Billiard Representation for Multidimensional Cosmology with Multicomponent Perfect Fluid near the Singularity

V. D. Ivashchuk and V. N. Melnikov

Center for Surface and Vacuum Research,
8 Kravchenko str., Moscow, 117331, Russia
e-mail: mel@cvsi.uucp.free.net

ABSTRACT

The multidimensional cosmological model describing the evolution of \( n \) Einstein spaces in the presence of multicomponent perfect fluid is considered. When certain restrictions on the parameters of the model are imposed, the dynamics of the model near the singularity is reduced to a billiard on the \((n-1)\)-dimensional Lobachevsky space \( H^{n-1} \). The geometrical criterion for the finiteness of the billiard volume and its compactness is suggested. This criterion reduces the problem to the problem of illumination of \((n-2)\)-dimensional sphere \( S^{n-2} \) by point-like sources. Some generalization of the considered scheme (including scalar field and quantum generalizations) are considered.

PACS numbers: 04.20, 04.40.

Moscow 1994
1 Introduction

Last years multidimensional classical and quantum cosmology (see, for example, [1-33] and references therein) became a rather popular object of investigations both from physical and mathematical points of view. A lot of interesting topics in multidimensional cosmology were considered: exact solutions and problem of integrability, superstring cosmology and problem of compactification, variation of constants, classical and quantum wormholes, chaotic behavior near the singularity etc.

In the present paper we deal with a stochastic behavior in multidimensional cosmological models [24-33]. This direction in higher-dimensional gravity was stimulated by well-known results for "mixmaster" model [34-37]. We note, that there is also an elegant explanation for stochastic behavior of scale factors of Bianchi-IX model suggested by Chitre [36,37] and recently considered in [38,39,40]. (For "history" of the problem see also [41].) In the Chitre’s approach the Bianchi-IX cosmology near the singularity is reduced to a billiard on the Lobachevsky space $H^2$ (see Fig. 4 below). The volume of this billiard is finite. This fact together with the well-known behavior (exponential divergences) of geodesics on the spaces of negative curvature leads to a stochastic behavior of the dynamical system in the considered regime [42,43].

It is quite natural to generalize the approach [36] to the multidimensional case. This program was started in [33]. The present paper is devoted to a construction of "billiard representation" for the multidimensional cosmological model describing the evolution of $n$ Einstein spaces in the presence of $(m + 1)$-component perfect fluid [21] (see Sec. 2). One of these components correspond to the cosmological constant term [20]. In some sense the model [21] may be considered as "universal" cosmological model: a lot of cosmological models (not obviously multidimensional) may be embedded into this model. (We note also, that in some special cases the model [21] was considered by many authors [4,7,8,10,12,14-17]).

We impose certain restrictions on the parameters of the model [21] and reduce its dynamics near the singularity to a billiard on the $(n-1)$-dimensional Lobachevsky space $H^{n-1}$ (Sec. 3). The geometrical criterion for the finiteness of the billiard volume and its compactness is suggested. This criterion reduces the considered problem to the geometrical (or topological) problem of illumination of $(n-2)$-dimensional unit sphere $S^{n-2}$ by $m \leq m_0$ point-like sources located outside the sphere [45,46]. These sources correspond to the components with $(u^{(a)})^2 > 0$ (Sec. 3). When these sources illuminate the sphere then, and only then, the billiard has a finite volume and the cosmological model possesses a stochastic behavior near the singularity. (We note, that, for cosmological and curvature terms $(u^{(a)})^2 < 0$ and these terms may be neglected near the singularity). For the case of an infinite billiard volume the cosmological model has a Kasner-like behavior near the singularity [18]. When the minimally coupled massless scalar field is added into consideration, the evolution in time is bounded: $t > t_0$ and the limit $t \to t_0$ corresponds to the approach to the singularity.

In Sec. 4 we illustrate the suggested approach on an example of the Bianchi-IX cosmology. In Sec. 5 the Wheeler-DeWitt equation for the considered model in a special time gauge is considered. Near the singularity this equation has an approximate solution generalizing that for Bianchi-IX case [40]. This solution may be considered as a starting point for the construction of the third-quantized cosmological models in the vicinity of the singularity (for
the "mixmaster" case one of such models was considered in [40]).

2 The model

In this paper we consider a cosmological model describing the evolution of \( n \) Einstein spaces in the presence of \( (m + 1) \)-component perfect-fluid matter [21]. The metric of the model

\[
g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^n \exp[2x^i(t)]g^{(i)},
\]

is defined on the manifold

\[
M = R \times M_1 \times \ldots \times M_n,
\]

where the manifold \( M_i \) with the metric \( g^{(i)} \) is an Einstein space of dimension \( N_i \), i.e.

\[
R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)},
\]

\( i = 1, \ldots, n; \ n \geq 2 \). The energy-momentum tensor is adopted in the following form

\[
T^M_N = \sum_{\alpha=0}^m T^M_{N}(\alpha),
\]

\[
(T^M_{N}(\alpha)) = \text{diag}(-\rho^{(\alpha)}(t), p_1^{(\alpha)}(t)\delta_{k1}^{m_1}, \ldots, p_n^{(\alpha)}(t)\delta_{kn}^{m_n}).
\]

\( \alpha = 0, \ldots, m \), with the conservation law constraints imposed:

\[
\nabla_M T^M_N(\alpha) = 0
\]

\( \alpha = 0, \ldots, m - 1 \). The Einstein equations

\[
R^M_N - \frac{1}{2} \delta^M_N R = \kappa^2 T^M_N
\]

(\( \kappa^2 \) is gravitational constant) imply \( \nabla_M T^M_N = 0 \) and consequently \( \nabla_M T^M_N(\alpha) = 0 \).

