A CLASS OF QUANTUM STATES WITH CLASSICAL-LIKE EVOLUTION

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Abstract

In the framework of the Stochastic Formulation of Quantum Mechanics we derive non-stationary states for a class of time-dependent potentials. The wave-packets follow a classical motion with constant dispersion. The new states define a possible extension of the harmonic-oscillator coherent states. As an explicit application we study a sestic oscillator potential.

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Coherent states were introduced for the first time by Schrödinger in his attempt at finding the quantum mechanical states that evolve according to a classical law without spreading of the wave-packet [1].

Starting from the seminal works of Glauber, Klauder and Sudarshan [2], coherent states have come to play a central role in many different areas of research, ranging from quantum mechanics and quantum optics to quantum field theory and condensed matter physics. In the course of time they have been thoroughly analysed in a group-theoretical framework [3], and extended in the direction of squeezing phenomena [4].

On the other hand, the original problem formulated by Schrödinger, namely whether and in what sense coherent structures can be introduced for general quantum systems, is still open.

Coherent states of arbitrary potentials should be such to retain as much as possible the basic property of the harmonic-oscillator coherent states, namely they should be non-stationary states without dispersion of the wave-packet width and with time evolution driven by a classical equation. Thus the search for a sensible definition of generalized coherent states should be based on the above physical requirement. This way of looking at the problem has been strongly advocated and pursued by Nieto and collaborators who constructed coherent states associated to arbitrary potentials ([5], [6]).

These coherent states, however, can only be constructed for classically integrable systems for which there exists a set of canonical coordinates and momenta \( \{Q, P\} \) such that the respective Hamiltonians can be reduced to quadratures. The canonical transformations from the original variables \( \{q, p\} \) to the “natural” [5] variables \( \{Q, P\} \) are energy-dependent.

Quantization of the variables \( \{Q, P\} \rightarrow \{\hat{Q}, \hat{P}\} \) allows to built the coherent states.

The energy-dependence of the classical “natural” [5] variables \( \{Q, P\} \) determines the level-dependence of \( \hat{Q} \) and \( \hat{P} \) through the Hamiltonian \( \hat{H} \) and prevents in general the exact solution of the equations defining the generalized coherent states. Approximate solutions can be obtained by letting \( \hat{H} \rightarrow \langle \hat{H} \rangle \). The more confining a potential is, and the closer to equally spaced its energy eigenstates are, then the better the generalized coherent state will evolve according to a classical law, with conserved spreading of the wave-packet, until, in the limit of systems with equally spaced energy levels, the approximation becomes exact [6].

The above scheme has been applied also to define the squeezed states.
of general potentials [7], while other classes of generalized squeezed states have been recently introduced by Nieto and Truax by an annihilation-operator method [8]. The different definitions of generalized coherent and squeezed states for non-harmonic potentials and the connections among them are nicely reviewed in a recent report by Nieto [9].

In this letter we consider again the problem of Schrödinger for non-harmonic potentials. Our strategy to construct generalized coherent states is based on imposing some conditions that have to be satisfied directly in the quantum regime and for the original variables $\{\hat{q}, \hat{p}\}$. More precisely, the requirements we impose are a constant spreading $\Delta \hat{q}$ and the following classical evolution equations

$$\frac{m}{2} \frac{d^2}{dt^2} \langle \hat{q} \rangle = -\nabla V(q, t) \big|_{q=\langle \hat{q} \rangle} ,$$

$$\langle \hat{p} \rangle = \frac{m}{dt} \langle \hat{q} \rangle .$$

Our construction is obtained exploiting the techniques of Nelson stochastic quantization and the relation among stochastic and quantum uncertainties. We select a class of potentials which satisfy Schrödinger’s conditions; these potentials in general depend from the classical trajectory described by the configurational mean value.

The Stochastic Formulation of Quantum Mechanics (S.F.Q.M.), originally introduced by Nelson, is a useful tool in the study of the relationships among classical and quantum evolutions [10]. It is indeed based on the replacement of the deterministic trajectories of classical mechanics with suitably chosen Brownian paths, and it is thus close in spirit to Feynman path integral quantization.

