Fully Electrified Neugebauer Spacetimes

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Abstract

Generalizing a method presented in an earlier paper, we express the complex potentials $\mathcal{E}$ and $\Phi$ of all stationary axisymmetric electrovac spacetimes that correspond to axis data of the form

$$\mathcal{E}(\xi, \eta) = \frac{U - W}{U + W}, \quad \Phi(\xi, \eta) = \frac{V}{U + W},$$

where

$$U = z^2 + U_1 z + U_2,$$

$$V = V_1 z + V_2,$$

$$W = W_1 z + W_2,$$

in terms of the complex parameters $U_1, V_1, W_1, U_2, V_2$ and $W_2$, that are directly associated with the various multipole moments.

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I. INTRODUCTION

This is the second paper of a short series of papers, all of which are concerned with those stationary axisymmetric solutions of the Einstein and the Einstein-Maxwell equations that are characterized by axis data of the form

\[ \mathcal{E} = \frac{U - W}{U + W}, \quad \Phi = \frac{V}{U + W}, \]  \hspace{1cm} (1.1)

with

\[ U = \sum_{a=0}^{n} U_a z^{n-a}, \]  \hspace{1cm} (1.2a)
\[ V = \sum_{a=1}^{n} V_a z^{n-a}, \]  \hspace{1cm} (1.2b)
\[ W = \sum_{a=1}^{n} W_a z^{n-a}, \]  \hspace{1cm} (1.2c)

where \( \mathcal{E} \) and \( \Phi \) are the complex potentials of Ernst, \( z \) is the Weyl canonical coordinate, and the coefficients in the polynomials are complex constants.

In the first paper\(^1\) of this series, to which we shall henceforth refer as Ref. 1, we found that it was extraordinarily simple to construct the general solution of the vacuum problem \( (V = 0) \) for the case \( n = 2 \). That is, we described a procedure that allows one to express the complex potential \( \mathcal{E} \) and the metrical fields \( \omega \) and \( \gamma \) of an exact vacuum solution \( (V = 0) \) directly in terms of the complex parameters \( U_a, W_a \) \((a = 1, \ldots, n)\) with \( U_0 = 1 \). In particular, we showed that the resulting family of solutions contained as a special case the vacuum limit of an electrovac solution published recently by Manko et al.\(^2\)

In the present paper, we undertake the extension of the methods of Ref. 1 to electrovac fields \((V \neq 0)\). This extension is far from trivial, and to construct it we found that it was necessary to attain first an understanding of why everything worked out so handily in the vacuum case.

An additional complexity of the electrovac problem arises from the fact that, while all the vacuum solutions could be constructed from Minkowski space by applying successive quadruple-Neugebauer Bäcklund transformations,\(^3\) or double-Harrison Bäcklund
transformations,\textsuperscript{4} there was no single known Kinnersley-Chitre transformation that could yield all the electrovac solutions with axis data of the type we are considering. This forced us to consider what we call a complexified Cosgrove transformation, which will be defined in this paper.

II. THE AXIS RELATION

Using the Hauser-Ernst axis relation\textsuperscript{5,6} one can always identify a Kinnersley-Chitre transformation that will, in principle, produce from Minkowski space a spacetime with any specified axis data. Of course, it may be difficult to solve in closed form the associated homogeneous Hilbert problem.

A. The vacuum case

Let us first review the application of the axis relation within the context of vacuum spacetimes, where we know that the quadruple-Neugebauer (double-Harrison) transformation does the job. It will suffice to consider the generation of the Kerr metric, for which

$$\mathcal{E} = \infty - \frac{\epsilon_{\parallel}}{\nabla - \epsilon_{\parallel} \cos \theta}.$$  \hfill (2.1)

On the axis, where $\theta = 0$, we have $\cos \theta = 1$. On the other hand, the Weyl canonical coordinates $z, \rho$ are given by

$$\rho^2 = (r^2 + a^2 - 2mr) \sin^2 \theta, \ z = (r - m) \cos \theta.$$  \hfill (2.2)

Therefore, the axis data for the Kerr metric assumes the form

$$\mathcal{E}(\frac{\parallel}{2}, t) = \frac{\parallel - \parallel - \parallel^\dagger}{\frac{\parallel}{2} + \frac{\parallel}{2} - \parallel^\dagger},$$  \hfill (2.3)

i.e., $U(z, 0) = z - i \alpha$ and $W(z, 0) = m$.

Now, the axis relation says\textsuperscript{5}

$$-i\mathcal{E}(\tau, t) = -i\mathcal{E}(\tau, t) \underline{UL}(\tau) + \underline{UR}(\tau)$$

$$-i\mathcal{E}(\tau, t) \underline{EL}(\tau) + \underline{LR}(\tau),$$  \hfill (2.4)
where $\mathcal{E}(t)$ is the complex potential of the seed metric. For Minkowski space, $\mathcal{E}(t) = \infty$. Plugging in $\mathcal{E}(\tau, t)$ for the Kerr metric, we are tempted to try

$$v(\tau) = \begin{pmatrix} \tau - m & -a \\ a & \tau + m \end{pmatrix}.$$  

However, $v(\tau)$ should be an matrix of the group $SU(1, 1) = SL(2, \mathbb{R})$. Since the determinant has the value

$$\det v(\tau) = z^2 - m^2 + a^2 ,$$

we should divide by the square root of this determinant, and select

$$v(\tau) = \frac{1}{\sqrt{z^2 - m^2 + a^2}} \begin{pmatrix} \tau - m & -a \\ a & \tau + m \end{pmatrix}.$$  \hspace{1cm} (2.5)

Suppose, for the moment, that $a^2 < m^2$. Then, using the identities

$$\tau = \frac{1}{2} \left[ (\tau + \sqrt{m^2 - a^2}) + (\tau - \sqrt{m^2 - a^2}) \right] , \quad 1 = \frac{1}{2} \left[ (\tau + \sqrt{m^2 - a^2}) - (\tau - \sqrt{m^2 - a^2}) \right] ,$$  \hspace{1cm} (2.6)

we can cast our expression for $v(\tau)$ into the form

$$v(\tau) = \frac{1}{2} (I + J) \sqrt{\frac{\tau + \sqrt{m^2 - a^2}}{\tau - \sqrt{m^2 - a^2}}} + \frac{1}{2} (I - J) \sqrt{\frac{\tau - \sqrt{m^2 - a^2}}{\tau + \sqrt{m^2 - a^2}}} ,$$  \hspace{1cm} (2.7)

where the matrix

$$J := \frac{1}{\sqrt{m^2 - a^2}} \begin{pmatrix} -m & -a \\ a & m \end{pmatrix}$$  \hspace{1cm} (2.8)

satisfies

$$\text{tr} J = 0 , \quad J^2 = I .$$  \hspace{1cm} (2.9)

This result can also be expressed in the exponential form

$$v(\tau) = \exp(J \eta(\tau)) ,$$  \hspace{1cm} (2.10)
where
\[ e^{i\alpha} = \sqrt{\frac{\tau + \sqrt{\tau^2 - a^2}}{\tau - \sqrt{\tau^2 - a^2}}} \]  
(2.11)

If, on the other hand, \( a^2 > m^2 \), this procedure would have yielded
\[ v(\tau) = \frac{1}{2}(I + J) \left[ \frac{\tau + i\sqrt{\tau^2 - m^2}}{\tau - i\sqrt{\tau^2 - m^2}} \right] + \frac{1}{2}(I - J) \left[ \frac{\tau - i\sqrt{\tau^2 - m^2}}{\tau + i\sqrt{\tau^2 - m^2}} \right] \]  
(2.12)

where
\[ iJ := \frac{1}{\sqrt{a^2 - m^2}} \begin{pmatrix} -m & -a \\ a & m \end{pmatrix} \]  
(2.13)

In either case, \( v(\tau) \) is an \( SU(1,1) = SL(2,R) \) matrix. In the case \( a^2 > m^2 \) we probably would have used the symbol \( J \) for the real matrix \( iJ \) and then would have had \( J^2 = -1 \) for that case.

