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The q -deformation of quantum mechanics of one degree of freedom

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Abstract

The q -quantum mechanics of the one-degree of freedom is studied. Among others the holomorphic representation of q -deformed Heisenberg-Weyl algebra and its realization by covariant Berezin symbols is described.

1 Introduction.

The notion of coherent states map is fruitful for the describing of a physical systems, see [11, 12]. In this paper we use it for the construction of q -deformation, where $-1 \leq q \leq 1$, of the quantum mechanics of one-degree of freedom. Starting from the map K_q of the disc $\mathbf{D}_q$ into complex Hilbert space $\mathcal{M}$ we find the link between q -commutation relation (see [5, 6, 7]) and q -analysis (see [8]). In effect we construct the q -deformed Bargmann-Fock representation and prove the q -reproducing property for this case, see formula (33) and [1]. It means that $\exp_q(\bar{z}v)$ is a reproducing kernel for the space $L^2\mathcal{O}(\mathbf{D}_q, d\mu_q)$ and allows us to find an integral representation for elements of the Wick algebra.

In §4 we define the q -analogs of Heisenberg and Schrödinger equations. In the same section the q -deformation of Heisenberg-Weyl group is presented in terms of the q -exponential function and two others relations, valid for q -commutative elements of Wick algebra are proved, see Proposition 1 and Proposition 2. The analytical formula for the $*_q$ product of the covariant Berezin symbols is calculated explicitly in §5. At the end of the same section the correspondence between quantum description of bosons and fermions is discussed.

Although the representations of generalized commutation relations were studied by many authors, see [4, 5, 6, 7, 13, 14, 15, 16], in our approach we try to present exhaustive description of the connection between algebraic, geometric, analytic and physical aspects of the considered model.

2 Coherent states map and q -canonical commutation relation.

Let $\mathbf{D}_q$ be the open disc in $\mathbf{C}$ of radius $\frac{1}{\sqrt{1-q}}$ centered at zero, where $-1 < q < 1$, and let $\mathcal{M}$ be a complex separable Hilbert space with orthonormal basis $\{e_n\}, n = 0, 1, 2, \dots$. We define analytic map (the coherent states map), see [11], $K_q : \mathbf{D}_q \rightarrow \mathcal{M}$ by

$$K_q(z) = \sum_{n=0}^{\infty} \left[\frac{(1-q)^n}{(1-q) \cdots (1-q^n)} \right]^{\frac{1}{2}} z^n e_n, \quad (1)$$

where for $n = 0$ one assumes that the coefficient in front of e_0 is equal one. Because of

$$\langle K_q(z) | K_q(z) \rangle = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(1-q) \cdots (1-q^n)} (\bar{z}z)^n =: \exp_q(\bar{z}z) < \infty, \quad (2)$$

for $\bar{z}z < \frac{1}{1-q}$, the definition is correct. The function $\exp_q$ is called q -deformation of the exponential function $\exp = \exp_1$ (see [8]). So for $q=1$ one obtains the standard coherent states map (Bargmann - Fock coherent states map) and for $q=0$ one gets the geometric sequence $\exp_0 \bar{z}z = \frac{1}{1-\bar{z}z}$.

Having introduced the coherent states map one can define annihilation operator by

$$AK_q(z) := zK_q(z), \quad \forall z \in \mathbf{D}_q. \quad (3)$$

This is a bounded operator with norm $\|A\| = \frac{1}{\sqrt{1-q}}$ for $0 \leq q < 1$ and $\|A\| = 1$ for $-1 < q < 0$, which in the given basis is expressed by

$$Ae_n = \left(\frac{1-q^n}{1-q} \right)^{\frac{1}{2}} e_{n-1} \quad (4)$$

for $n \geq 1$ and $Ae_0 = 0$. Its hermitian conjugate $A^\dagger$ is given by

$$A^\dagger e_n = \left(\frac{1 - q^{n+1}}{1 - q} \right)^{\frac{1}{2}} e_{n+1}, \quad (5)$$