We suppose that for any \( \alpha \)-th component of matter the pressures in all spaces are proportional to the density

\[
p^{(\alpha)}_i(t) = (1 - \frac{u^{(\alpha)}_i}{N_i})\rho^{(\alpha)}(t),
\]

where \( u^{(\alpha)}_i = \text{const}, \ i = 1, \ldots, n; \ \alpha = 0, \ldots, m \).

Non-zero components of the Ricci-tensor for the metric (2.1) are the following

\[
R_{00} = -\sum_{i=1}^n N_i[\ddot{x}^i - \gamma \dot{x}^i + (\dot{x}^i)^2],
\]

\[
R_{m_i n_i} = g_{m_i n_i}^{(i)} \lambda^i + \exp(2x^i - 2\gamma)(\ddot{x}^i + \dot{x}^i(\sum_{i=1}^n N_i\dot{x}^i - \dot{\gamma}))],
\]

\( i = 1, \ldots, n \).
The conservation law constraint (1.6) for $\alpha \in \{0, \ldots, m\}$ reads

$$\rho^{(\alpha)} + \sum_{i=0}^{n} N_i \dot{x}^i (\rho^{(\alpha)} + \rho_i^{(\alpha)}) = 0. \quad (2.11)$$

From eqs. (2.8), (2.11) we get

$$\rho^{(\alpha)}(t) = A^{(\alpha)} \exp[-2 N_i x^i(t) + u_i^{(\alpha)} x^i(t)], \quad (2.12)$$

where $A^{(\alpha)} = const$. Here and below the summation over repeated indices is understood. We define

$$\gamma_0 \equiv \sum_{i=1}^{n} N_i x^i \quad (2.13)$$
in (2.1).

Using relations (2.8), (2.9), (2.10), (2.12) it is not difficult to verify that the Einstein equations (2.7) for the metric (2.1) and the energy-momentum tensor from (2.4), (2.5) are equivalent to the Lagrange equations for the Lagrangian

$$L = \frac{1}{2} \exp(-\gamma + \gamma_0(x)) G_{ij} \dot{x}^i \dot{x}^j - \exp(\gamma - \gamma_0(x)) V(x). \quad (2.14)$$

Here

$$G_{ij} = N_i \delta_{ij} - N_i N_j \quad (2.15)$$

are the components of the minisuperspace metric,

$$V = V(x) = -\frac{1}{2} \sum_{i=1}^{n} \lambda^i N_i \exp(-2 x^i + 2 \gamma_0(x)) + \sum_{\alpha=0}^{m} \kappa^2 A^{(\alpha)} \exp(u_i^{(\alpha)} x^i). \quad (2.16)$$

is the potential. This relation may be also presented in the form

$$V = \sum_{\alpha=0}^{m} A_{\alpha} \exp(u_i^{(\alpha)} x^i), \quad (2.17)$$

where $n = m + n$; $A_{\alpha} = \kappa^2 A^{(\alpha)}$, $\alpha = 0, \ldots, m$; $A_{m+i} = -\frac{4}{3} \lambda^i N_i$ and

$$u_j^{(m+i)} = 2(-\delta^i_j + N_j), \quad (2.18)$$

$i, j = 1, \ldots, n$. We also put $A_0 = \Lambda$ and

$$u_j^{(0)} = 2 N_j, \quad (2.19)$$

$j = 1, \ldots, n$. Thus the zero component of the matter describe a cosmological constant term ($\Lambda$-term).

**Diagonalization.** We remind [14,15] that the minisuperspace metric

$$G = G_{ij} dx^i \otimes dx^i \quad (2.20)$$
has a pseudo-Euclidean signature \((-,+,\ldots,+)\), i.e. there exist a linear transformation
\begin{equation}
z^a = e_i^a x^i, \tag{2.21}
\end{equation}
diagonalizing the minisuperspace metric (2.20)
\begin{equation}
G = \eta_{ab} dz^a \otimes dz^b = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{2.22}
\end{equation}
where
\begin{equation}
(\eta_{ab}) = (\eta^{ab}) \equiv diag(-1, +1, \ldots, +1), \tag{2.23}
\end{equation}
a, b = 0, \ldots, n - 1. The matrix of the linear transformation \((e^i_a)\) satisfies the relation
\begin{equation}
\eta_{ab} e^a_i e^b_j = G_{ij} \tag{2.24}
\end{equation}
or equivalently
\begin{equation}
\eta^{ab} = e_i^a G_{ij} e^b_j = <e^a, e^b>. \tag{2.25}
\end{equation}
Here
\begin{equation}
G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2 - D} \tag{2.26}
\end{equation}
are components of the matrix inverse to the matrix (2.15) \([15]\), \(D = 1 + \sum_{i=1}^{n} N_i\) is the dimension of the manifold \(M\) (2.2) and
\begin{equation}
<u, v> \equiv G^{ij} u_i v_j \tag{2.27}
\end{equation}
defines a bilinear form on \(R^n\) \((u = (u_i), v = (v_i))\). Inverting the map (2.21) we get
\begin{equation}
x^i = e^i_a x^a, \tag{2.28}
\end{equation}
where for the components of the inverse matrix \((e^i_a) = (e^a_i)^{-1}\) we obtain from (2.25)
\begin{equation}
e^i_a = G^{ij} e^b_j \eta_{ba}. \tag{2.29}
\end{equation}