Alternatively, we can unify these two cases by considering members of the larger group \( SL(2,C) \), temporarily setting aside the reality condition on \( v(\tau) \) and allowing general complex values for the parameters. It turns out\(^5\) that the Hauser-Ernst homogeneous Hilbert problem works for members of \( SL(2,C) \) as well as \( SU(1,1) = SL(2,R) \), so there is no need to solve the HHP twice. The solutions for both \( a^2 < m^2 \) and \( a^2 > m^2 \) can be inferred from the complexified spacetime that results from an application of the \( SL(2,C) \) transformation.

**B. The electrovac case**

Now let us turn our attention to the charged Kerr metric, where
\[ \mathcal{E} = \infty - \frac{\hat{\Phi}}{\nabla - \hat{\Theta} \cos \theta}, \quad \Theta = \frac{1}{\nabla - \hat{\Theta} \cos \theta} \]  
(2.14)

Thus,
\[ \mathcal{E}(\tau, i) = \frac{\tau - \hat{\Phi} - \hat{\Theta}}{\tau + \hat{\Phi} - \hat{\Theta}}, \quad \Theta(\tau, i) = \frac{1}{\tau + \hat{\Phi} - \hat{\Theta}}. \]  
(2.15)

In the electrovac case the axis relation can be expressed in the form\(^6\)
\[ X(\tau)v(\tau)Y(\tau) = 0 \,, \quad X(\tau)v(\tau)Z(\tau) = 0 \,, \quad (2.16) \]

where

\[ X(\tau) := \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \mathcal{E}(\tau, \theta) & \Phi(\tau, 0) \end{pmatrix} \,, \quad (2.17) \]

and

\[
Y(\tau) := \begin{pmatrix} -i \mathcal{E}^{(0)}(\tau, \theta) \\ 1 \\ 0 \end{pmatrix}, \quad Z(\tau) := \begin{pmatrix} -\frac{i}{\sqrt{2}} \Phi^{(0)}(\tau, 0) \\ 0 \\ 1 \end{pmatrix}. \quad (2.18)
\]

In the present case

\[ X(\tau) = \begin{pmatrix} -\frac{i}{\sqrt{2}}(\tau + m - ia) & \frac{1}{\sqrt{2}}(\tau - m - ia) & e \end{pmatrix}, \quad (2.19) \]

and

\[ Y(\tau) = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \quad Z(\tau) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.20) \]

The resulting conditions upon the \( SU(2,1) \) matrix \( v(\tau) \) are not as simple as in the vacuum case, but there is a certain resemblance nevertheless. Guided by how the HHP was solved in the vacuum case, we look for a \( v(\tau) \) of the form

\[ v(\tau) = \exp(J \eta(\tau)) \,, \quad (2.21) \]

where two of the three eigenvalues of \( J \) are degenerate. Because \( \text{tr} J = 0 \), these two eigenvalues are \( \lambda \) and \( -2\lambda \), respectively, where \( \lambda \) is a constant. If \( \mathcal{P} \) is a projection operator onto the subspace corresponding to the nondegenerate eigenvector, we may express \( v(\tau) \) in the form

\[ v(\tau) = (I - \mathcal{P})^{\lambda \eta(\tau)} + \mathcal{P}^{-\lambda \eta(\tau)}, \quad (2.22) \]

where the projection operator \( \mathcal{P} \) can be written in term of the nondegenerate eigenvector \( h \) of \( J \) as
\[ \mathcal{P} = (\infty/\mathcal{E}) \langle \ell \otimes \mathcal{E} \rangle, \quad \mathcal{E} := \langle \ell \otimes \mathcal{E} \rangle, \quad (2.23) \]

where
\[
\Omega := \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.24)
\]

This way of writing \( \mathcal{P} \) implies that
\[
\mathcal{P}^\dagger \otimes = \otimes \mathcal{P}. \quad (2.25)
\]

On the other hand,
\[ J = \lambda (I - 3\mathcal{P}) , \quad (2.26) \]

so
\[ \lambda J^\dagger \Omega - \lambda^* \Omega J = 0 . \quad (2.27) \]

On the other hand, the \( SU(2,1) \) conditions
\[
\det v = 1 , \quad v^\dagger \Omega v = \Omega , \quad (2.28)
\]

require
\[ \eta(\tau)^* J^\dagger \Omega + \eta(\tau) \Omega J = 0 , \quad (2.29) \]

which is satisfied if and only if \( \lambda \eta(\tau) \) is imaginary.

We may now express the axis relations in the form
\[
X(\tau) \left[ I + \mathcal{P} \left( \frac{1}{\lambda} - 3 \lambda \eta(\tau) \right) \right] Y(\tau) = 0 , \quad (2.30a)
\]
\[
X(\tau) \left[ I + \mathcal{P} \left( \frac{1}{\lambda} - 3 \lambda \eta(\tau) \right) \right] Z(\tau) = 0 . \quad (2.30b)
\]

However, in our case
\[ X(\tau) Y(\tau) = -\sqrt{2} m \text{ and } X(\tau) Z(\tau) = e \quad (2.31) \]
are both constants. Thus, our pair of equations reduces to

\[
\begin{align*}
(e^{-3\lambda_0\tau} - 1) X(\tau) h h^\dagger \Omega Y &= \sqrt{2} E m, \\
(e^{-3\lambda_0\tau} - 1) X(\tau) h h^\dagger \Omega Z &= -E e.
\end{align*}
\] (2.32a, 2.32b)

Because

\[
h^\dagger \Omega Y = i(h_1^* + i h_2^*) , \ h^\dagger \Omega Z = h_3^*,
\] (2.33)

it is immediately apparent that

\[
Q := \sqrt{\varepsilon} \langle \psi_3 / (\langle \psi_\infty + \rangle \langle \zeta \rangle) = \mid \Psi \rangle.
\] (2.34)

We also find that

\[
X(\tau) h = -i \left\{ \left( \frac{h_1 + i h_2}{\sqrt{2}} \right)(\tau - i a) + \left( \frac{h_1 - i h_2}{\sqrt{2}} \right)m + i h_3 e \right\}.
\] (2.35)

Of course, we do not expect the eigenvector to be determined completely. Rather, only the ratios of its components will be determined. Therefore, we shall impose an additional condition; for example,

\[
\frac{h_1 + i h_2}{\sqrt{2}} = 1.
\] (2.36)

Thus, we end up with the simple result

\[
X(\tau) h = -i \left\{ (\tau - i a) + \frac{i h_3}{e} (e^2 - m^2) \right\}.
\] (2.37)

