultraviolet and infrared subgraphs of the diagram. For an ultraviolet divergences the renormalization theory sets the number and the type of subtraction counterterms to get the finite contribution. There is a convenient prescription to deal with ultraviolet divergences directly on the graphical level - the Bogolyubov R -operation.¹¹ In a large scale calculation, as a rule at the four-loop level^{13,2} this step involves

long manipulations with Feynman graphs. Some simple computer implementation of the Bogolyubov R -operation has been made in the four-loop calculation of the total hadronic decay rate of the Z -boson.¹³ For a simplification of the topology of Feynman integral and to deal with an infrared divergent subgraphs the so called infrared rearrangement prescription has been suggested.⁵ This technique has been successfully used in the recent high order calculations (see the Ref. 2 and references therein). At the final step the relevant Feynman diagrams should be evaluated. This issue will be discussed in more detail.

One considers the dimensionally regularized³ Feynman integral, which corresponds to one-particle irreducible oriented graph, having two external lines (e.g., the graph in Fig. 1). The D -dimensional Euclidean momentum P_i and the num-

Fig. 1. One-particle irreducible oriented graph.

ber α_i correspond to the i -th line of the diagram. The Euclidean momentum Q correspond to the external lines. A formal expression for arbitrary diagram is:

$$I = \int \frac{\mathcal{P}_n(P_i, Q)}{\prod_i P_i^{2\alpha_i}}, \quad (1)$$

where $\mathcal{P}_n(P_i, Q)$ is the n -th polynomial in P_i and Q . The integration should be performed with respect of linear independent internal momenta according to the prescription of dimensional regularization.³ From a dimensional analyses one obtains:

$$I = (Q^2)^{D/2+n/2-\sum_i \alpha_i} f(D), \quad (2)$$

where l is the number of loops equal to the number of integrations over the D -dimensional space-time. Depending on the number of loops and the type of problem, for $f(D)$ it is sufficient to know the first few coefficients in its Laurent expansion in terms of $\epsilon = (4 - D)/2$, including the pole parts:

$$f(D) = \sum_k f_k \epsilon^{-k}. \quad (3)$$

The evaluation of a diagram (integral) means the evaluation of the coefficients f_k .

There is only one relevant topological type of the diagram at the one-loop level (Fig. 2). Note that in the present paper one considers only a massless fermion limit and a so called tadpoles vanish identically. The corresponding Feynman integral

Fig. 2. A one-loop diagram ($P_1 \equiv P$, $P_2 \equiv P - Q$)

can be evaluated via the following basic equation:⁶

$$I = \int \frac{d^D P}{(2\pi)^D} \frac{\mathcal{P}_n(P, Q)}{P^{2\alpha}(P-Q)^{2\beta}} = \frac{1}{(4\pi)^2} (Q^2)^{D/2-\alpha-\beta} \sum_{\sigma \geq 0} G(\alpha, \beta, n, \sigma) Q^{2\sigma} \left[\frac{1}{\sigma!} \left(\frac{\square}{4} \right)^\sigma \mathcal{P}_n(P) \right]_{P=Q}, \quad (4)$$

where $\square = \partial^2 / \partial P_\mu \partial P^\mu$ and

$$G(\alpha, \beta, n, \sigma) = (4\pi)^{2-D/2} \frac{\Gamma(\alpha + \beta - \sigma - D/2)}{\Gamma(\alpha)\Gamma(\beta)} B(D/2 - \alpha + n - \sigma, D/2 - \beta + \sigma). \quad (5)$$

It is trivial to obtain the expansions of the above G -functions in a Laurent series in ϵ . On the other hand, a convenient recursion relations for G -functions has been obtained⁶ (see also the Ref. 14), which allow simple computer implementation within various algebraic programming systems. Thus, one has to obtain the expansion for only a few types of G -functions. For example, for the $G(1, 1, 0, 0)$, which corresponds to the trivial one-loop scalar integral, one has in the standard $\overline{MS}$ ¹⁵-normalization:

$$G(1, 1, 0, 0) = (4\pi)^\epsilon \left[\frac{1}{\epsilon} + 2 + 4\epsilon + \left(8 - \frac{7}{3}\zeta(3) \right) \epsilon^2 + \left(16 - \frac{14}{3}\zeta(3) - \frac{13}{4}\zeta(4) \right) \epsilon^3 + O(\epsilon^4) \right] \quad (6)$$