So defined operators satisfy q -Heisenberg commutations relation

$$[A, A^\dagger]_q := AA^\dagger - qA^\dagger A = 1 \quad (6)$$

which was studied, e.g. in the papers [4, 5, 6, 7, 13, 14, 15, 16]. Further we define the operators

$$Qe_n := q^n e_n, \quad (7)$$

$$Ke_n := [(1 - q)(1 - q^{n+1})]^{\frac{1}{2}} e_{n+1}, \quad (8)$$

$$Je_n := \left(\frac{1 - q}{1 - q^{n+1}} \right)^{\frac{1}{2}} e_{n+1}, \quad (9)$$

which will be important for the subsequent considerations. The following relations hold

$$Q = Q^\dagger = AA^\dagger - A^\dagger A, \quad (10)$$

$$K = (1 - q)A^\dagger, \quad (11)$$

$$K = J(1 - qQ). \quad (12)$$

The operators Q, K , and J are bounded with norms $\|Q\| = 1$ and $\|K\| = \|J\| = \sqrt{1 - q}$, for $0 \leq q < 1$. For the case $-1 < q < 0$, one has $\|K\| = \|J\| = 1 - q$. Let us also notice that

$$QK_q(z) = K_q(qz). \quad (13)$$

The formula $J = K + qJQ$ after iteration gives

$$J = K(1 + qQ + q^2Q^2 + \dots). \quad (14)$$

Because of $\|Q\| = 1$ and $|q| < 1$ the above sequence is summable. Therefore, the operators Q, K , and J belong to the C^* - algebra generated by the q -creation $A^\dagger$ and the q -annihilation A operators, i.e. q -Heisenberg - Weyl algebra W_q , or according to [7] the Wick algebra.

One has the relations

$$AJ = 1, \quad (15)$$

$$[A, Q]_q = 0, \quad (16)$$

$$[Q, A^\dagger]_q = 0, \quad (17)$$

$$[Q, J]_q = 0. \quad (18)$$

The relation (15) shows that J is a right inverse to annihilation operator. This is the reason why below we shall call it q -integration operator. Using relations (17) and (11) one obtains from (14) the formula

$$JK^n = K^{n+1}(1 + q^{n+1}Q + (q^{n+1})^2Q^2 + \cdots + (q^{n+1})^kQ^k + \cdots), \quad (19)$$

which we shall use in the next section.

3 Holomorphic representation

In this section we represent the Wick algebra by the operators acting in the space $L^2\mathcal{O}(\mathbf{D}_q, d\mu_q)$ of holomorphic functions on disc $\mathbf{D}_q$, which are square integrable with respect to the q -Gauss measure. This allows us to interpret $A, A^\dagger, Q$, and J as q -differentiation, multiplication by z , dilatation and q -integration operators, respectively. At the end we will show that $\langle K_q(z) | K_q(v) \rangle = \exp_q(\bar{z}v)$ is reproducing kernel function for the Hilbert space $L^2\mathcal{O}(\mathbf{D}_q, d\mu_q)$. We shall start from the embedding

$$I : \mathcal{M} \hookrightarrow \mathcal{O}(\mathbf{D}_q),$$

which is defined by

$$I(v)(z) := \langle v | K_q(z) \rangle, \quad (20)$$

where $v \in \mathcal{M}$. Since coherent states $K_q(z)$, $z \in \mathbf{D}_q$, form a linearly dense subset in $\mathcal{M}$, the map is an anti-linear monomorphism of complex vector spaces. Therefore $I(\mathcal{M})$ inherits Hilbert space structure with the scalar product given by $\langle I(v) | I(v) \rangle := \langle v | v \rangle$. After simple calculation one finds