Moreover,

\[
E = 1 + (|h_3|/e)^2 (e^2 - m^2),
\] (2.38)

so the axis relation yields

\[
\begin{align*}
(e^{-3\lambda_0\tau} - 1) \left[ \tau - ia + \frac{ih_3}{e} (e^2 - m^2) \right]
&= - \left[ \frac{ie}{h_3^*} + \frac{ih_3}{e} (e^2 - m^2) \right],
\end{align*}
\] (2.39)

or
\[ e^{\lambda \eta(\tau)} = \frac{\tau - ia + i(h_3/c)(c^2 - m^2)}{\tau - ia - i(e/h_3^2)}. \]  

To complete the discussion we must fully determine the eigenvector \( h \), and hence the projection operator \( \mathcal{P} \). Only \( h_3 \) remains to be determined, for once \( h_3 \) is known, we shall also know \((h_1 - ih_2)/\sqrt{2}\), and \((h_1 + ih_2)/\sqrt{2} = 1\) by our convention for the selection of the representative eigenvector \( h \). Suppose now that \( e/h_3 \) is a root of the equation

\[ (e/h_3)^2 + 2a(e/h_3) - (e^2 - m^2) = 0. \]  

If \( a^2 + e^2 > m^2 \), there are two real roots

\[ e/h_3 = -a \pm \sqrt{a^2 + e^2 - m^2}, \]  

and, if \( a^2 + e^2 < m^2 \), there are two complex conjugate roots

\[ e/h_3 = -a \pm i\sqrt{m^2 - a^2 - e^2}. \]  

Finally, substituting the first of these roots back into Eq. (2.40), we obtain

\[ e^{3\lambda \eta(\tau)} = \frac{\tau + i\sqrt{a^2 + e^2 - m^2}}{\tau - i\sqrt{a^2 + e^2 - m^2}}, \]  

and substituting the second of these roots into Eq. (2.40), we obtain

\[ e^{3\lambda \eta(\tau)} = \frac{\tau + \sqrt{m^2 - a^2 - e^2}}{\tau - \sqrt{m^2 - a^2 - e^2}}. \]  

In the first case, \( \lambda \eta(\tau) \) is imaginary, while in the second case, \( \lambda \eta(\tau) \) is real. Thus, only in the first case is the \( SU(2, 1) \) condition satisfied by \( \psi(\tau) \).

One may infer from this disappointing result that an \( SU(2, 1) \) matrix \( \psi(\tau) \) for the case \( a^2 + e^2 < m^2 \) must not have two degenerate eigenvectors. That complicates the identification of a suitable transformation. As far as we know, no one has yet worked out an \( SU(2, 1) \) transformation matrix \( \psi(\tau) \) that accomplishes our purposes when \( a^2 + e^2 < m^2 \), let alone solved the associated HHP. On the other hand, the \( SU(2, 1) \) transformation matrix \( \psi(\tau) \) that we found for the case \( a^2 + e^2 > m^2 \) corresponds to a transformation that was discovered by Alekseev\(^7\) and by Cosgrove\(^8\) many years ago.
Like the double-Harrison transformation, the Cosgrove transformation can be complexified, i.e., the \( SU(2,1) \) transformation matrix \( \psi(\tau) \) can be replaced by an \( SL(3,C) \) matrix. The solution corresponding to \( a^2 + e^2 > m^2 \) will then correspond to an obvious real cross section of the complexified spacetime that results from the application of the complexified Cosgrove transformation to Minkowski space. The big question is, however, “Can one identify another real cross section that corresponds to the case \( a^2 + e^2 < m^2 \)?

In the case of the charged Kerr metric it is fairly trivial to infer the \( \mathcal{E} \) and \( \Phi \) potentials and the metric fields for the case \( a^2 + e^2 < m^2 \) from the corresponding potentials and fields for the case \( a^2 + e^2 > m^2 \). The author has always believed that this type of construction would be possible for all electrovac spacetimes that belong to the Cosgrove family, but only recently, after recognizing the formal similarity of a five-parameter electrovac solution obtained by Manko et al.\(^2\) to a particular specialization of a twelve-parameter solution that was generated many years ago by Guo and Ernst\(^9\) using the Cosgrove transformation, has he actually tried to prove that this is indeed possible.

### III. COMPLEXIFIED COSGROVE TRANSFORMATION

The form of \( U, V \) and \( W \) that we shall present for the spacetime that results from applying a succession of \( n \) Cosgrove transformations of Minkowski space is new, and was derived in the following way from expressions the reader can find in earlier work of Cosgrove,\(^8\) Guo and Ernst,\(^9\) Chen, Guo and Ernst\(^10\) and Wang, Guo and Wu.\(^11\)

For the first Cosgrove transformation, Guo and Ernst expressed the complex potentials in the form\(^12\)

\[
\mathcal{E} = \infty - i\epsilon \frac{\mathcal{N}}{D}, \quad \Theta = -\frac{\mathcal{N}^2}{D}, \tag{3.1}
\]

where \( D \) was written as the determinant of a \( 3 \times 3 \) matrix, the columns of which were denoted by \( \psi_1, \psi_2 \) and \( \psi_3 \), and which were proportional, respectively, to

\[
P^{(0)}(K^*)h, \quad P^{(0)}(K)h', \quad \text{and} \quad P^{(0)}(K)h'', \tag{3.2}
\]
where

\[ P^{(0)}(\tau) := \frac{1}{\sqrt{2r(\tau)}} \begin{pmatrix} -[r(\tau) - (\tau - z)] & i[r(\tau) + (\tau - z)] & 0 \\ -i & 1 & 0 \\ 0 & 0 & \sqrt{2r(\tau)} \end{pmatrix} \]  

(3.3)

is the \( P \)-potential\(^6\) of Minkowski space,

\[ r(\tau) := \sqrt{(z - \tau)^2 + \rho^2}, \]  

(3.4)

and \( K \) is a complex parameter. The elements of the vectors \( h \) are arbitrary complex parameters, but only ratios of these components are significant. The vectors \( h' \) and \( h'' \) are linearly independent vectors that are “orthogonal” to \( h \) in the sense

\[ h^\dagger \Omega h' = 0 = h^\dagger \Omega h'', \]  

(3.5)

where

\[ \Omega := \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]  

(3.6)

Explicit expressions for the components of the vectors \( \psi_i^{(k)} \) were given by Chen, Guo and Ernst in Eqs. (8). However, to obtain simpler expressions, we shall select \( h'' \) differently than they did. If one chooses the vectors

\[ h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \quad h' = \begin{pmatrix} h_1^* \\ h_2^* \\ 0 \end{pmatrix}, \quad h'' = \begin{pmatrix} h_3^* \\ ih_3^* \\ h_1^* + ih_2^* \end{pmatrix}, \]  

(3.7)

then one can select

\[ \psi_1 = \begin{pmatrix} Q_1 \\ 1 \\ Q^* \mathcal{S}_\infty \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} Q_2 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} iQ \\ 0 \\ 1 \end{pmatrix}, \]  