At the two-loop level there are three topologically distinct Feynman diagrams. One of them is pictured in Fig. 1 and the others are shown in Fig. 3. The diagrams in Fig. 3 can be evaluated by an iterative use of the one-loop formula (4). Below these type of diagrams will be referred as a recursively one-loop ones. Using the momentum conservation at the vertices, the polynomial $\mathcal{P}_n(P_i, Q)$ can be reduced to the polynomial in P_i^2 and Q^2 . Each exponent of P_i^2 can be included in the denominator, simultaneously changing the corresponding α_i . As a result, the numerator of the integrand can be reduced to 1. Thus, using the above procedure, one can reduce the initial two-loop integral of the type in Fig. 1, with the polynomial $\mathcal{P}_n(P_i, Q)$ in the numerator of the integrand, to a linear combination of the diagrams

Fig. 3. Recursively one-loop type two-loop diagrams

of the following type:

$$F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \int \frac{1}{\prod_{i=1}^5 P_i^{2\alpha_i}}. \quad (7)$$

If one of the exponents α_i turns out to be a nonpositive integer then the corresponding line “shrinks” and the integral is of the topological type of diagram obtained from the initial one by shrinking this line into a point. Thus, shrinking of any of the lines from 1 to 5 in the diagram of Fig. 1 converts it to one of the diagrams in Fig. 3. However, if all of the $\alpha_i > 0$ then there are no shrunk lines in the diagram in Fig. 1. The diagrams of such a type provide the greatest difficulty while being calculated. The method for calculation of such diagrams was developed by Chetyrkin and Tkachov⁶. The method is based on the so-called “rule of triangle”, which states the following. If the diagram contains the subgraph of the type in Fig. 4, where the integration with respect to a single D -dimensional momentum

Fig. 4. A key triangle subgraph

is implied, then the following relation is true within the dimensional regularization:

$$\left(\frac{\partial}{\partial P_1^\alpha} + \frac{\partial}{\partial P_4^\alpha} + \frac{\partial}{\partial P_5^\alpha} \right) P_i^\alpha \{Triangle\} = 0, \quad i = 1, 4, 5. \quad (8)$$

After applying that differential operator to the above mentioned subgraph (in the case of the diagram in Fig. 1 one takes $i = 5$) the scalar products of vectors P_j , $j = 1, 4, 5$ appear in the numerator of the corresponding integrand. Using

the momentum conservation at the vertices, they can be expanded into a linear combination of standard invariants P_j^2 , $j = 1, 2, \dots$. These invariants cancel the corresponding powers in the denominator of the integrand. As a result, the initial integral is expressed in terms of the integrals with some other indices α_i . The above described procedure leads, for example, to the following typical recursion relation for the diagrams of the type in Fig. 1:

$$F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (D - \alpha_1 - \alpha_4 - 2\alpha_5)^{-1} \\ \{ \alpha_1 [F(\alpha_1 + 1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 - 1) - F(\alpha_1 + 1, \alpha_2 - 1, \alpha_3, \alpha_4, \alpha_5)] \\ + \alpha_4 [F(\alpha_1, \alpha_2, \alpha_3, \alpha_4 + 1, \alpha_5 - 1) - F(\alpha_1, \alpha_2, \alpha_3 - 1, \alpha_4 + 1, \alpha_5)] \}. \quad (9)$$

Obviously, the application of the eq.(9) to the diagram in Fig. 1 with an integer α_i will eventually express it via recursively one-loop diagrams shown in Fig. 3. A similar recursion relations can be obtained when not all of the α_i are integers. Such a diagrams appear in the three- and higher loop calculations. The three-loop algorithm is also based on the above described procedure. However, there are more topologically distinct Feynman diagrams and the corresponding algorithm is more complicated. The above algorithm up to three-loop level⁶ was fully implemented within various algebraic programming systems.

3. Status of existing packages

Several programs were written during the last decade for analytical evaluation of multiloop Feynman diagrams. The algebraic programming systems used are: REDUCE,⁷ SCHOONSCHIP⁸ and FORM.⁹

The first program for the SCHOONSCHIP system has been worked out in a somewhat primitive form.¹⁶ This program evaluates traces of products of Dirac γ matrices and does some other simplifications, to reduce the initial diagram to a sum of scalar integrals. Then it uses a database of the precalculated scalar integrals in the form of trivial substitutions. This program is very fast for the very restricted number of problems it was written for. However, a large number of problems are out of the range of its capacity, since too many scalar integrals have to be precalculated manually. In any case, more complicated problems cannot be dealt with unless calculation of two- and more loop integrals is made fully automatic. The above program has been used in the first three-loop calculation of QCD β -function.¹⁷