Like in \([15,21]\) we put
\begin{equation}
z^0 = e^0_i x^i = q^{-1} N_i x^i, \quad q = [(D - 1)/(D - 2)]^{1/2}. \tag{2.30}
\end{equation}
In this case the 00-component of eq. (2.25) is satisfied and the set \((e^a, a = 1, \ldots, n - 1)\) is defined up to \(O(n - 1)\)-transformation. A special example of the diagonalization with the relations (2.30) and
\begin{equation}
z^a = e^a_i x^i = [N_a/(\sum_{j=a}^{n} N_j)(\sum_{j=a+1}^{n} N_j)]^{1/2} \sum_{j=a+1}^{n} N_j (x^j - x^i), \tag{2.31}
\end{equation}
a = 1, \ldots, n - 1, was considered in \([14,15]\).
In $z$-coordinates (2.21) with $z^0$ from (2.30) the Lagrangian (2.14) reads
\[ L = L(z^a, \dot{z}^a, N) = \frac{1}{2} \mathcal{N}^{-1} \eta_{ab} \dot{z}^a \dot{z}^b - \mathcal{N} V(z), \]  
(2.32)
where
\[ \mathcal{N} = \exp(\gamma - \gamma_0(x)) > 0 \]  
(2.33)
is the Lagrange multiplier (modified lapse function) and
\[ V(z) = \sum_{a=0}^{\infty} A_a \exp(u^a z^a) \]  
(2.34)
is the potential. Here we denote
\[ u^a = e^a_i u^{(i)}_i = \langle u^{(a)}, e^b \rangle > \eta_{ba}, \]  
(2.35)
\[ a = 0, \ldots, n - 1, \text{ (see (2.27) and (2.29)). From (2.35) we get (see (2.26) and (2.30))} \]
\[ u^a_0 = - \langle u^{(a)}, e^0 \rangle = (\sum_{i=1}^{n} u^{(a)}_i) / (D - 2). \]  
(2.36)
For $\Lambda$-term and curvature components (see (2.19) and (2.18)) we have
\[ u^0_0 = 2q > 0, \quad u^m_{0+j} = 2/q > 0, \]  
(2.37)
\[ j = 1, \ldots, n. \]
The calculation of
\[ (u^a)^2 = \eta^{ab} u^a_0 u^a_b = \langle u^{(a)}, u^{(a)} \rangle = (u^{(a)})^2, \]  
(2.38)
for these components gives
\[ (u^0)^2 = 4(D - 1) / (2 - D) < 0, \quad (u^m_{0+j})^2 = 4(1 / N_j - 1) < 0, \]  
(2.39)
for $N_j > 1, j = 1, \ldots, n$. For $N_j = 1$ we have $\lambda^j = A_{m+j} = 0$. 

### 3 Billiard representation

Here we consider the behavior of the dynamical system, described by the Lagrangian (2.32) for $n \geq 3$ in the limit
\[ z^0 \to -\infty, \quad z = (z^0, \vec{z}) \in \mathcal{V}_-, \]  
(3.1)
where $\mathcal{V}_- \equiv \{ (z^0, \vec{z}) \in \mathbb{R}^n : z^0 < -|\vec{z}| \}$ is the lower light cone. For the volume scale factor
\[ v = \exp \left( \sum_{i=1}^{n} N_i x^i \right) = \exp(q z^0) \]  
(3.2)
(see (2.30)) we have in this limit \( v \to 0 \). Under certain additional assumptions the limit (3.1) describes the approaching to the singularity. We impose the following restrictions on the parameters \( u^a \) in the potential (2.34) for components with \( A_a \neq 0 \):

\[
1) A_a > 0 \ if \ (u^a)^2 = -(u_0^a)^2 + (\bar{u}^a)^2 > 0; \\
2) u_0^a > 0 \ for \ all \ \alpha.
\]

We note that due to (2.37) the second condition is always satisfied for \( \Lambda \)-term and curvature components (i.e. for \( \alpha = 0, m + 1, \ldots, m + n = \tilde{n} \)).