(3.8)
where $K_1 := K^*$, $K_2 := K$, 

$$Q_k := i[X_k r_k + (K_k - z)] , \ S_k := X_k r_k ,$$  \hspace{1cm} (3.9) 

and 

$$X_1 := -\frac{h_1 - ih_2}{h_1 + ih_2} , \ X_2 := -\frac{h_1^* - ih_2^*}{h_1^* + ih_2^*} , \ Q := \sqrt{\varepsilon} \left\langle \frac{\phi}{\phi} \right\rangle_{x} .$$  \hspace{1cm} (3.10) 

We note, in particular, that 

$$K_2 = K_1^* \text{ and } X_1^*X_2 = 1 .$$  \hspace{1cm} (3.11) 

Guo and Ernst also constructed the complex potentials $E$ and $\Phi$ for the spacetime that results when Minkowski space is subjected to two successive Cosgrove transformations. Wang, Guo and Wu then showed that $D$, $N$ and $N'$ could be reexpressed as $6 \times 6$ determinants, while, for higher values of $n$, they could be expressed as $3n \times 3n$ determinants. In our present gauge the Wang-Guo-Wu expression for $D$ assumes the simple form 

$$D = \begin{vmatrix} D_{11} & \cdots & D_{1n} \\ \vdots & \ddots & \vdots \\ D_{n1} & \cdots & D_{nn} \end{vmatrix} ,$$  \hspace{1cm} (3.12) 

where the $3 \times 3$ submatrices $D_{jk}$ are given by 

$$D_{jk} := \begin{pmatrix} (K_{2k-1})^{j-1}Q_{2k-1} & (K_{2k})^{j-1}Q_{2k} & i(K_{2k})^{j-1}Q_\parallel \\ (K_{2k-1})^{j-1} & (K_{2k})^{j-1} & 0 \\ (K_{2k-1})^{j-1}Q_\parallel S_{\varepsilon||-\infty} & 0 & (K_{2k})^{j-1} \end{pmatrix} .$$  \hspace{1cm} (3.13) 

The determinants $N$ and $N'$ can be constructed from $D$ by replacing, respectively, the $(3n)$-th and $(3n - 2)$-nd rows by 

$$K_{1n}^n \ K_{2n}^n \ 0 \ \cdots \ K_{2n-1}^n \ K_{2n}^n \ 0 .$$

The fields $U$, $V$ and $W$ are defined (up to a common factor) by 

$$U := D - iN , \ V := -N' \ \text{and} \ W := iN ,$$  \hspace{1cm} (3.14)
each of which can obviously be written as a single determinant. By using elementary row
operations it can be shown that the term \( i(K_k - z) \) in \( Q_k(k = 1, \ldots, 2n) \) contributes nothing
to any of the determinants. Therefore, each \( Q_k \) can be replaced by \( iS_k \). In conclusion, the
complex potentials \( \mathcal{E} \) and \( \Phi \) of the electrovac solution that results from applying a succession
of \( n \) Cosgrove transformations to Minkowski space are given by

\[
\mathcal{E} = \frac{U - W}{U + W}, \quad \oplus = \frac{V}{U + W},
\]

(3.15)

where \( U \) is the \( 3n \times 3n \) determinant

\[
U = \begin{vmatrix}
U_{11} & \cdots & U_{1n} \\
\vdots & \ddots & \vdots \\
U_{n1} & \cdots & U_{nn}
\end{vmatrix},
\]

(3.16)

in which occur the \( 3 \times 3 \) submatrices

\[
U_{jk} := \begin{pmatrix}
(K_{2k-1})^{j-1}X_{2k-1}r_{2k-1} & (K_{2k})^{j-1}X_{2k}r_{2k} & (K_{2k})^{j-1}Q_{\parallel} \\
(K_{2k-1})^{j-1} & (K_{2k})^{j-1} & 0 \\
(K_{2k-1})^{j-1}Q_{\parallel} & \nabla_{\parallel} & 0 & (K_{2k})^{j-1}
\end{pmatrix},
\]

(3.17)

where

\[
r_a := \sqrt{(z - K_a)^2 + \rho^2}.
\]

(3.18)

The \( 3n \times 3n \) determinants \( -V \) and \( W \) are constructed from \( U \) by replacing, respectively,
the \( (3n) \)-th and the \( (3n - 2) \)-nd row of the latter determinant by

\[
(K_1)^n \quad (K_2)^n \quad 0 \quad \cdots \quad (K_{2n-1})^n \quad (K_{2n})^n \quad 0.
\]

We have seen that the Cosgrove transformation, which is characterized by a \( v(\tau) \) with one
non-degenerate eigenvector \( h \), and a pair of degenerate eigenvectors \( h' \) and \( h'' \), cannot cover
all cases. A Kinnersley-Chitre transformation \( v(\tau) \) with three non-degenerate eigenvectors
is needed as well, but such a transformation has not yet been identified, and the associated
HHP has not yet been solved.
By the *complexified Cosgrove transformation* we shall mean the $SL(3,C)$ matrix $v(\tau)$ with eigenvectors such that Eqs. (3.8) through (3.11) are valid, but $X_1^*, X_2^*$ and $Q^*$ are complex parameters that are no longer identified as the complex conjugates of $X_1, X_2$ and $Q$, respectively. Similarly, $K_1^* = K_2$ is no longer the complex conjugate of $K_1$, and $K_2^* = K_1$ is no longer the complex conjugate of $K_2$. The $n$-fold complexified Cosgrove transformation is characterized by $6n$ complex parameters, while the ordinary Cosgrove transformation is characterized by only $3n$ complex parameters. The complexified Cosgrove transformation for $n = 2$ has sufficiently many complex parameters to cover all axis data of the form

$$
U(z, 0) = z^2 + U_1 z + U_2, \quad (3.19a)
$$

$$
V(z, 0) = V_1 z + V_2, \quad (3.19b)
$$

$$
W(z, 0) = W_1 z + W_2, \quad (3.19c)
$$

$$
U^*(z, 0) = z^2 + U_1^* z + U_2^*, \quad (3.19d)
$$

$$
V^*(z, 0) = V_1^* z + V_2^*, \quad (3.19e)
$$

$$
W^*(z, 0) = W_1^* z + W_2^*, \quad (3.19f)
$$

where the complex fields $U^*$, $V^*$ and $W^*$ are regarded as independent of the complex fields $U$, $V$ and $W$, and where $U_1, U_2, V_1, V_2, W_1, W_2, U_1^*, U_2^*, V_1^*, V_2^*, W_1^*$ and $W_2$ are twelve independent complex constants. These fields $U, V, W$ and $U^*, V^*, W^*$ satisfy the field equations

$$
(U^* U + V^* V - W^* W) \nabla^2 \begin{pmatrix} U \\ V \\ W \end{pmatrix} = 2(U^* \nabla U + V^* \nabla V - W^* \nabla W) \cdot \nabla \begin{pmatrix} U \\ V \\ W \end{pmatrix}, \quad (3.20a)
$$

$$
(U^* U + V^* V - W^* W) \nabla^2 \begin{pmatrix} U^* \\ V^* \\ W^* \end{pmatrix} = 2(U \nabla U^* + V \nabla V^* - W \nabla W^*) \cdot \nabla \begin{pmatrix} U^* \\ V^* \\ W^* \end{pmatrix}. \quad (3.20b)
$$

One should be aware of the fact that the metric field $f$, which is defined by

$$
f := \frac{U^* U + V^* V - W^* W}{(U^* + V^*)(U + W)}, \quad (3.21)
$$
is, in general, complex, since $U^*$ is no longer the complex conjugate of $U$, etc. For this reason, we refer to these as *complexified spacetimes*, even though the $z$ and $\rho$ coordinates remain real.