A full computer implementation of algorithms⁶ at the two-loop level has been done in 1984 for the REDUCE system on the IBM mainframes in a form of procedures. The program calculates one- and two-loop arbitrary massless propagator type Feynman integrals and allows arbitrary tensor structure in the numerator of the integrand. No database of the precalculated integrals was used. Several two-loop calculations have been done with the help of this program. Among them one mentions the calculation of coefficient functions of gluon and quark condensates in the sum rule method;¹⁸ the calculation of two-loop anomalous dimensions of baryonic currents¹⁹ etc. Later this program was published with the name *LOOPS*.²⁰

Presently the unpublished IBM/PC version of this program is also available. The program *LOOPS*²⁰ has also been used in the calculation of five-loop renormalization group functions in ϕ^4 -theory and critical exponents^{21,22} and in the calculation of the decay rate of the Z-boson to three-²³ and four-loop¹³ levels. However, in a very large scale calculations *LOOPS* uses comparatively large computer resources, which in fact, conditioned by the specific features of the REDUCE system.⁷

An implementation of three-loop algorithms was first done²⁴ for the CDC version of SCHOONSCHIP. This program calculates arbitrary massless scalar (with no free Lorenz indices) one-, two- and three-loop Feynman diagrams of the propagator type. However, unfortunately the physical quantities which have been calculated by this program - the three-loop total decay width of the Higgs boson,²⁵ the four-loop QED β function²⁶ and the four-loop $\Gamma(Z \rightarrow \text{hadrons})$ ²⁷ turned out to be incorrect (see below).

The next program,²⁸ which fully implements the three-loop algorithm,⁶ was written for the IBM and CDC versions of SCHOONSCHIP. This program uses programming experience and some of the ideas of the previous program²⁴ and includes the number of original solutions. The structure of the program has totally been changed according to the needs of the IBM version of SCHOONSCHIP.⁸ As a test, the recalculation of three-loop total decay width of Higgs boson has been made²⁹ with help of the new program.²⁸ It was shown²⁹ that the results of the previous calculation²⁵ by the old program²⁴ was incorrect. Later, with help of test runs on the new program²⁸ the source of error was discovered in the old program.²⁴ The error was insufficient expansions in Laurent series of one-loop "basic" scalar integrals (one of them is given in eq. (6)). The other independent source of errors in the old program²⁴ was an incorrect logical structure of the subprogram for trace calculations. Due to those errors, the previous results^{26,27} also turned out to be incorrect. Fortunately, the new program²⁸ was initially free of those errors and without changes was published with the name of *MINCER*.¹⁴ The recalculations of four-loop QED β function³⁰ and $\Gamma(Z \rightarrow \text{hadrons})$ ¹³ were done with help of program *MINCER*.^{28,14} The result for $\Gamma(Z \rightarrow \text{hadrons})$ ¹³ is smaller by an order of magnitude and has an opposite sign as compared to the incorrect one.²⁷ The new result for the QED beta function³⁰ also differs from the incorrect one.²⁶ The results of the new calculations^{30,13} have been confirmed in the Refs. 31,32, where the corrected computer program was used. The following notes are in order. First of all one should mention that the evaluation of the four-loop QED β function^{30,31} has been done independently and the results were published in a joint publication.³³ It is also important to know that the above complicated four-loop calculations were completely independent. However, one should stress that the calculation in Refs. 31,32 has been done with the old program²⁴ with substituted of significant part from the new program.^{28,14} Only from this point of view the independence of the calculations in Refs. 31,32 from the calculations in Refs. 30,13 is doubtful. If the above mentioned "program dependence" is disregarded then those calculations are completely independent and the confirmation in Refs. 31,32 of the results of Refs.

30,13 is still strong evidence that both of the results are correct and the program *MINCER*^{28,14} is reliable.

About five years ago a new SCHOONSCHIP - like algebraic programming system FORM has been worked out.⁹ This system is operable on different types of computers. The FORM system has combined most of the positive features of both REDUCE and SCHOONSCHIP systems and is efficient enough to deal with very large scale analytical computations. The only restrictions that apply are the particular computer CPU speed and a hard disk capacity.

Recently a new program - "The FORM version of MINCER" has been worked out.³⁴ This program is written in procedure form and calculates three-loop massless propagator type scalar (with no free Lorenz indices) Feynman diagrams. Some useful optimizations have been done. However, the claimed super efficiency of this program is restricted by the builded in database of precalculated scalar integrals.

4. External specifications of the program *HEPLoops*

A new program *HEPLoops* for high order analytical perturbative calculations for the FORM system has been written recently³⁵ and was briefly announced in a review paper.³⁶ A brief description of external specifications of this program will be given here.