We restrict the Lagrange system (2.32) on \( \mathcal{V}_- \), i.e. we consider the Lagrangian

\[
L_- = L|_{\mathcal{V}_-}, \quad M_- = \mathcal{V}_- \times R_+,
\]

where \( T\mathcal{M}_- \) is tangent vector bundle over \( M_- \) and \( R_+ \equiv \{ \mathcal{N} > 0 \}. \) (Here \( F|_A \) means the restriction of function \( F \) on \( A \).) Introducing an analogue of the Misner-Chitre coordinates in \( \mathcal{V}_- \) [36,37]

\[
z^0 = -\exp(-y_0^0) \frac{1 + \bar{y}^2}{1 - \bar{y}^2},
\]

\[
z = -2\exp(-y_0^0) \frac{\bar{y}}{1 - \bar{y}^2},
\]

\(|\bar{y}| < 1\), we get for the Lagrangian (2.32)

\[
L_- = \frac{1}{2} \mathcal{N}^{-1} e^{-2y_0^0} [-(y_0^0)^2 + h_{ij}(\bar{y}) \bar{y}^i \bar{y}^j] - \mathcal{N} V.
\]

Here

\[
h_{ij}(\bar{y}) = 4\delta_{ij}(1 - \bar{y}^2)^{-2},
\]

\(|i, j = 1, \ldots, n - 1, \) and

\[
V = V(y) = \sum_{\alpha=0}^n A_\alpha \exp \Phi(y, u^\alpha),
\]

where

\[
\Phi(y, u) \equiv -e^{-y_0^0} (1 - \bar{y}^2)^{-1} [u_0(1 + \bar{y}^2) + 2u \bar{y}],
\]

\(|u| < 1\)

We note that the \( (n - 1) \)-dimensional open unit disk (ball)

\[
D^{n-1} \equiv \{ \bar{y} = (y^1, \ldots, y^n) ||\bar{y}| < 1 \} \subset R^{n-1}
\]

with the metric \( h = h_{ij}(\bar{y}) dy^i \otimes dy^j \) is one of the realization of the \( (n - 1) \)-dimensional Lobachevsky space \( H^{n-1} \).

We fix the gauge

\[
\mathcal{N} = \exp(-2y_0^0) = -z^2.
\]

Then, it is not difficult to verify that the Lagrange equations for the Lagrangian (3.8) with the gauge fixing (3.13) are equivalent to the Lagrange equations for the Lagrangian

\[
L_* = -\frac{1}{2}(y_0^0)^2 + \frac{1}{2} h_{ij}(\bar{y}) \bar{y}^i \bar{y}^j - V_*
\]

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with the energy constraint imposed
\[ E_* = -\frac{1}{2}(\dot{y}_0)^2 + \frac{1}{2}h_{ij}(\vec{y})\dot{y}^i\dot{y}^j + V_* = 0. \]  
(3.15)

Here
\[ V_* = e^{-2y^0} V = \sum_{\alpha=0}^{m} A_{\alpha} \exp(\Phi(y, u^{\alpha})), \]  
(3.16)

where
\[ \Phi(y, u) = -2y^0 + \Phi(y, u). \]  
(3.17)

Now we are interested in the behavior of the dynamical system in the limit \( y^0 \to -\infty \)
(or, equivalently, in the limit \( z^2 = -(z^0)^2 + (\vec{z})^2 \to -\infty, \ z^0 < 0 \) implying (3.1). Using the relations \( (u_0 \neq 0) \)
\[ \Phi(y, u) = -u_0 \exp(-y^0) \frac{A(\vec{y}, -\vec{u}/u_0)}{1 - \vec{y}^2} - 2y^0, \]  
(3.18)
\[ A(\vec{y}, \vec{v}) \equiv (\vec{y} - \vec{v})^2 - \vec{v}^2 + 1 \]  
(3.19)

we get
\[ \lim_{y^0 \to -\infty} \exp \Phi(y, u) = 0 \]  
(3.20)

for \( u^2 = -u_0^2 + (\vec{u})^2 \leq 0, \ u_0 > 0 \) and
\[ \lim_{y^0 \to -\infty} \exp \Phi(y, u) = \theta_{\infty}(\Phi(y, u_0)) \]  
(3.21)

for \( u^2 > 0, \ u_0 > 0 \). In (3.21) we denote
\[ \theta_{\infty}(x) \equiv + \infty, \ x \geq 0, \]  
\[ 0, \ x < 0. \]  
(3.22)

Using restrictions (3.3), (3.4) and relations (3.16), (3.20), (3.21) we obtain
\[ V_\infty(\vec{y}) \equiv \lim_{y^0 \to -\infty} V_*(y^0, \vec{y}) = \sum_{\alpha \in \Delta_+} \theta_{\infty}(A(\vec{y}, -\vec{u}/u_0^\alpha)). \]  
(3.23)

Here we denote
\[ \Delta_+ \equiv \{ \alpha | (u^\alpha)^2 > 0 \}. \]  
(3.24)

We note that due to (2.39) \( \Lambda \)-term and curvature components do not contribute to \( V_\infty \) (i.e. they may be neglected in the vicinity of the singularity).