In the case of the complexified Cosgrove transformation, there are expressions for the independent complex potentials $\mathcal{E}^*$ and $\Phi^*$ that precisely parallel the expressions we have already given for $\mathcal{E}$ and $\Phi$. We shall not state these relations explicitly, since they can be constructed quite easily by the reader. When $\mathcal{E}^*$ is the complex conjugate of $\mathcal{E}$ and $\Phi^*$ is the complex conjugate of $\Phi$, we shall say that we have a *real cross section* of the complexified spacetime.

When the complex constants $Q_\parallel$ and $Q_\perp^*$ all vanish, we obtain a vacuum solution, the real cross sections of which are the vacuum metrics of the Neugebauer family, the $n = 2$ exemplars of which were studied in Ref. 1.

**IV. THE $N = 2$ SOLUTION**

In this paper, as in Ref. 1, we shall restrict our attention to the case $n = 2$, where $U$, $V$ and $W$ are given by

$$U = (K_1 - K_4)(K_2 - K_3) [(1 - |Q_\infty|^\varepsilon)\mathcal{X}_\infty \nabla_\infty - \mathcal{X}_\varepsilon \nabla_\varepsilon] [(1 - |Q_\varepsilon|^\varepsilon)\mathcal{X}_\varepsilon \nabla_\varepsilon - \mathcal{X}_\Delta \nabla_\Delta]$$

$$+ (K_1 - K_2)(K_3 - K_4) [(1 - Q_\infty^* Q_\varepsilon)\mathcal{X}_\infty \nabla_\infty - \mathcal{X}_\varepsilon \nabla_\varepsilon] [(1 - Q_\varepsilon^* Q_\varepsilon)\mathcal{X}_\varepsilon \nabla_\varepsilon - \mathcal{X}_\Delta \nabla_\Delta] , \quad (4.1a)$$

$$V = - \Delta(K_2, K_3, K_4) Y_1 r_1 + \Delta(K_3, K_4, K_1) Y_2 r_2$$

$$- \Delta(K_4, K_1, K_2) Y_3 r_3 + \Delta(K_1, K_2, K_3) Y_4 r_4 , \quad (4.1b)$$

$$W = - \Delta(K_2, K_3, K_4) Z_1 r_1 + \Delta(K_3, K_4, K_1) Z_2 r_2$$

$$- \Delta(K_4, K_1, K_2) Z_3 r_3 + \Delta(K_1, K_2, K_3) Z_4 r_4 , \quad (4.1c)$$

where

$$Y_1 := \left[ Q_\varepsilon (\infty - |Q_\infty|^\varepsilon) + \frac{\mathcal{K}_\infty - \mathcal{K}_\varepsilon}{\mathcal{K}_\Delta - \mathcal{K}_\varepsilon} (Q_\infty - Q_\varepsilon) \right] X_1 , \quad (4.2a)$$

$$Y_2 := Q_\varepsilon \mathcal{X}_\varepsilon , \quad (4.2b)$$

15
\[ Y_3 = \left[ Q_\infty (\infty - |Q_\infty|^\epsilon) - \frac{K_{\Delta} - K_3}{K_{\Delta} - K_\epsilon} (Q_\infty - Q_\epsilon) \right] X_3 , \]  
(4.2c)

\[ Y_4 := Q_\infty K_\Delta , \]  
(4.2d)

and

\[ Z_1 := \left[ (1 - |Q_\infty|^\epsilon) + \frac{K_{\infty} - K_\epsilon}{K_{\Delta} - K_\epsilon} Q_\infty^* (Q_\infty - Q_\epsilon) \right] X_1 , \]  
(4.3a)

\[ Z_2 := X_2 , \]  
(4.3b)

\[ Z_3 := \left[ (1 - |Q_\epsilon|^\epsilon) - \frac{K_{\Delta} - K_3}{K_{\Delta} - K_\epsilon} Q_\epsilon^* (Q_\infty - Q_\epsilon) \right] X_3 , \]  
(4.3c)

\[ Z_4 := X_4 , \]  
(4.3d)

and \( \Delta \) is the Vandermonde determinant introduced in Ref. 1. (We have divided out a common factor \( K_4 - K_2 \) from \( U, V \) and \( W \).)

From the above expression for \( U \), one easily identifies the complex constants

\[ U_0 = (K_1 - K_4)(K_2 - K_3) \left[ (1 - |Q_\infty|^\epsilon) X_\infty - X_\epsilon \right] \left[ (1 - |Q_\epsilon|^\epsilon) X_\epsilon \right] \]  
\[ + (K_1 - K_2)(K_3 - K_4) \left[ (1 - Q_\infty^* Q_\epsilon) X_\infty - X_\Delta \right] \left[ (1 - Q_\infty^* Q_\epsilon) X_\epsilon \right] , \]  
(4.4a)

\[ U_1 = - (K_1 - K_4)(K_2 - K_3) \left\{ \left[ (1 - |Q_\infty|^\epsilon) X_\infty - X_\epsilon \right] \left[ (1 - |Q_\epsilon|^\epsilon) K_3 X_\epsilon - K_\Delta X_\Delta \right] \right\]  
\[ + \left[ (1 - |Q_\infty|^\epsilon) K_\infty X_\infty - K_\epsilon X_\epsilon \right] \left[ (1 - |Q_\epsilon|^\epsilon) X_\epsilon - X_\Delta \right] \} , \]  
(4.4b)

\[ U_2 = (K_1 - K_4)(K_2 - K_3) \left[ (1 - |Q_\infty|^\epsilon) K_\infty X_\infty - K_\epsilon X_\epsilon \right] \left[ (1 - |Q_\epsilon|^\epsilon) K_3 X_\epsilon - K_\Delta X_\Delta \right] \]  
\[ + (K_1 - K_2)(K_3 - K_4) \left[ (1 - Q_\infty^* Q_\epsilon) K_\infty X_\infty - K_\Delta X_\Delta \right] \left[ (1 - Q_\infty^* Q_\epsilon) K_3 X_\epsilon - K_\epsilon X_\epsilon \right] , \]  
(4.4c)

while from the expressions for \( V \) and \( W \) one obtains the complex constants

\[ V_1 = -V^{(0)} , \quad V_2 = V^{(1)} , \quad W_1 = -W^{(0)} , \quad W_2 = W^{(1)} , \]  
(4.5)

where

\[ V^{[a]} := - \Delta(K_2, K_3, K_4) K_1^a Y_1 + \Delta(K_3, K_4, K_1) K_2^a Y_2 \]
\[ - \Delta(K_1, K_2, K_3, K_4) K_3^a Y_3 + \Delta(K_1, K_2, K_3) K_4^a Y_4, \quad (4.6a) \]

\[ W^{(a)} := - \Delta(K_2, K_3, K_4) K_1^a Z_1 + \Delta(K_3, K_4, K_1) K_2^a Z_2 \]
\[ - \Delta(K_1, K_2, K_3) K_4^a Z_3 + \Delta(K_1, K_2, K_3) K_4^a Z_4. \quad (4.6b) \]

Our immediate objective is to determine the parameters \(X_a, Y_a,\) and \(Z_a\) \((a = 1, 2, 3, 4)\) in terms of the axis data \(U_1, U_2, V_1, V_2, W_1\) and \(W_2\) (where \(U_0 = 1\)), and the \(K\)’s.