The program *HEPLoops* (version 2.0) is intended for analytical computation of dimensionally regularized massless Feynman diagrams depending on a single external momentum, with the number of loops $l \leq 3$. The program implements the three-loop algorithms⁶ within the algebraic programming system FORM.⁹ The program calculates expansions of Feynman integrals in terms of the parameter of dimensional regularization $\epsilon = 2 - D/2$ up to $O(\epsilon^{4-l})$, which is sufficient for a direct evaluation of four-loop integrals. The program *HEPLoops* (version 2.0) consists of 10 blocks. (The version 3. contains also additional blocks for ultraviolet renormalization procedure and for integration of trivial 4th loop.) The blocks are linked in a proper order which should not be changed by the user. However, in a large scale calculations it is possible to execute block by block in several runs. The total number of lines in the package is 3380.

The file DECLARE contains all necessary declarations of symbols, functions, indices etc. This file is included in each program block by the *#include* command. The file Form.set defines all appropriate system parameters, which usually requires some trivial adjustments for the particular computer configuration (see FORM manual⁹). The file INPUT contains the input for initial diagrams. The user will mainly deal with only this file to put the corresponding input expressions for the Feynman diagrams. The file INPUT is included in the block START by the *#include* command. The block SCHEME defines a renormalization scheme in which the result will be printed. The basic version of the program gives the result in the normalizations of momentum integrals corresponding to the $\overline{MS}$ -scheme¹⁵ (see eq.(6)). The other program blocks step-by-step implement the calculational algorithm.⁶

One-, two- and three-loop basic diagrams are shown in Fig. 2, Fig. 1 and

Fig. 5 correspondingly. Note, that the program *HEPLoops* integrates over the D -

Fig. 5. Three-loop basic scalar Feynman diagrams (from left to right: Ladder, Benz, Nonplanar)

dimensional ($D = 4 - 2\epsilon$) Euclidean space. The directions of the momenta in the program are fixed as in the figures 1,2,5. It is not difficult to see that arbitrary one-, two- and three-loop diagrams can be turned by a proper choice of the exponents of the propagator powers to one of the types of diagrams shown in Figs. 1,2,5.

The file *INPUT* contains five calculable $G(\text{lobal})$ expressions: Ladder; Benz; Nonplanar; Twoloop and Oneloop. The r.h.s of each of them must contain an input data (or zero) for the corresponding topology of the diagram and the number of loops. Any of the data in the r.h.s. can contain following factors.

- Numerical coefficient.
- Product (including an empty one) of the integer powers of scalar products $P1.P1$ and $P2.P2$ for Oneloop; $P1.P1, \dots, P5.P5$ for Twoloop and $P1.P1, \dots, P8.P8$ for Ladder, Benz and Nonplanar (the momentum numerations correspond to figs. 1,2,5).
- Product (including an empty one) of scalar products $P_i.P_j$ and $P_i.QQ$, where $i, j = 1, 2$ for Oneloop; $i, j = 1, \dots, 5$ for Twoloop and $i, j = 1, \dots, 8$ for all three-loop G -expressions (QQ codes an external momentum for all diagrams). Product of Dirac γ -matrices (corresponding to the product of the fermionic propagators and vertices along the fermionic loop), which should be coded within the FORM as follows: $g_-(1, a, P1, m, P2, \dots) * g_-(2, n, P3, k, P4, \dots) * \dots$, where the first argument indicates the particular fermionic circle (see FORM manual⁹), $a, m, n, k, \dots$ correspond to $\gamma^a, \gamma^m, \gamma^n, \gamma^k, \dots$ and $P1, P2, \dots$ correspond to $P1_\alpha \gamma^\alpha, P2_\alpha \gamma^\alpha, \dots$ (In the block *START* the trace calculations will be performed for $g_-(1, \dots), g_-(2, \dots), \dots$)
- Products of four-momenta $P_i(\alpha)$ and $QQ(\alpha)$ where $\alpha = m, n, k, l, j, a, r$ and $i = 1, 2$ for Oneloop, $i = 1, \dots, 5$ for Twoloop and $i = 1, \dots, 8$ for all three-loop diagrams. (Note, that only two uncontracted Lorenz indices m and n are allowed. In this case the common factor $PRO(m, n)$ should be used at the input.)