Consider, without loss of generality, a one-dimensional system. At the kinematical level S.F.Q.M. associates to the classical configurational variable a diffusion process $q(t)$ governed by Itô’s stochastic differential equation

$$dq(t) = v_+(q(t), t) dt + \left( \sqrt{\frac{\hbar}{2m}} \right) dw(t), \quad dt > 0 .$$

Here $v_+(q(t), t)$ is the forward drift, and $dw(t)$ is the time-increment of the Gaussian white noise $w(t)$, superimposed on the otherwise deterministic
evolution, with expectation $E(dw(t)) = 0$ and covariance $E(dw^2(t)) = 2dt$. The probability density $\rho(x, t)$ associated to the process satisfies the forward and backward Fokker-Planck equations. The forward and the backward drifts $v_+(x, t)$ and $v_-(x, t)$ are defined as

$$v_+(x, t) = \lim_{\Delta t \to 0^+} E \left( \frac{q(t + \Delta t) - q(t)}{\Delta t} \mid q(t) = x \right),$$

$$v_-(x, t) = \lim_{\Delta t \to 0^+} E \left( \frac{q(t) - q(t - \Delta t)}{\Delta t} \mid q(t) = x \right).$$

(3)

From definitions (3) it is clear that the forward (backward) drift is the mean forward (backward) velocity field.

The relation between $v_+$ and $v_-$ is:

$$v_- = -\frac{\hbar}{m} \frac{\partial_x \rho}{\rho}. \tag{4}$$

We define also the osmotic velocity $u(x, t)$ and the current velocity $v(x, t)$ as

$$u = \frac{v_+ - v_-}{2} = \frac{\hbar}{2m} \frac{\partial_x \rho}{\rho}, \tag{5}$$

$$v = \frac{v_+ + v_-}{2}.$$

The osmotic velocity $u(x, t)$ then “measures” the non-differentiability of the random trajectories; in the classical limit, $\hbar \to 0$, $u$ vanishes and the current velocity $v(x, t)$ goes to the classical velocity.

As a last consequence of Fokker-Planck equation we have for the probability density $\rho(x, t)$ the continuity equation

$$\partial_t \rho = -\partial_x \left( \rho v \right). \tag{6}$$

At the dynamical level each single-particle quantum state $\Psi(x, t)$ written in the form

$$\Psi(x, t) = \rho^{\frac{i}{\hbar}} \exp \left[ \frac{i}{\hbar} S(x, t) \right], \tag{7}$$
where $\rho$ and $S$ are two real functions, corresponds in S.F.Q.M. to the diffusion process $q(t)$ with

$$\rho(x, t) = |\Psi(x, t)|^2,$$  \hspace{1cm} (8)

and

$$v(x, t) = \frac{1}{m} \partial_x S(x, t),$$  \hspace{1cm} (9)

where $m$ is the mass of the particle.

The Schrödinger equation with potential $V(x, t)$ for the complex wavefunction $\Psi$ is then equivalent to two coupled real equations for the probability density $\rho$ and the phase $S$.

The first equation is the continuity equation (6), while the second one is the hydrodynamical Hamilton-Jacobi-Madelung (HJM) equation

$$\partial_t S + \frac{\partial_x S}{2m} = -V,$$  \hspace{1cm} (10)

or in terms of the osmotic velocity $u$

$$\partial_t S + \frac{mv^2}{2} - \frac{m}{2} u^2 - \frac{\hbar}{2} \partial_x u = -V.$$  \hspace{1cm} (11)

The gradient of the HJM equation yields

$$m \partial_t v + m v \partial_x v - m u \partial_x u - \frac{\hbar}{2} \partial_x^2 u = -\partial_x V.$$  \hspace{1cm} (12)

This equation can be seen as the quantum analogue of Newton’s equation in a force field [10].