**A. Determination of \(X_a\) \((a = 1, 2, 3, 4)\)**

The simplest case is when \(V_2 W_1 - V_1 W_2 = 0\). This corresponds to \(\mathcal{Q}_\infty = \mathcal{Q}_\varepsilon =: \mathcal{Q}\), where \(V_1 = \mathcal{Q} W_\infty\) and \(V_2 = \mathcal{Q} W_\varepsilon\). In this case one gets the same expressions for \((1 - |\mathcal{Q}|^2) X_\infty\), \(X_2\), \((1 - |\mathcal{Q}|^2) X_3\) and \(X_4\) as one got for \(X_1, X_2, X_3\) and \(X_4\), respectively, in the vacuum case. This solution is merely the electrovac solution that is generated by the old electrification transformation of Harrison, as reformulated by Ernst. In this paper, we are concerned primarily with the case when \(V_2 W_1 - V_1 W_2 \neq 0\), i.e., \(\mathcal{Q}_\infty \neq \mathcal{Q}_\varepsilon\).

In the vacuum problem it was surprisingly easy to solve for the \(X\)’s in terms of the axis data and the \(K\)’s. One had to solve nothing but linear and quadratic algebraic equations. The solution itself revealed a most intriguing structure, to which the simplicity of the solution can be attributed, and a knowledge of which we found to be indispensable for solving the electrovac problem. On the one hand, one had the hierarchy of linear equations (for the \(X\)’s)

\[ W^{(0)} = -W_1, \quad (4.7a) \]
\[ W^{(1)} = W_2, \quad (4.7b) \]
\[ W^{(2)} = \frac{U_1 W_2 - U_2 W_1}{U_0}, \quad (4.7c) \]
\[ W^{(3)} = \frac{U_1 [(U_1 W_2 - U_2 W_1)/U_0] - U_2 W_2}{U_0}, \quad (4.7d) \]

and, on the other hand,

\[ \Delta(K_1, K_2, K_3, K_4) = -\frac{W_1 [(U_1 W_2 - U_2 W_1)/U_0] - W_2^2}{U_0}. \quad (4.8) \]
In particular, the last equation allowed one to determine $U_0$ in terms of the axis data and the $K$’s, a critical step in the complete determination of the $X$’s.

After spending a considerable amount of time trying to identify the $X$’s and the $Q$’s in the electrovac case, we abandoned that effort, and approached the problem of determining $U$, $V$ and $W$ in a new way that avoids the determination of the $X$’s and $Q$’s (although these objects can be calculated at the very end, if they are really desired). In the electrovac case we have two sets of four linear equations,

$$V^{(0)} = -V_1,$$  
$$V^{(1)} = V_2,$$  
$$V^{(2)} = \frac{U_2 V_1 - U_1 V_2}{U_0},$$  
$$V^{(3)} = -\frac{U_2 V_2 + U_1 [(U_2 V_1 - U_1 V_2)/U_0]}{U_0} + \frac{V_2 W_1 - V_1 W_2}{U_0} \left(\frac{W_1}{U_0}\right)^*, \quad (4.9d)$$

and

$$W^{(0)} = -W_1,$$  
$$W^{(1)} = W_2,$$  
$$W^{(2)} = \frac{U_2 W_1 - U_1 W_2}{U_0},$$  
$$W^{(3)} = -\frac{U_2 W_2 + U_1 [(U_2 W_1 - U_1 W_2)/U_0]}{U_0} + \frac{V_2 W_1 - V_1 W_2}{U_0} \left(\frac{W_1}{U_0}\right)^*, \quad (4.10d)$$

As in the vacuum case, the right side of each of the four equations is equal to $U_0$ times a quantity that can be easily expressed in terms of the axis data alone, while the left side of each of the four equations is equal to a linear combination of $Y_a$ ($a = 1, 2, 3, 4$) or $Z_a$ ($a = 1, 2, 3, 4$).

**B. Determination of $U$, $W$ and $V$ up to a common factor**

The four linear equations (4.9a)—(4.9d) for $Y_a$ ($a = 1, 2, 3, 4$) and the four linear equations (4.10a)—(4.10d) for $Z_a$ ($a = 1, 2, 3, 4$) are easily solved. One obtains
\[ D \mathcal{Y}_\infty = \{ U_2 V_2 + U_1 (U_2 V_1 - U_1 V_2) - (V_2 W_1 - V_1 W_2) W^*_1 \} + (K_2 + K_3 + K_4) (U_2 V_1 - U_1 V_2) - (K_2 K_3 + K_2 K_4 + K_3 K_4) V_2 - (K_2 K_3 K_4) V_1, \] (4.11)

and

\[ D \mathcal{Z}_\infty = \{ U_2 W_2 + U_1 (U_2 W_1 - U_1 W_2) - (V_2 W_1 - V_1 W_2) V^*_1 \} + (K_2 + K_3 + K_4) (U_2 W_1 - U_1 W_2) - (K_2 K_3 + K_2 K_4 + K_3 K_4) W_2 - (K_2 K_3 K_4) W_1, \] (4.12)

where \( U, V \) and \( W \) have been adjusted so that \( U_0 = 1 \). The expressions for \( Y_2, Y_3 \) and \( Y_4 \) can be inferred from the expression for \( Y_1 \) and the expressions for \( Z_2, Z_3 \) and \( Z_4 \) can be inferred from the expression for \( Z_1 \) by permuting indices on the \( K \)'s. \( D \) is given by

\[ D := -\frac{\cdot (\mathcal{K}_\infty, \mathcal{K}_\varepsilon, \mathcal{K}_\varnothing, \mathcal{K}_\Delta)}{U_0}, \] (4.13)

where \( U_0 \) is the original value of \( U_0 \), not 1.

Using Eqs. (4.1b) and (4.1c), these expressions for \( D \mathcal{Y}_+, D \mathcal{Z}_+ (\dot{t} = \infty, \varepsilon, \varnothing, \Delta) \) permit us to evaluate \( D \mathcal{V} \) and \( D \mathcal{W} \) without further ado, for we have

\[ D \mathcal{V} = -\Delta(K_2, K_3, K_4) D \mathcal{Y}_\infty \nabla \mathcal{Y}_\infty + \cdot (K_\varnothing, K_\Delta, K_\infty) D \mathcal{Y}_\varepsilon \nabla \mathcal{Y}_\varepsilon - \Delta(K_4, K_1, K_2) D \mathcal{Y}_\varepsilon \nabla \mathcal{Y}_\varepsilon + \cdot (K_\infty, K_\varepsilon, K_\varnothing) D \mathcal{Y}_\Delta \nabla \mathcal{Y}_\Delta, \] (4.14a)

\[ D \mathcal{W} = -\Delta(K_2, K_3, K_4) D \mathcal{Z}_\infty \nabla \mathcal{Z}_\infty + \cdot (K_\varnothing, K_\Delta, K_\infty) D \mathcal{Z}_\varepsilon \nabla \mathcal{Z}_\varepsilon - \Delta(K_4, K_1, K_2) D \mathcal{Z}_\varepsilon \nabla \mathcal{Z}_\varepsilon + \cdot (K_\infty, K_\varepsilon, K_\varnothing) D \mathcal{Z}_\Delta \nabla \mathcal{Z}_\Delta, \] (4.14b)