The potential \( V_\infty \) may be also written as following
\[ V_\infty(\vec{y}) = V(\vec{y}, B) \equiv 0, \ \vec{y} \in B, \]  
\[ +\infty, \ \vec{y} \in D^{n-1} \backslash B; \]  
(3.25)

where
\[ B = \bigcap_{\alpha \in \Delta_+} B(u^\alpha) \subset D^{n-1}, \]  
(3.26)
\[ B(u^\alpha) = \{ \vec{y} \in D^{n-1} \mid |\vec{y} + \frac{\vec{v}^\alpha}{u_0^\alpha}| > \sqrt{(\vec{v}^\alpha)^2 - 1} \}, \]

\( \alpha \in \Delta_+ \). \( B \) is an open domain. Its boundary \( \partial B = \hat{B} \setminus B \) is formed by certain parts of \( m_+ = |\Delta_+| \) (\( m_+ \) is the number of elements in \( \Delta_+ \)) of \((n-2)\)-dimensional spheres with the centers in the points

\[ \vec{v}^\alpha = -\vec{u}^\alpha / u_0^\alpha, \quad \alpha \in \Delta_+, \]

\(|v^{\vec{v}^\alpha}| > 1 \) and radii

\[ r_\alpha = \sqrt{(v^{\vec{v}^\alpha})^2 - 1} \]

respectively (for \( n = 3, m_+ = 1 \), see Fig. 1).

Fig. 1

So, in the limit \( y^0 \to -\infty \) we are led to the dynamical system

\[ L_\infty = -\frac{1}{2}(y^0)^2 + \frac{1}{2}h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V_\infty(\vec{y}), \]

\[ E_\infty = -\frac{1}{2}(y^0)^2 + \frac{1}{2}h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V_\infty(\vec{y}) = 0, \]

which after the separating of \( y^0 \) variable

\[ y^0 = \omega(t - t_0), \]

\( \omega \neq 0 \), \( t_0 \) are constants) is reduced to the Lagrange system with the "Lagrangian"

\[ L_B = \frac{1}{2}h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V(\vec{y}, B). \]

Due to (3.32)

\[ E_B = \frac{1}{2}h_{ij}(\vec{y}) \dot{y}^i \dot{y}^j - V(\vec{y}, B) = \frac{\omega^2}{2}. \]

We put \( \omega > 0 \), then the limit \( t \to -\infty \) describes the approach to the singularity. When the set (3.24) is empty \( (\Delta_+ = \emptyset) \) we have \( B = D^{n-1} \) and the Lagrangian (3.33) describes the geodesic flow on the Lobachevsky space \( H^{n-1} = (D^{n-1}, h_{ij}dy^i \otimes dy^j) \). In this case there are two families of non-trivial geodesic solutions (i.e. \( y(t) \neq const \)):

1. \( \vec{y}(t) = \vec{n}_1[\sqrt{v^2 - 1} \cos \varphi(\vec{\bar{l}}) - v] + \vec{n}_2\sqrt{v^2 - 1} \sin \varphi(\vec{\bar{l}}), \)

   \[ \varphi(\vec{\bar{l}}) = 2 \arctan[(v - \sqrt{v^2 - 1}) \tanh(\omega \vec{\bar{l}})], \]

2. \( \vec{y}(t) = \vec{n} \tanh(\omega \vec{\bar{l}}). \)

Here \( n^2 = n_1^2 = n_2^2 = 1, \ n_1 n_2 = 0, \ v > 1, \ \omega > 0, \ \vec{\bar{l}} = t - t_0, \ t_0 = const. \)

Graphically the first solution corresponds to the arc of the circle with the center at point \((-v\vec{n}_1)\) and the radius \( \sqrt{v^2 - 1} \). This circle belongs to the plane spanned by vectors \( \vec{n}_1 \) and \( \vec{n}_2 \) (the centers of the circle and the ball \( D^{n-1} \) also belong to this plane). We note, that the solution (3.35)-(3.36) in the limit \( v \to \infty \) coincides with the solution (3.37).

We note, that the boundary of the billiard \( \partial B \) is formed by geodesics. For some billiards this fact may be used for "gluing" certain parts of boundaries.
When $\Delta_+ \neq \emptyset$ the Lagrangian (3.33) describes the motion of the particle of unit mass, moving in the $(n - 1)$-dimensional billiard $B \subset D^{n-1}$ (see (3.26)). The geodesic motion in $B$ (3.35)-(3.37) corresponds to a "Kasner epoch" and the reflection from the boundary corresponds to the change of Kasner epochs. For $n = 3$ some examples of (2-dimensional) billiards are depicted in Figs. 2-4.

Fig. 2-4

The billiard $B$ in Fig. 2, has an infinite volume: $\text{vol} B = +\infty$. In this case there are three open zones at the infinite circle $|\vec{y}| = 1$. After a finite number of reflections from the boundary the particle moves toward one of these open zones. For corresponding cosmological model we get the "Kasner-like" behavior in the limit $t \rightarrow -\infty$ [18].

For billiards depicted in Figs. 3 and 4 we have $\text{vol} B < +\infty$. In the first case (Fig. 3) the closure of the billiard $\bar{B}$ is compact (in the topology of $D^{n-1}$) and in the second case (Fig. 4) $\bar{B}$ is non-compact. In these two cases the motion of the particle is stochastic.