$$(I \circ A \circ I^{-1}\varphi)(z) = (\partial_q \varphi)(z), \quad (21)$$

$$(I \circ A^\dagger \circ I^{-1}\varphi)(z) = z\varphi(z), \quad (22)$$

$$(I \circ Q \circ I^{-1}\varphi)(z) = \varphi(qz), \quad (23)$$

$$(I \circ J \circ I^{-1}\varphi)(z) = (J_q\varphi)(z), \quad (24)$$

where

$$(\partial_q\varphi)(z) := \frac{\varphi(z) - \varphi(qz)}{(1-q)z} \quad (25)$$

is the q -derivative and

$$(J_q\varphi)(z) := (1-q)z(\varphi(z) + q\varphi(qz) + \cdots + q^n\varphi(q^n z) + \cdots) \quad (26)$$

is q -integral, see [8] for a detailed discussion. The q -exponential function could be defined as the solution of the equation

$$\partial_q \exp_q = \exp_q, \quad (27)$$

which gives

$$\exp_q(z) = \frac{1}{1 - (1-q)z} \exp_q(qz). \quad (28)$$

Iterating (28) one obtains the formula

$$\exp_q(z) = \prod_{n=0}^{\infty} \frac{1}{1 - (1-q)q^n z}, \quad (29)$$

useful for further considerations.

The formula

$$\int_0^{\frac{1}{1-q}} x^n \exp_q(qz)^{-1} d_q x = \frac{(1-q) \cdots (1-q^n)}{(1-q)^n} \quad (30)$$

follows from (19) and (24).

Now, let us define the scalar product of holomorphic functions $\varphi, \psi \in \mathcal{O}(\mathbf{D}_q)$ by the q -integration over the disc $\mathbf{D}_q$

$$\begin{aligned} \langle \psi | \varphi \rangle_q &:= \int_{\mathbf{D}_q} \overline{\psi(z)} \varphi(z) (\exp_q(qz\bar{z}))^{-1} d\mu_q(z, \bar{z}) \\ &= \frac{1}{\pi} \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{1-q}}} \overline{\psi(\varrho e^{i\varphi})} \varphi(\varrho e^{i\varphi}) (\exp_q(q\varrho^2))^{-1} \varrho d_q \varrho \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{\frac{1}{1-q}} \overline{\psi(\sqrt{x} e^{i\varphi})} \varphi(\sqrt{x} e^{i\varphi}) (\exp_q(qx))^{-1} d_q x, \end{aligned} \quad (31)$$

where the radial integration is q -integration and $z = \varrho e^{i\varphi}$, $x = \varrho^2$. The integral with respect to the angle is taken in the Lebesgue sense. One can consider the scalar product $\langle \psi | \varphi \rangle_q$ as the standard integral over $\mathbf{D}_q$ with respect to the measure (the q -deformed Gauss measure)

$$d\mu_q(z, \bar{z}) = \frac{1}{2\pi} \sum_{n=0}^{\infty} q^n \delta(x - \frac{q^n}{1-q}) dx d\varphi. \quad (32)$$

Using formula (30) one obtains from (31) the equality

$$\langle I(e_n) | I(e_n) \rangle_q = \delta_{nm}$$

and thus the Hilbert spaces isomorphism

$$I : \mathcal{M} \xrightarrow{\sim} L^2 \mathcal{O}(\mathbf{D}_q, d\mu_q).$$

This fact could be expressed also by the resolution of identity

$$\int_{\mathbf{D}_q} P_q(z, \bar{z}) \frac{1}{1 - (1-q)z\bar{z}} d\mu_q(z, \bar{z}) = 1 \quad (33)$$

where

$$P_q(z, \bar{z}) = \frac{|K_q(z)\rangle \langle K_q(z)|}{\langle K_q(z) | K_q(z) \rangle}$$

is the orthogonal projector on the coherent states $K_q(z)$. The above means that $\exp_q(\bar{z}v)$ is the reproducing kernel for the Hilbert space $L^2 \mathcal{O}(\mathbf{D}_q, d\mu_q)$.

$$\varphi(v) = \int_{\mathbf{D}_q} \varphi(z) \exp_q(\bar{z}v) [\exp_q(qz\bar{z})]^{-1} d\mu_q(\bar{z}, z). \quad (34)$$