The correspondence between expectations and correlations defined in the stochastic and in the canonic pictures are
\[\langle \hat{q} \rangle = E(q), \quad \langle \hat{p} \rangle = mE(v), \]

\[\Delta \hat{q} = \Delta q, \quad (\Delta \hat{p})^2 = m^2[(\Delta u)^2 + (\Delta v)^2], \tag{13}\]

and furthermore the chain inequality holds

\[ (\Delta \hat{q})^2(\Delta \hat{p})^2 \geq m(\Delta q)^2(\Delta u)^2 \geq \frac{\hbar^2}{4}, \tag{14}\]

where \(\hat{q}\) and \(\hat{p}\) are the position and momentum observables in the Schrödinger picture, \(\langle \cdot \rangle\) denotes the expectation value of the operators in the given state \(\Psi\), \(E(\cdot)\) is the expectation values of the stochastic variable associated in the Nelson picture to the state \(\{\rho, S\}\), and \(\Delta(\cdot)\) denotes the root mean square deviation.

The chain inequality (14), i.e. the osmotic velocity-position stochastic uncertainty and its equivalence with the momentum-position quantum uncertainty, were proven in [11].

The diffusion processes that minimize the stochastic uncertainty product have been recently derived [12]. From eqs. (13), (14) and the analysis exploited in ref. [12], it follows that the Nelson minimum uncertainty states are comprehensive both of the standard Glauber and Klauder-Sudarshan harmonic oscillator coherent states (Heisenberg minimum uncertainty states), and of all the Schrödinger minimum uncertainty states.

Through saturation of the osmotic velocity-position stochastic uncertainty we have

\[ u(x, t) = -\frac{\hbar}{2m\Delta q} \left( \frac{x - E(q)}{\Delta q} \right); \tag{15}\]

if we insert eq. (15) in the continuity equation (6) we obtain, after simple calculations, the following form of the current velocity

\[ v(x, t) = \frac{d}{dt} E(q) + \left( \frac{x - E(q)}{\Delta q} \right) \frac{d}{dt} \Delta q. \tag{16}\]
The minimum uncertainty stochastic states (MUSSs) are thus determined by a current velocity \( v \) and an osmotic velocity \( u \) both linear in the argument \( (x - E(q))/\Delta q \). It can be shown that the MUSSs can be divided in two classes, the first with constant \( \Delta q \) and the second with time-dependent \( \Delta q \). In the actual context it is convenient to consider \( \Delta q \) constant (for the more general case see [12]). Inserting, relations (15) and (16) into eq. (11) we find that

\[
V(x, t) = \frac{m}{2} \omega^2 x^2, \tag{17}
\]

\[
\frac{d^2}{dt^2} E(q) = -\omega^2 E(q), \tag{18}
\]

and we obtain the standard Glauber coherent states wave functions. We note that, while \( u(x, t) \) of the form (15) implies \( v(x, t) \) of the form (16), the contrary is not true. In fact, this is an interesting general feature of the S.F.Q.M.: a given choice of the current velocity \( v \) determines a whole class of osmotic velocities \( u \) and thus selects a class of quantum states with the same dynamical evolution. A simple but relevant example [13] is given by the choice \( v = 0 \); in this case it is immediately seen that eq. (11) becomes the standard eigenvalue equation for the Hamiltonian operator.

Now, we use this property to construct general coherent states in the sense of Schrödinger. To this aim, it is natural to search for the class of quantum states selected by a current velocity of the minimum-uncertainty form (16) with constant \( \Delta q \):

\[
v = \frac{d}{dt} E(q). \tag{19}
\]

Exploiting the continuity equation (6) written in terms of \( v \) and \( u \) and substituting for \( v \) the form (16) we have that \( u \) satisfies the equation

\[
\partial_t u = -E(v) \partial_x u, \tag{20}
\]

whose general solution is

\[
u = \frac{1}{\Delta q} G(\xi), \tag{21}\]

with

\[
\xi = \frac{x - E(q)}{\Delta q}. \tag{22}\]
The function $G(\xi)$ can be arbitrary, only restricted by the condition that $u$ must yield a normalizable probability density $\rho(\xi)$.