Analogous arguments may be applied to the to the case $n > 3$. So, we are interested in the configurations with finite volume of $B$. We propose a simple geometric criterion for the finiteness of the volume of $B$ and compactness of $\bar{B}$ in terms of the positions of the points (3.28) with respect to the $(n - 2)$-dimensional unit sphere $S^{n-2}$ $(n \geq 3)$. We say that the point $\vec{y} \in S^{n-2}$ is (geometrically) illuminated by the point-like source located at the point $\vec{v}$, $|\vec{v}| > 1$ if and only if $|\vec{y} - \vec{v}| \leq \sqrt{|\vec{v}|^2 - 1}$. In Fig. 3 the source $P$ illuminates the closed arc $[P_1, P_2]$. We also say that the point $\vec{y} \in S^{n-2}$ is strongly illuminated by the point-like source located at the point $\vec{v}$, $|\vec{v}| > 1$ if and only if $|\vec{y} - \vec{v}| < \sqrt{|\vec{v}|^2 - 1}$. In Fig. 3 the source $P$ strongly illuminates the open arc $(P_1, P_2)$. The subset $N \subset S^{n-2}$ is called (strongly) illuminated by point-like sources at $\{\vec{v}_\alpha, \alpha \in \Delta_+\}$ if and only if any point from $N$ is (strongly) illuminated by some source at $\vec{v}_\alpha$ ($\alpha \in \Delta_+$).

Proposition. The billiard $B$ (3.26) has a finite volume if and only if the point-like sources of light located at the points $\vec{v}_\alpha$ (3.28) illuminate the unit sphere $S^{n-2}$. The closure of the billiard $\bar{B}$ is compact (in the topology of $D^{n-1}$) if and only if the sources at points (3.28) strongly illuminate $S^{n-2}$.

Proof. We consider the set $\partial^c B \equiv B^c \setminus \bar{B}$, where $B^c$ is the completion of $B$ (or, equivalently, the closure of $B$ in the topology of $R^{n-1}$). We remind that $\bar{B}$ is the closure of $B$ in the topology of $D^{n-1}$. Clearly that $\partial^c B$ is a closed subset of $S^{n-2}$, consisting of all those points that are not strongly illuminated by sources (3.28). There are three possibilities: i) $\partial^c B$ is empty; ii) $\partial^c B$ contains some interior point (i.e. the point belonging to $\partial^c B$ with some open neighborhood); iii) $\partial^c B$ is non-empty finite set, i.e. $\partial^c B = \{\vec{y}_1, \ldots, \vec{y}_k\}$. The first case i) takes place if and only if $\bar{B}$ is compact in the topology of $D^{n-1}$. Only in this case the sphere $S^{n-2}$ is strongly illuminated by the sources (3.28). Thus the second part of proposition is proved. In the case i) $\text{vol} B$ is finite. For the volume we have

$$\text{vol} B = \int_B d^{n-1} \vec{y} \sqrt{h} = \int_0^1 dr (1 - r^2)^{1-n} S_r. \quad (3.38)$$

The "area" $S_r \rightarrow C > 0$ as $r \rightarrow 1$ in the case ii) and, hence, the integral (3.38) is divergent. In the case iii)

$$S_r \sim C_1 (1 - r)^{2(n-2)} \text{ as } r \rightarrow 1 \quad (3.39)$$
\( C_1 > 0 \) and, so, the integral (3.38) is convergent. Indeed, in the case iii), when \( r \to 1 \), the "area" \( S_r \) is the sum of \( l \) terms. Each of these terms is the \((n - 2)\)-dimensional "area" of a transverse side of a deformed pyramid with a top at some point \( \tilde{y}_k, \ k = 1, \ldots, l \). This multidimensional pyramid is formed by certain parts of spheres orthogonal to \( S^{n-2} \) in the point of their intersection \( \tilde{y}_k \). Hence, all lengths of the transverse section \( r = \text{const} \) of the "pyramid" behaves like \((1 - r)^2\), when \( r \to 1 \), that justifies (3.39). But the unit sphere \( S^{n-2} \) is illuminated by the sources (3.28) only in the cases i) and iii). This completes the proof.

The problem of illumination of convex body in multidimensional vector space by point-like sources for the first time was considered in \([45, 46]\). For the case of \( S^{n-2} \) this problem is equivalent to the problem of covering the spheres with spheres \([47, 48]\). There exist a topological bound on the number of point-like sources \( m_+ \) illuminating the sphere \( S^{n-2} \) \([16]\):

\[
m_+ \geq n.
\]  

(3.40)

Remark 1. Let the points (3.26) form an open convex polyhedron \( P \subset R^{n-1} \). Then the sources at (3.26) illuminate \( S^{n-2} \), if \( D^{n-1} \subset P \), and strongly illuminate \( S^{n-2} \), if \( \overline{D^{n-1}} \subset P \).