Thus one obtains additionally

$$(f(\partial_q)(v)) = \int_{\mathbf{D}_q} \varphi(z) f(\bar{z}) [\exp_q(qz\bar{z})]^{-1} d\mu_q(z, z) \quad (35)$$

where $f \in \mathcal{O}(\mathbf{D}_q)$ is such that the integral of the right side of (35) exists. If $X \in W_q$ and $\|X\| \leq \frac{1}{\sqrt{1-q}}$, then from (34) one gets an integral formula for the operator $\varphi(X)$

$$\varphi(X) = \int_{\mathbf{D}_q} \varphi(z) \exp_q(\bar{z}X) \cdot [\exp_q(qz\bar{z})]^{-1} d\mu_q(\bar{z}, z). \quad (36)$$

If $q \rightarrow 1$ the (34) becomes reproducing property for Bargmann-Fock representation. If $q = 0$, (3.4) becomes Cauchy formula and $L^2 \varphi(\mathbf{D}_0, d\mu_0)$ is the Hardy-Lebesgue space.

4 q -deformation of the Weyl-Heisenberg group and q -evolution

The purpose of this section is an explanation of what is the q -analog of the Weyl-Heisenberg group. We shall start with the following statement.

Proposition 1

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function on $\mathbf{D}_q$, and let X and Y be q -commuting elements of the Wick algebra, i.e. $[X, Y]_q = XY - qYX = 0$, and such that $\exp_q(X + Y), \exp_q(X), \exp_q(Y), f(X + Y)$ and $f(Y) \in W_q$. Then

$$f(X + Y) = [\exp_q(Y \partial_q) f](X) \quad (37)$$

and

$$\exp_q(X + Y) = \exp_q(X) \exp_q(Y) \quad (38)$$

Proof.

From $[X, Y]_q = 0$ one obtains

$$(X + Y)^n = \sum_{k=0}^n \binom{n}{k}_q Y^k X^{n-k}, \quad (39)$$

where

$$\binom{n}{k}_q = \frac{(1 - q) \cdots (1 + q + \cdots + q^{n-1})}{(1 - q) \cdots (1 + q + \cdots + q^{k-1}) (1 - q) \cdots (1 + q + \cdots + q^{n-k-1})}.$$

Because of

$$\partial_q^k z^n = (1 + q + \cdots + q^{n-1}) \cdots (1 + q + \cdots + q^{n-k}) z^{n-k}$$

the formula (39) can be rewritten in the form

$$(X + Y)^n = \sum_{k=0}^n \frac{(1 - q)^k}{(1 - q) \cdots (1 - q^k)} Y^k \partial_q^k (X^n) = \exp_q(Y \partial_q) X^n. \quad (40)$$

Multiplying by a_n and summing with respect to the index n one comes to (37). Then (38) follows from (37) if one puts $f = \exp_q$. $\square$

A similar consideration gives

$$Af(A^\dagger) = f(qA^\dagger)A + (\partial_q f)(A^\dagger). \quad (41)$$

Putting $f(z) = \exp_q(tz)$ one gets

$$A \exp_q(\bar{t}A^\dagger) = \exp_q(\bar{t}qA^\dagger)A + \bar{t} \exp_q(\bar{t}A^\dagger). \quad (42)$$

Because of (28) one has

$$\exp_q(q\bar{t}A^\dagger) = (1 - (1 - q)tA^\dagger) \exp_q(tA^\dagger). \quad (43)$$

Combining (42) and (43) one obtains

$$A^n \exp_q(\bar{t}A^\dagger) = \exp_q(\bar{t}qA^\dagger)(A + \bar{t}Q)^n \quad (44)$$

and thus

$$\exp_q(sA) \exp_q(tA^\dagger) = \exp_q(\bar{t}A^\dagger) \exp_q(sA + s\bar{t}Q). \quad (45)$$

Taking account of $[A, Q]_q = 0$ and (38), we see that we have proved the following:

Proposition 2

The q -deformed Weyl-Heisenberg relation is given by

$$\exp_q(sA) \exp_q(tA^\dagger) = \exp_q(\bar{t}A^\dagger) \exp_q(s\bar{t}Q) \exp_q(sA) \quad (46)$$

for $s, t \in \mathbf{D}_q$ and $-1 < q < 1$.