The minimum-uncertainty osmotic velocity (15) linear in $\xi$ is then just a particular case of the general form (22).

Inserting (21) and (22) into eq. (12) we obtain the identity

$$m \frac{d^2}{dt^2} E(q) = \frac{m}{(\Delta q)^3} \left( \frac{\hbar}{2m} G''(\xi) + G(\xi)G'(\xi) \right) - \partial_x V(x, E(q)),$$  \hspace{1cm} (23)

where the primes denote derivation respect to $\xi$. If one compute relation (23) in $x = E(q) \equiv \langle \hat{q} \rangle$ (which corresponds also to $\xi = 0$), it is then evident that the classical equations of motion (1) for $E(q)$ and then for $\langle \hat{q} \rangle$ is guaranteed by the choice (19) and the condition

$$\frac{m}{(\Delta q)^3} \left( \frac{\hbar}{2m} G''(0) + G(0)G'(0) \right) \bigg|_{\xi=0} = 0.$$  \hspace{1cm} (24)

Now, a particular choice of $G(\xi)$ which satisfies condition (24) completely determines, through the relations and definitions (8), (9), (11), (12) the potential $V$ and the wave-function of the "coherent-like" state, which has namely the general form

$$\Psi_\xi(x,t) = \frac{1}{\Delta q} \exp \left\{ \frac{2m}{\hbar \Delta q} \int_0^\xi G(\xi') d\xi' + \frac{i}{\hbar} (x\langle \hat{p} \rangle + S_0(t)) \right\},$$  \hspace{1cm} (25)

where $S_0(t)$ is calculated in $x = x_0 = 0$.

To clarify our structure, we now choose a particular form of the function $G(\xi)$, and namely of the osmotic velocity $u$:

$$u = -\frac{\hbar}{2m \Delta q} \xi^3.$$  \hspace{1cm} (26)

The choice of the coefficient $\hbar/2m$ is dictated by dimensional considerations, while the minus sign is fixed by requiring normalization of the density $\rho$. Notice that it is the same coefficient appearing in the Heisenberg minimum-uncertainty $u$ . In fact, $-\hbar/2m$ is the universal coefficient for any polynomial choice of $u(\xi)$. From eq. (27) and the first of eqs. (5) one obtains the normalized probability density

$$\rho(x,t) = \sqrt{\frac{2}{(\Delta q)^2}} \left[ \Gamma \left( \frac{1}{4} \right) \right]^{-1} \exp \left[ -\frac{(x - E(q))^4}{4(\Delta q)^4} \right],$$  \hspace{1cm} (27)
where \( \Gamma(1/4) \) is the Gamma-function \( \Gamma(s) \) evaluated in \( s = 1/4 \). Inserting the choice (26) for \( u \) and the expression (19) for \( v \) in the HJM equation (11) we obtain that the potential must be of the form:

\[
V(x,t) = \sum_{i=0}^{6} a_i(E(q))x^i.
\]  

(28)

This is the expression of a sextic oscillator potential, whose coefficients \( \{a_i\} \) are time-dependent as functions of the expectation \( E(q) \). Their explicit expressions are computed through the identity (11) with the choices (26), (19), and are given by:

\[
a_0 = \frac{\hbar^2}{8m(\Delta q)^8}, \quad a_4 = -\frac{3\hbar^2}{4m(\Delta q)^8}E(q),
\]

\[
a_1 = \frac{15\hbar^2}{8m(\Delta q)^8}E^2(q) - \frac{3\hbar^2}{2m(\Delta q)^8}E^3(q),
\]

\[
a_2 = \frac{15\hbar^2}{8m(\Delta q)^8}E^4(q) - \frac{3\hbar^2}{4m(\Delta q)^4},
\]

\[
a_3 = \frac{15\hbar^2}{8m(\Delta q)^8}E^5(q) - \frac{3\hbar^2}{2m(\Delta q)^8}E^6(q),
\]

\[
a_4 = -m\frac{d^2}{dt^2}E(q) + \frac{3\hbar^2}{2m(\Delta q)^8}E^2(q) - \frac{3\hbar^2}{4m(\Delta q)^4}E^3(q),
\]

\[
a_5 = \frac{\hbar^2}{2m(\Delta q)^8}E^4(q) - \frac{3\hbar^2}{4m(\Delta q)^4}E^5(q).
\]