**Scalar field generalization.** Let us assume that an additional \( m + 1 \)-component with the equation of state \( p^{(m+1)} = \rho^{(m+1)} \) is considered, \( i = 1, \ldots, n \). This component describes Zeldovich matter \([49]\) in all spaces and is equivalent to homogeneous massless free minimally coupled scalar field \([50]\). In this case \( u_i^{(m+1)} = 0, \ i = 1, \ldots, n \) and the potential (2.17) is modified by the addition of constant \( A_{m+1} > 0 \). Then the potential \( V_\ast \) (3.16) is modified by the addition of the following term

\[
\Delta V = A_{m+1} \exp(-2y^0).
\]  

(3.41)

This do not prevent from the formation of the billiard walls but change the time dependence of \( y^0 \)-variable:

\[
\exp(2y^0) = 2A_{m+1} \sinh^2[\omega(t - t_0)]/\omega^2,
\]  

(3.42)

\( \omega > 0 \) instead of (3.32). In the limit \( t \to t_0 + 0 \) we have \( y^0 \to -\infty \) and \( \tilde{y}(t) \to \tilde{y}_0 \in B \).

4 Bianchi-IX cosmology

Here we consider the well-known mixmaster model \([34, 35]\) with the metric

\[
g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{3} \exp[2x^i(t)]e^i \otimes e^i,
\]  

(4.1)

where 1-forms \( e^i = e^i(\zeta)d\zeta^i \) satisfy the relations

\[
d e^i = \frac{1}{2} \varepsilon_{ijk} e^j \wedge e^k,
\]  

(4.2)

\( i, j, k = 1, 2, 3 \). The Einstein equations for the metric (4.1) lead to the Lagrange system (2.14)-(2.17) with (see, for example, \([35]\)) \( n = 3, \ N_1 = N_2 = N_3 = 1, \ m = 6, \ A_1 = A_2 = A_3 = 1/4, \ A_4 = A_5 = A_6 = -1/2, \ A_0 = A_r = A_8 = A_9 = 0 \), and

\[
u^{(a)}_i = 4\delta^a_i, \quad u^{(3+a)}_i = 2(1 - \delta^a_i),
\]  

(4.3)
\( \alpha = 1, 2, 3 \). In the \( z \)-coordinates (2.30), (2.31) we have for 3-vectors (2.35)

\[
   u^1 = \frac{4}{\sqrt{6}}(1, 1, -\sqrt{3}), \quad u^2 = \frac{4}{\sqrt{6}}(1, 1, +\sqrt{3}), \quad u^3 = \frac{4}{\sqrt{6}}(1, -2, 0),
\]

\( u^4 = \frac{1}{2}(u^1 + u^2), \quad u^5 = \frac{1}{2}(u^1 + u^3), \quad u^6 = \frac{1}{2}(u^2 + u^3), \)

and, consequently,

\[
   (u^\alpha)^2 = 8, \quad (u^{3+\alpha})^2 = 0,
\]

\( \alpha = 1, 2, 3 \). Thus the conditions (3.3), (3.4) are satisfied. The components with \( \alpha = 4, 5, 6 \) do not survive in the approaching to the singularity. For the vectors (3.28) we have

\[
   \vec{v}^1 = (1, -\sqrt{3}), \quad \vec{v}^2 = (1, +\sqrt{3}), \quad \vec{v}^3 = (-2, 0),
\]

i.e. a triangle from Fig. 4 (see also [38]). In this case the circle \( S^1 \) is illuminated by sources at points \( \vec{v}^i \), \( i = 1, 2, 3 \), but not strongly illuminated. In agreement with Proposition the billiard \( B \) has finite volume, but \( \bar{B} \) is not compact.

### 5 Quantum case

The quantization of zero-energy constraint (3.15) leads to the Wheeler-DeWitt (WDW) equation in the gauge (3.13) [15,52]

\[
   (-\frac{1}{2}\Delta[\vec{G}] + a_n R[\vec{G}] + V_\varphi)\Psi = 0.
\]

Here \( \Psi = \Psi(y) \) is "the wave function of the Universe", \( V_\varphi = V_\varphi(y) \) is the potential (3.16), \( a_n = (n - 2)/8(n - 1) \), \( \Delta[\vec{G}] \) and \( R[\vec{G}] \) are the Laplace-Beltrami operator and the scalar curvature of the minisuperspace metric

\[
   \vec{G} = -dy^0 \otimes dy^0 + h, \quad h = h_{ij}(\vec{y})dy^i \otimes dy^j.
\]

(We remind that, \( h \) is the metric on Lobachevsky space \( D^{n-1} \).) The form of WDW eq. (5.1) follows from the demands of minisuperspace invariance and conformal covariance [51,52,15]. Using relations

\[
   \Delta[\vec{G}] = -\left(\frac{\partial}{\partial y^0}\right)^2 + \Delta[h], \quad R[\vec{G}] = R[h] = -(n - 1)(n - 2),
\]

we rewrite (5.1) in the form

\[
   \left(\frac{1}{2}\left(\frac{\partial}{\partial y^0}\right)^2 - \frac{1}{2}\Delta[h] - \frac{(n - 2)^2}{8} + V_\varphi\right)\Psi = 0.
\]