Now, let us propose the q -evolution in time of the system described by selfadjoint Hamiltonian $H^\dagger = H \in W_q$ with the norm $\|H\| \leq \frac{1}{\sqrt{1-q}}$ (i.e. bounded by the radius of the disc. We will assume also that the time is bounded by $-\frac{1}{\sqrt{1-q}} < t < \frac{1}{\sqrt{1-q}}$. Therefore the state

$$\psi(t) = \exp_q(itH)\psi_0, \quad (47)$$

for $\psi_0 \in \mathcal{M}$ is well defined and satisfies q -analog of the Schrödinger equation

$$-i\partial_q \psi(t) = H\psi(t). \quad (48)$$

Let $B \in W_q$ and let the operator $B(t)$ be defined by the condition

$$\langle \psi(t) | B \psi(t) \rangle = \langle \psi_0 | B(t) \psi_0 \rangle, \quad \psi_0 \in \mathcal{M}. \quad (49)$$

Thus, one has

$$B(t) = \exp_q(-itH) B \exp_q(itH)$$

and $B(t)$ satisfies the q -analogue of the Heisenberg equation

$$i\partial_q B(t) = [H, B(t)]_q - (1 - q)(B(t) - it\mathbf{1})H. \quad (50)$$

Because of $[H, H]_q \neq 0$ the q -evolution operators $\exp_q(itH)$ do not satisfy the one - parameter group property, which holds only when $q \rightarrow 1$.

Example.

A standard example of the physical system is the harmonic oscillator. In order to formulate its q -version one defines the operators

$$X = \frac{1}{\sqrt{2}}(A + A^\dagger), \quad (51)$$

$$P = \frac{-i}{\sqrt{2}}(A - A^\dagger),$$

which describe position and momentum of the considered system. The Hamiltonian of the q -harmonic oscillator is given by

$$H = \frac{1}{2}(P^2 + X^2) = \frac{1}{2}(AA^\dagger + A^\dagger A). \quad (52)$$

Now, the q -canonical commutation relation can be expressed by

$$XP - PX = \frac{2i}{1+q}\mathbf{1} - 2i\frac{1-q}{1+q}H. \quad (53)$$

One has

$$H = \frac{1}{1-q}\mathbf{1} - \frac{1}{2}\frac{1+q}{1-q}Q. \quad (54)$$

Thus from (7) one finds

$$He_n = E_n e_n,$$

where the spectrum is given by

$$E_n = \frac{1-q^n}{1-q} + \frac{1}{2}q^n \quad (55)$$

for $n = 0, 1, 2, \dots$. If $q \rightarrow 0$ one obtains $E_n = n + \frac{1}{2}$.

5 The star product of Berezin symbols. Concluding remarks.

The mean values of the observable $B \in W_q$ on the coherent states $K_q(z)$ where $z \in \mathbf{D}_q$, define function

$$\langle B \rangle(z, \bar{z}) := \frac{\langle K_q(z) | B K_q(z) \rangle}{\langle K_q(z) | K_q(z) \rangle}, \quad (56)$$

which is called covariant Berezin symbol of the operator B (see [2, 3]). Because $K_q : \mathbf{D}_q \rightarrow \mathcal{M}$ is holomorphic, the operator B is determined by its covariant symbol. The product of two operators $B, C \in W_q$ defines the product of their symbols

$$\langle B \rangle *_q \langle C \rangle(z, \bar{z}) = \frac{\langle K_q(z) | B C K_q(z) \rangle}{\langle K_q(z) | K_q(z) \rangle} \quad (57)$$

and thus the Wick algebra W_q could be completely described by $*_q$ -product algebra of covariant Berezin symbols.