but what about \( U \), which is given by Eq. (4.1a)? Interestingly, \( U \) can be expressed directly in terms of the \( Y \)'s and \( Z \)'s as follows:

\[ U = -\frac{1}{2} \left( \frac{K_4 - K_2}{Q_1 - Q_2} \right) \sum_{i,j,k,l} \epsilon_{ijkl} (K_i - K_j) Z_k Y_l r_k r_l, \] (4.15)

where \( \epsilon_{ijkl} \) is Levi-Civita's permutation symbol. On the other hand,

\[ \frac{K_4 - K_2}{Q_\infty - Q_\varepsilon} = -\frac{\Delta(K_1, K_2, K_3, K_4) U_0}{V_2 W_1 - V_1 W_2} = D \left( \frac{U_0}{V_\varepsilon W_\infty - V_\infty W_\varepsilon} \right). \] (4.16)
Hence, with $U_0 = 1$, $DU$ can be expressed in the final form

$$DU = -\sum_{\epsilon \in \{V_\epsilon W_\infty - V_\infty W_\epsilon\}}^\infty \epsilon ||| (K_i - K_i) (DZ_i)(DY_i) \nabla ||| \nabla .$$

(4.17)

This is most remarkable, since it means that $DU$ involves only the axis data, the known $DY_i$'s and $DZ_i$'s and the K's. Since one is only interested in ratios of $U$, $V$ and $W$, it suffices to know $DU$, $DY$ and $DW$. One does not have to evaluate $D$ itself!

C. Determination of $K_a$ ($a = 1, 2, 3, 4$)

Using Eqs. (3.11) and defining $|Z|^2 := Z^*Z$ even when $Z^*$ is not just the complex conjugate of $Z$, we find that, on the symmetry axis,

$$|U(z, 0)|^2 + |V(z, 0)|^2 - |W(z, 0)|^2 = |U_0|^2 (K_1 - z)(K_2 - z)(K_3 - z)(K_4 - z),$$

(4.18)

from which it follows that each $K_a$ ($a = 1, 2, 3, 4$) satisfies the quartic equation

$$0 = K_a^4 + 2(Re U_1) K_a^3 + (|U_1|^2 + |V_1|^2 - |W_1|^2 + 2Re U_2) K_a^2$$

$$+ 2Re (U_2 U_1^* + V_2 V_1^* - W_2 W_1^*) K_a + (|U_2|^2 + |V_2|^2 - |W_2|^2),$$

(4.19)

where $Re Z := (Z + Z^*)/2$ even when $Z^*$ is not just the complex conjugate of $Z$.

Assuming that $z$ has been chosen so that $Re U_1 = 0$ (or, equivalently, $K_1+K_2+K_3+K_4 = 0$), we have

$$K_a^4 - AK_a^2 - BK_a + C = 0,$$

(4.20)

where

$$A = |W_1|^2 - |V_1|^2 - |U_1|^2 - 2Re U_2 ,$$

(4.21a)

$$B = -2Re (U_2 U_1^* + V_2 V_1^* - W_2 W_1^*),$$

(4.21b)

$$C = |U_2|^2 + |V_2|^2 - |W_2|^2 .$$

(4.21c)

The general solution $K_a$ ($a = 1, 2, 3, 4$) of this quartic equation is given by Eqs. (2.9a) through (2.9d) or Eqs. (2.14a) through (2.14d) of Ref. 1.
In conclusion, the determination of the $K$’s is no more difficult in the electriving case than it was in the vacuum case. Of course, when $U(z,0)^*$, $V(z,0)^*$ and $W(z,0)^*$ are the complex conjugates of $U(z,0)$, $V(z,0)$ and $W(z,0)$, respectively, the quartic equation, like its vacuum analog, has solutions in which the $K$’s are real, rather than occurring in complex conjugate pairs. Such $K$’s cannot be used with the ordinary Cosgrove transformation. It is instead necessary to employ the complexified Cosgrove transformation.

We would be the first to admit that the picture we have painted using broad brushstrokes requires further refinement, which we hope to supply in a future paper. However, we shall turn now to an application that already demonstrates the practical value of this approach.

\textbf{V. A SIMPLE BUT CONVINCING APPLICATION}

Suppose we select the axis data

\[ U_1 = -ia, \ U_2 = b, \ V_1 = e, \ V_2 = ic, \ W_1 = m, \ W_2 = 0, \]  

(5.1)

where the parameters $a, b, e, c, m$ are real. In this case, one has

\[ U_2 U_1^* + V_2 V_1^* - W_2 W_1^* = b(-ia)^* + (ic)(e)^* - 0(m)^* = i(ab + ce), \]

(5.2)

so $\text{Re}(U_2 U_1^* + V_2 V_1^* - W_2 W_1^*) = 0$. Therefore, as in Ref. 1, we may write

\[ K_1 = -K_2 = \frac{1}{2}(\kappa_+ + \kappa_-), \ K_3 = -K_4 = \frac{1}{2}(\kappa_+ - \kappa_-), \]

(5.3)

where $\kappa_+$ and $\kappa_-$ are given by

\[ \kappa_\pm := \sqrt{m^2 - a^2 - c^2 + 2(d-b)}, \quad d := \sqrt{b^2 + c^2}. \]

(5.4)

With the selected axis data, Eqs. (4.11), (4.12) and their analogs reduce to

\begin{align*}
\mathcal{D} \mathcal{Y}_\infty &= (e/m) \mathcal{D} Z_\infty - \frac{\infty}{e} \left[ \left[ (\mathcal{E} - d)^{\mathcal{E}} - (\mathcal{E}) + \kappa_+ \kappa_- \right] + (\kappa_+ + \kappa_-) \right], \quad (5.5a) \\
\mathcal{D} \mathcal{Y}_e &= (e/m) \mathcal{D} Z_e - \frac{\infty}{e} \left[ \left[ (\mathcal{E} - d)^{\mathcal{E}} - (\mathcal{E}) + \kappa_+ \kappa_- \right] - (\kappa_+ + \kappa_-) \right], \quad (5.5b) \\
\mathcal{D} \mathcal{Y}_3 &= (e/m) \mathcal{D} Z_3 - \frac{\infty}{e} \left[ \left[ (\mathcal{E} - d)^{\mathcal{E}} - (\mathcal{E}) - \kappa_+ \kappa_- \right] + (\kappa_+ - \kappa_-) \right], \quad (5.5c) \\
\mathcal{D} \mathcal{Y}_\Delta &= (e/m) \mathcal{D} Z_\Delta - \frac{\infty}{e} \left[ \left[ (\mathcal{E} - d)^{\mathcal{E}} - (\mathcal{E}) - \kappa_+ \kappa_- \right] - (\kappa_+ - \kappa_-) \right], \quad (5.5d)
\end{align*}
\[DZ_{\infty} = -m \left\{ i(ab + ce) + \frac{1}{2} \left[ \kappa_+(d + b) - \kappa_-(d - b) \right] \right\}, \quad (5.6a)\]
\[DZ_{v} = -m \left\{ i(ab + ce) - \frac{1}{2} \left[ \kappa_+(d + b) - \kappa_-(d - b) \right] \right\}, \quad (5.6b)\]
\[DZ_{\gamma} = -m \left\{ i(ab + ce) + \frac{1}{2} \left[ \kappa_+(d + b) + \kappa_-(d - b) \right] \right\}, \quad (5.6c)\]
\[DZ_{\Delta} = -m \left\{ i(ab + ce) - \frac{1}{2} \left[ \kappa_+(d + b) + \kappa_-(d - b) \right] \right\}, \quad (5.6d)\]