In the limit \( y^0 \to -\infty \) the WDW eq. reduces to the relations

\[
   ((\frac{\partial}{\partial y^0})^2 - \Delta_\varphi[h])\Psi_\infty = 0, \quad \Psi_\infty|_{\partial B} = 0,
\]
where $\partial B = \tilde{B} \setminus B$ is the boundary of the billiard $B$ (3.26) (in $D^{n-1}$) and

$$
\Delta_n [h] = \Delta [h] + \frac{(n - 2)^2}{4}.
$$

(5.6)

Now, we suppose that $\tilde{B}$ is compact and the operator (5.6) with the boundary condition has a negative spectrum, i.e.

$$
\Delta_n [h] \Psi_n = - E_n^2 \Psi_n,
$$

(5.7)

$E_n > 0$, $n = 0, 1, \ldots$ (this is valid at least for "small enough" $B$). Using (5.7) we get the general solution of the asymptotic WDW eq. (5.5)

$$
\Psi_\infty(y^0, \vec{y}) = \sum_{n=0}^{\infty} [c_n \exp(-iE_n y^0) \Psi_n (\vec{y}) + c_n^* \exp(iE_n y^0) \Psi_n^* (\vec{y})],
$$

(5.8)

that may be considered as a starting point for the construction of third quantized models in the vicinity of the singularity.

### 6 Discussions

Thus, we obtained the "billiard representation" for the cosmological model [21] and proved the geometrical criterion for finiteness of the billiard volume and the compactness of the billiard (Proposition, Sec. 3). This criterion may be used as a rather effective (and universal) tool for selection of the cosmological models with a stochastic behavior near the singularity.

For an "isotropic" component: $p_i^{(\alpha)} = (1 - h) \rho^{(\alpha)}, \ i = 1, \ldots, n$, with $h \neq 0$ we have $(u^{(\alpha)})^2 = h^2(D - 1)/(2 - D) < 0$ and, hence, this component may be neglected near the singularity. Only "anisotropic" components with $(u^{(\alpha)})^2 > 0$ take part in the formation of billiard walls near the singularity. According to the topological bound (3.10) [16] the stochastic behavior near the singularity in the considered model may occur only if the number of components with $(u^{(\alpha)})^2 > 0$ is not less than the minisuperspace dimension.

We also note that here, like in the Bianchi-IX case [36,37], the considered reduction scheme uses a special time gauge (or parametrization of time). As it was pointed in [38] one should be careful in the interpretations of the results of computer experiments for other choices of time.

**Restrictions on parameters.** Here we discuss the physical sense of the restrictions on parameters of the model (3.3) and (3.4). The condition (3.3) means that the densities of the "anisotropic" components with $(u^{(\alpha)})^2 > 0$ should be positive. Using (2.8) and (2.36) we rewrite the restriction (3.4) in the equivalent form

$$
\sum_{i=1}^{n} N_i p_i^{(\alpha)} \frac{\rho^{(\alpha)}}{\rho^{(\alpha)}} > 0,
$$

(6.1)

$(\rho^{(\alpha)} \neq 0)$ $\alpha = 1, \ldots, m$ (for curvature and $\Lambda$-terms (3.4) is satisfied). For

$$
\rho^{(\alpha)} > 0, \quad p_i^{(\alpha)} < \rho^{(\alpha)},
$$

(6.2)
α = 1, . . . , m, i = 1, . . . , n, (6.1) is satisfied identically.

Remark 2. It may be shown [53] that the condition (3.4) may be weakened by the following one

\[ u_0^a > 0, \quad if \quad (u^a)^2 \geq 0. \]  

(6.3)

In this case there exists a certain generalization of the set \( B(u^a) \) from (3.27) for arbitrary \( u_0^a \) \((u^a)^2 < 0) [53]\). The Proposition (Sec. 3) should be modified by including into consideration the sources at infinity (for \( u_0^a = 0 \)) and "anti-sources" (for \( u_0^a < 0 \)). For "anti-source" the shadowed domain coincides with the illuminated domain for the usual source (with \( u_0^a > 0 \)). In this case we deal with the kinematics of tachyons. (We may also consider a covariant and slightly more general condition instead of (6.3)

\[ \text{sign} u_0^a = \varepsilon, \quad \text{for all} \quad (u^a)^2 \geq 0, \quad \varepsilon = \pm 1. \]  

(6.4)

Acknowledgments

The authors would like to thank our colleagues K.A.Bronnikov, A.A.Kirillov, M.Yu.Konstatinov and A.G.Radynov for useful discussions. One of us (V.D.I) is grateful to R.V.Galiullin for pointing out the attention to ref. [46].

This work was supported in part by the Russian Ministry of Science.

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List of captions for illustrations

Fig. 1. An example of billiard for \( n = 3, m_+ = 1 \).

Figs. 2-4. Examples of billiards for \( n = 3, m_+ = 3 \) with: 2) \( volB = +\infty \); 3) \( \tilde{B} \) is compact; 4) \( volB < +\infty \) and \( \tilde{B} \) is not compact (Bianchi-IX).