The coefficients are thus all time-dependent but that of the highest power, which is constant. The coefficient of the quadratic power is indeed made up of two contributions, a constant and a time-dependent one. It is straightforward to verify that the condition (24) is identically satisfied. We are thus assured that the states corresponding to the osmotic velocity (26) follow the classical evolution (1). From the coefficient of the linear term in the potential, we read off the actual equation of motion obeyed by the expectation \( \langle \dot{q} \rangle \equiv E(q) \) and so determine the corresponding time-evolution of the state. By choosing without loss of generality \( a_0 = a_1 = 0 \), one can easily verify that the time-dependent part of the potential (28) gives no contribution to the classical equation of motion (1), which reads

\[
m\frac{d^2}{dt^2}\langle \dot{q} \rangle = -\frac{3\hbar^2}{4m(\Delta q)^8}\langle \dot{q} \rangle^5 + \frac{3\hbar^2}{2m(\Delta q)^8}\langle \dot{q} \rangle.
\]  

(30)
The state thus evolves in a classical potential field \( V_{cl} \) which is obtained by retaining only the time-independent part of the expression \( V \), and reads

\[
V_{cl}(x) = \frac{\hbar^2}{8m(\Delta \dot{q})^2} x^6 - \frac{3\hbar^2}{4m(\Delta \dot{q})^4} x^2. \tag{31}
\]

The classical part \( S_0(t) \) of the action \( S(x, t) \) is readily determined from the last of eqs. (29) by exploiting eq. (30). Collecting things together, reminding definitions (7), (13) and expression (27) for the density \( \rho \), one finally obtains the explicit expression for the states:

\[
\Psi_c(x, t) = \left[ \frac{\Delta \dot{q} \Gamma(1/4)}{2} \right]^{-\frac{1}{2}} \exp \left\{ -\frac{(x - \langle \dot{q} \rangle)^4}{8(\Delta \dot{q})^4} + \frac{i}{\hbar} \left( \langle \hat{p} \rangle x - \frac{1}{2} \langle \dot{q} \rangle \langle \hat{p} \rangle \right) \right\}. \tag{32}
\]

The wave-functions (32) appear as a natural extension of the standard Glauber coherent states to non-harmonic potentials. The modulus is modulated by the actual interaction. The phase retains the same structure as in the case of the Glauber wave-functions; namely, all these phases in fact encode the classical-like dynamics of the wave packet.

Note that we give here the sestic polynomial as the simplest solvable example besides the well known harmonic one. However, there are many other potentials with classical motion that can be selected through our approach; for instance, more complex polynomials, periodic potentials, an extension of the Morse potential, etc.

In conclusion, we have found a class of potentials and states compatible with condition (1) and constant \( \Delta \dot{q} \).

Our class of states contains as a subset the standard Glauber coherent states.

It is interesting to observe, as it can be seen from the above polynomial example (26)-(32), that these states are associated to potentials which are time-dependent through \( \dot{\dot{q}} \). This last quantity is in general governed by condition (1) which, in the case of the sestic oscillator potential, specializes to eq. (30).

We think that these new states show some intriguing features and should deserve further study and understanding. A possible future line of research might be connected with the fact that coherent states are a natural bridge for studying the quantum-classical correspondence. From this point of view, the construction of coherent-like states for non-harmonic potentials might be
useful to understand the manifestation of classical chaos in quantum systems [14].
References