respectively. Eqs. (4.17), (4.1b) and (4.1c) then yield the following expressions for \(DU\), \(DV\) and \(DW\):

\[
DU = \kappa_+^2 \left\{ d \left[ (m^2 - a^2 - e^2) + c^2 - a(ab + ce) \right] (r_1 r_3 + r_2 r_4) \\
+ i\kappa_+ (ab + ce + ad) (r_1 r_3 - r_2 r_4) \right\} \\
+ \kappa_0^2 \left\{ \left[ d(m^2 - a^2 - e^2) - c^2 + a(ab + ce) \right] (r_1 r_4 + r_2 r_3) \\
+ i\kappa_- (ab + ce - ad) (r_1 r_4 - r_2 r_3) \right\} \\
- 4d \left[ b(m^2 - e^2) + c(ac + b) \right] (r_2 r_1 + r_4 r_3), \quad (5.7a)
\]

\[
DV = \kappa_+ \kappa_- \left\{ d \left[ c(m^2 - a^2 - e^2) - 2ac \right] (r_1 + r_3 - r_2 - r_4) + de \kappa_+ \kappa_- (r_2 + r_1 + r_4 + r_3) \\
+ i\kappa_+ \left[ (\kappa_+ + \kappa_-)(r_2 - r_1) + (\kappa_+ - \kappa_-)(r_3 - r_4) \right] \\
+ ic \left[ \kappa_+ \kappa_- (r_3 - r_4) + (\kappa_+ - \kappa_-)(r_2 - r_1) \right] \right\}, \quad (5.7b)
\]

\[
DW = m \kappa_+ \kappa_- \left\{ d \left[ (m^2 - a^2 - e^2)(r_4 + r_3 - r_2 - r_1) + \kappa_+ \kappa_- (r_2 + r_1 + r_4 + r_3) \right] \\
+ i(ab + ce) \left[ \kappa_+ \kappa_- (r_3 - r_4) + (\kappa_+ - \kappa_-)(r_2 - r_1) \right] \right\}. \quad (5.7c)
\]

Of course, there are similar expressions for \((DU)^* := DU^*\), etc.

The real cross sections of the complexified spacetime are easily identified. They correspond to real values of \(\kappa_+^2\) and \(\kappa_-^2\). The solution given in Eqs. (5.7a), (5.7b) and (5.7c) is valid not only when \(0 > \kappa_+^2 > \kappa_-^2\), but for other values of \(\kappa_+\) and \(\kappa_-\) as well. When both \(\kappa_+\) and \(\kappa_-\) are real, the solution is identical to the five-parameter electrovac solution published recently by Manko et al.\(^2\) in which the parameters \(m, a, b, c, e\) were associated, respectively, with the mass, the rotation, the mass quadrupole moment, the electric charge and the magnetic dipole moment.
It should be observed that \((DU)^*, (DV)^*\) and \((DW)^*\) have the same functional form if \(\kappa_\pm\) and \(r_a\) \((a = 1, 2, 3, 4)\) are treated as real as they have if \(\kappa_\pm\) are treated as imaginary, with \(r_1^* = r_2\) and \(r_3^* = r_4\). This means that the expression obtained by Manko et al. for the metric fields \(f\), \(\gamma\) and \(\omega\) in

\[
ds^2 = f^{-1} \left\{ e^{2\gamma} \left( dz^2 + d\rho^2 \right) + \rho^2 d\varphi^2 \right\} - f \left( dt - \omega d\varphi \right)^2 .
\] (5.8)

will hold for the other cases as well. In a later paper concerned with the complexified Cosgrove transformation, we shall develop a completely general formula for the field \(\omega\). At this time, we merely remark that the field \(\gamma\) is given by

\[
\epsilon^{2\gamma} = \frac{|U|^2 + |V|^2 - |W|^2}{|U|^2 r_1 r_2 r_3 r_4},
\] (5.9)

and the field \(f\) is given by

\[
f = \text{Re} \mathcal{E} + |\Phi|^2 = \frac{|U|^2 + |V|^2 - |W|^2}{|U + W|^2}.
\] (5.10)

Thus, the infinite red shift surface corresponds to

\[
|U|^2 + |V|^2 - |W|^2 = 0,
\] (5.11)

and the curvature singularities occur at \(U + W = 0\).

VI. TOWARD A PURELY ALGEBRAIC DERIVATION

It was Kinnersley\(^1^4\) who first pointed out that \(U\), \(V\) and \(W\) could always be selected so that the field equations

\[
(|U|^2 + |V|^2 - |W|^2) \nabla^2 \begin{pmatrix} U \\ V \\ W \end{pmatrix} = 2(U^* \nabla U + V^* \nabla V - W^* \nabla W) \cdot \nabla \begin{pmatrix} U \\ V \\ W \end{pmatrix}
\] (6.1)

are satisfied. The reader will find it instructive to work out the \(n = 1\) solution of these equations, where
\[ U = \sum_i u_i r_i , \]  
\[ V = v , \]  
\[ W = w , \]  
(6.2a)  
(6.2b)  
(6.2c)

and \( u_1, u_2, v \) and \( w \) are complex constants. This is not difficult to do, if one observes that

\[ \nabla^2 r_i = \frac{2}{r_i} , \]  
(6.3)

and

\[ \nabla r_i \cdot \nabla r_j = \frac{r_i^2 + r_j^2 - (K_i - K_j)^2}{2r_i r_j} , \]  
(6.4)

and one uses the relation\(^{15}\)

\[ |U(z, 0)|^2 + |V(z, 0)|^2 - |W(z, 0)|^2 = |U_0|^2 \Pi_{a=1}^{2n} (z - K_a) . \]  
(6.5)

In the present paper we have been interested in \( n = 2 \) solutions, in which

\[ U = \sum_{i < j} u_{ij} r_i r_j , \]  
(6.6a)

\[ V = \sum_i v_i r_i , \]  
(6.6b)

\[ W = \sum_i w_i r_i , \]  
(6.6c)

where the \( u_{ij} \), the \( v_i \) and the \( w_i \) are complex constants. For all values of \( n \) the mechanism of solution remains the same as that illustrated by the \( n = 1 \) case, but the algebra becomes increasingly more difficult as \( n \) increases.

It would be nice if one could formulate a simple strictly algebraic derivation of the general solution of Eqs. (6.1) corresponding to rational axis data by using Eqs. (6.3), (6.4) and (6.5). We shall postpone further consideration of this approach until a later paper, where we shall be concerned primarily with \( n > 2 \).

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REFERENCES


12 The conventions of Guo and Ernst differed slightly from ours. Compare, for example, the respective identifications of the $P$-potential and the respective expressions for $\Phi$.

13 Manko et al., unpublished work. Special cases are to be found in Ref. 2.