- Scalar variable EP^A , where $A \geq -1$ and EP codes the quantity $\varepsilon = 2 - D/2$.
- Function $PRO(m, n)$, which allows one to calculate integrals with two uncontracted Lorenz indices m and n in the numerator of the integrand. Thus, if the integrand contains uncontracted Lorenz indices m and n , then the corresponding input should be multiplied by $PRO(m, n)$.
- Any symbols admitted in FORM which are not used in the program. (The list of symbols and functions, which are used in the program can be found in the file `DECLARE`.)

After the execution of the program is finished the expression *Result* in the log-file of the block `FINAL` will contain the result of calculation in a form of polynomial in $1/EP$ (maximum exponent is 3 for a three-loop calculations) plus finite part - a rational number and Riemann zeta functions $\zeta(3)$, $\zeta(4)$ and $\zeta(5)$ with rational coefficients. The *Result* may contain an additional symbolic multiplicative factor which was used as a symbolic factor at the input and was transferred to the block `FINAL`. In the case when the input contains two uncontracted Lorenz indices m and n the result for the diagram should be proportional to the momentum factor: $g^{mn}Q^2A(\varepsilon) - Q^mQ^nB(\varepsilon)$. In this case the output of the program will contain a term which is multiplied by the symbolic factor `VA` and the other term with symbolic factor `VB`. These terms correspond to the g^{mn} and Q^mQ^n parts of the result. Furthermore, in the working directory the file *Result.res* will contain the result of calculation, which can be used as an input for another FORM package. Note again, that the standard version of the program prints the result in a normalization corresponding to the $\overline{MS}$ scheme,¹⁵ without taking into account gauge group weights, symmetry factors and the factor $(4\pi)^{-2}(\alpha/4\pi)^{(l-1)}(\mu^2/Q^2)^{(l\varepsilon)}$, where $\alpha = g^2/4\pi$ is the coupling, l is the number of loops and μ is 't Hooft's unit of mass.

A test runs of the program *HEP*Loops and the experience of using it in the calculations show that the program is fast enough to perform any three- and four-loop (with help of infrared rearrangement procedure⁵) calculations from the class of problems enumerated in the introduction. The program is much faster then the program *MINCER*.^{28,14} Some of the complicated four-loop diagrams contributing to $\Gamma(Z \rightarrow \text{hadrons})$ ¹³ have been recalculated by the program *HEP*Loops. The calculations of $O(\alpha_s^2)$ corrections to the decay rates of the Standard Model and the Minimal Supersymmetric Standard Model Higgs particles have been completed recently³⁷ with help of the program *HEP*Loops.³⁵

5. Test Run Output

Below the fragments of the test run output of the program *HEP*Loops for the diagram shown in Fig. 6 are presented. The total execution time is 188.2 sec. on the *Sun SPARC* station. (Compare the input, result and the CPU time to the ones from the Ref. 14 for the calculation of the same diagram by the program *MINCER*.)

Fig. 6. Three-loop diagram contributing to the $\Gamma(Z \rightarrow \text{hadrons})$. Solid lines correspond to the massless quark propagators, wave lines correspond to the gluon propagators in the Feynman gauge and crosses correspond to the electromagnetic current vertices.

Note, that the time shown on the Test Run Output is the time only of the execution of the last block FINAL.

```

FORM by J.Vermaseren. Version 1.1 3-jun-1992
*                                     PROGRAM HEPLoops
*                                     by Levan R. Surguladze
* ----- Version 2.0 -----
*           --- Block "START" ---
* ----- INPUT DATA -----
G Ladder    = PRO(m,n)
              *g_(1,P1,k,P2,l,P3,n,P4,l,P5,k,P6,m)
              /P1.P1/P2.P2/P3.P3/P4.P4/P5.P5/P6.P6/P7.P7/P8.P8;
G Benz      = 0;
G Nonplanar = 0;
G Twoloop   = 0;
G Oneloop   = 0;
* ----- END of INPUT -----
*           --- Block "FINAL" ---
*-
FINAL loaded
*           ---- R E S U L T ----

Time =      0.10 sec   Generated terms =      138
Result      Terms in output =      16
              Bytes used      =      554

Result =
+ VA * ( 226261/486 + 5903/81*EP^-1 + 254/27*EP^-2 + 8/9*EP^-3 -
320/3*dzeta(5) - 48*dzeta(4) - 1360/9*dzeta(3) - 32*dzeta(3)*
EP^-1 )
+ VB * ( 206539/486 + 5321/81*EP^-1 + 230/27*EP^-2 + 8/9*EP^-3 -
320/3*dzeta(5) - 48*dzeta(4) - 1360/9*dzeta(3) - 32*dzeta(3)*
EP^-1 );

```

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