Let f, g be real analytic functions on the open domains $\Omega \supset \overline{\mathbf{D}_q}$. Because of $\|A\| = \|A^\dagger\| = \frac{1}{\sqrt{1-q}}$, or $\|A\| = \|A^+\| = 1$ the operators

$$f(A^\dagger, A) = \sum_{\bar{n}, m=0}^{\infty} f_{\bar{n}m} A^{\dagger \bar{n}} A^m, \quad (58)$$

$$g(A^\dagger, A) = \sum_{\bar{k}, l=0}^{\infty} g_{\bar{k}l} A^{\dagger \bar{k}} A^l,$$

expressed in the Wick ordering, belong to W_q . Their covariant Berezin symbols are given by $\langle f(A^\dagger, A) \rangle = f$ and by $\langle g(A^\dagger, A) \rangle = g$, respectively. In this case the product formula takes the form

$$(f *_q g)(\bar{z}, z) = \sum_{\bar{n}, m=0}^{\infty} f_{\bar{n}m} \sum_{\bar{k}, l=0}^{\infty} g_{\bar{k}l} \bar{z}^{\bar{n}} \frac{\langle K_q(z) | A^m A^{\dagger \bar{k}} K_q(z) \rangle}{\langle K_q(z) | K_q(z) \rangle} z^l. \quad (59)$$

In view of the identity

$$\frac{\langle K_q(z) | A^m A^{\dagger \bar{k}} K_q(z) \rangle}{\langle K_q(z) | K_q(z) \rangle} = \frac{1}{\langle K_q(z) | K_q(z) \rangle} \bar{\partial}_q^m \partial_q^{\bar{k}} \langle K_q(z) | K_q(z) \rangle, \quad (60)$$

one obtains the following expression for product of two covariant Berezin symbols

$$(f *_q g)(\bar{z}, z) = \frac{1}{\exp_q(z\bar{z})} f(\bar{z}, \bar{\partial}_q) [g(\bar{z}, z) \exp_q(z\bar{z})]. \quad (61)$$

Specifying (61) one finds

$$z^n *_q g(\bar{z}) = \sum_{k=0}^n \binom{n}{k}_q \bar{z}^{n-k} (1 - (1-q)z\bar{z})(1 - (1-q)qz\bar{z}) \cdots (1 - (1-q)q^{k-1}z\bar{z}) (\partial_g^k)(z).$$

The q -deformed Heisenberg-Weyl relation expressed in the terms of the covariant Berezin symbols assumes the form

$$\begin{aligned} \exp_q(sz) *_q \exp_q(\bar{t}\bar{z}) &= \exp_q(sz) \exp_q(\bar{t}z) \cdot \sum_{n=0}^{\infty} \frac{(1-q)^n}{(1-q) \cdots (1-q^n)} (\bar{s}t)^n \times \\ &\times (1 - (1-q)z\bar{z})(1 - (1-q)qz\bar{z}) \cdots (1 - (1-q)q^{n-1}z\bar{z}), \end{aligned} \quad (62)$$

where $s, t \in \mathbf{D}_q$. Recalling the identity

$$\exp_q(\bar{t}\bar{z}) *_q \exp_q(sz) = \exp_q(sz) \exp_q(\bar{t}z),$$

we see that the non-commutativity of the multiplication of q -Heisenberg-Weyl group is determined by the covariant Berezin symbol of the operator $\exp_q(stQ)$.

We could extend our considerations to the case $|q| < 1, q \in \mathbf{C}$, or to the case $q^k = 1, k \in \mathbf{N}$. But for the complex q , the scalar product $\langle K_q(z) | K_q(z) \rangle$ of the coherent states is not expressed by the q -exponent function. In the consequence the model loses its elegance and its physical interpretation is unclear. However, the case $q^k = 1$ is of physical interest. Namely, if $q \rightarrow 1, k = 1$ the q -quantum mechanics corresponds to standard Heisenberg-Schrödinger boson quantum mechanics and all relations and formulae considered in the previous sections correspond to their quantum mechanical counterparts.

For $q \rightarrow -1, k = 2$, the formulae (4) and (5) preserve their meaning and give

$$Ae_{2n} = 0, \quad Ae_{2n+1} = e_{2n}, \quad (63)$$

$$A^\dagger e_{2n} = e_{2n+1}, \quad A^\dagger e_{2n+1} = 0,$$

Thus, one obtains fermionic anti-commutation relations

$$AA^\dagger + A^\dagger A = 1 \quad A^2 = A^{\dagger 2} = 0. \quad (64)$$

Because of $A^2 = 0$, we can preserve the notion of the coherent states map $K_{-1} : \mathbf{D}_{-1} \rightarrow \mathcal{M}$ in this case if and only if $z^2 = 0$. That is, one has to assume that z is a Grassmann variable. Then

$$K_{-1} = e_0 + ze_1 \quad (65)$$

and

$$\langle K_{-1}(z) | K_{-1}(v) \rangle = \exp_{-1}(\bar{z}v) = 1 + \bar{z}v, \quad (66)$$

for $z^2 = v^2 = \bar{z}^2 = \bar{v}^2 = 0$. The representation in the Hilbert space $\mathcal{M}$ will be irreducible and $K_{-1}(\mathbf{D}_{-1})$ will be linearly dense in $\mathcal{M}$ iff $\dim_{\mathbb{C}} \mathcal{M} = 2$. So the holomorphic representation makes the sense, too. Hence, one can interpret annihilation operator A and creation operator $A^\dagger$ as the ∂_{-1} -derivative and multiplication by z respectively.

The Heisenberg-Weyl relation (46), q -Schrödinger equation and q -Heisenberg equation for $q = -1$ also are valid. So we conclude that the model of q -quantum mechanics presented in this letter gives a one-parameter deformation $-1 \leq q \leq 1$ of the boson quantum mechanics to the fermion quantum mechanics, which are singular cases of the model. Also the case $q^k = 1$, $k > 2$, after assuming that $z^k = 0$, can be described in the way proposed above.

In the result one comes to q -quantum mechanics of the systems which satisfy parastatistics different from bosonic and fermionic ones.

If one takes $K_{q,h} : \mathbf{D}_{q,h} \rightarrow \mathcal{M}$, where $\mathbf{D}_{q,h} = \{z \in \mathcal{C} : |z| < \sqrt{\frac{h}{1-q}}\}$ and $K_{q,h}(z) = K_q(\frac{z}{\sqrt{h}})$ for $h > 0$, the annihilation and creation operators change by the factor $\sqrt{h}$ and deformed Heisenberg commutation relation assumes the form

$$AA^\dagger - qA^\dagger A = h.$$

In the consequence all formulae in the paper will be deformed by h in the way determined by dimensional argument. Among the others the deformation of the $\star$ -product formula (58) will be given by

$$(f \star_{(q,h)} g)(\bar{z}, z) = \frac{1}{\exp_q(\frac{\bar{z}z}{h})} f(\bar{z}, h\bar{\partial}_q) [g(\bar{z}, z) \exp_q(\frac{z\bar{z}}{h})]. \quad (67)$$

Thus, one has

$$f \star_{(q,h)} g \xrightarrow{q,h \rightarrow 0} f_g \quad (68)$$

and

$$\frac{1}{h} [f \star_{(q,h)} g - g \star_{(q,h)} f] \xrightarrow{q,h \rightarrow 0} \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial \bar{z}}. \quad (69)$$

From the above we see that the model discussed here is a (q, h) -deformation of the Hamiltonian mechanics of one-degree of freedom. In order to have $\mathbf{D}_{q,h} \xrightarrow{q,h \rightarrow 0} \mathcal{C}$, where $\mathcal{C}$ is interpreted as the classical phase space, one assumes that $h, q \rightarrow 0$ in such way that $\sqrt{\frac{h}{1-q}} \rightarrow \infty$.

The generalization to n -dimensional case is possible and we plan to study it in the further publications.

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