A nonlocal, covariant generalisation of the NJL model

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We solve a nonlocal generalisation of the NJL model in the Hartree approximation. This model has a separable interaction, as suggested by instanton models of the QCD vacuum. The choice of form factor in this interaction is motivated by the confining nature of the vacuum. A conserved axial current is constructed in the chiral limit of the model and the pion properties are shown to satisfy the Gell-Mann–Oakes–Renner relation. For reasonable values of the parameters the model exhibits quark confinement.

I. INTRODUCTION

The Nambu–Jona-Lasinio (NJL) model [1] is often used as a model to study the hidden chiral symmetry of QCD [2]. It describes fermions interacting via a local, chirally invariant four-point coupling. For large enough coupling strengths it leads to a vacuum state with a quark condensate and pions which are approximate Goldstone bosons. The local nature of the interaction considerably simplifies the calculations compared with ones based on finite-range forces. However it also means that loop diagrams are divergent and must be regularised in some way.

This regularisation scheme forms part of the model and introduces a cut-off as an extra parameter. Many procedures have been used in the literature, for example: three- and four-momentum cut-offs, Pauli–Villars, proper-time regularisation [3]. All of these yield qualitatively similar results. None however has any clear interpretation in terms of the
underlying theory, QCD. Cutting off the quark propagator at high momenta, as most of these schemes do, does violence to the Hilbert space. In loop diagrams with two internal quark propagators, such as are needed for calculating mesonic bound states, one has to be careful about how the cut-off is imposed on both propagators if axial Ward identities are not to be violated. Furthermore, while a cut-off is needed to make the nonanomalous part of the effective action finite, cutting off the anomalous part means that, for example, the amplitude for $\pi^0 \to 2\gamma$ is not given correctly.\(^1\)

An attractive alternative to the usual local NJL model is suggested by the instanton picture of the QCD vacuum studied by Diakonov and Petrov [5,6]. The interactions of the quarks with the instantons generate an effective four-point coupling (in the case of two light quark flavours). The nonlocality of this interaction provides a natural cut-off on the loop integrals. Moreover the nonlocality a separable form. This considerably simplifies the Schwinger–Dyson and Bethe–Salpeter equations in comparison with models based on quark-quark forces motivated by gluon exchange.\(^2\)

In Diakonov and Petrov’s original version the form factor in this interaction is given by the zero mode of a quark in the presence of an instanton. Here we consider more general choices, motivated in part by the fact that the confining nature of the QCD vacuum will modify the long-range behaviour of the form factor. We explicitly construct a conserved axial current for this type of model and demonstrate that PCAC, implicitly assumed by Diakonov and Petrov, does indeed hold. In addition we show that a rather natural choice of form factor can lead to quark confinement. We also compare our model with other nonlocal NJL-like models based on separable interactions.

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\(^1\) For a review of these problems, see [4].

\(^2\) See [7] for a recent example of such a model and many further references.
II. NONLOCAL MODEL

The action for a generalised NJL model with a nonlocal four-point interaction and approximate SU(2) × SU(2) chiral symmetry is

\[ S = \int d^4x \ \overline{\psi}(x)(i\slashed{D} - m_c)\psi(x) + \int d^4x_1 \ldots d^4x_4 \ \alpha(x_1, x_2, x_3, x_4) \times \left[ \overline{\psi}(x_1)\psi(x_3)\overline{\psi}(x_2)\psi(x_4) + \overline{\psi}(x_1)i\gamma_5\tau^a\psi(x_3)\overline{\psi}(x_2)i\gamma_5\tau^a\psi(x_4) \right]. \]

The symmetry is explicitly broken by the current quark mass \( m_c \). For a separable interaction the smearing function may be written in momentum space as

\[ \bar{\alpha}(p_1, p_2, p_3, p_4) = \frac{1}{2}(2\pi)^8 G f(p_1) f(p_2) f(p_3) f(p_4) \ \delta(p_1 + p_2 - p_3 - p_4), \]

where we take the form factors \( f(p) \) to be normalised such that \( f(0) = 1 \). In the limit of a local interaction, \( f(p) = 1 \) for all \( p \), this reduces to the ordinary NJL model with coupling strength \( G \).

As in the ordinary NJL model, for a sufficiently large coupling the quarks acquire a dynamical mass. With a nonlocal interaction this dynamical mass \( M(p) \) is momentum-dependent. It can be determined, in the Hartree approximation, from the gap equation:

\[ M(p) = m_c + G v f(p)^2 \int \frac{d^4k}{(2\pi)^4} \frac{M(k) f^2(k)}{k^2 + M(k)^2}, \]

where \( v = 4N_c N_f \), \( N_c \) and \( N_f \) being the numbers of colours and flavours respectively. The momentum integral has been Wick rotated so that the four-momentum \( k \) is in Euclidean space. From now on all momenta will be Euclidean, unless explicitly stated otherwise. The separable nature of our interaction means that the loop integral is independent of the external momentum \( p \) and so this equation can be solved in exactly the same way as in the NJL model.

In choosing the form factor \( f(p) \), we obviously want it to be Lorentz invariant. We assume that the interaction is generated by nonperturbative features of the QCD vacuum.
and so does not have the divergent short-distance behaviour of one-gluon exchange. Hence we want a form that falls off sufficiently fast at high momenta to keep the loop integrals finite. At low momenta we want it to be more regular than the instanton form used by Diakonov and Petrov since in coordinate space it corresponds to a function that falls off rapidly with distance. Such a fall-off is to be expected from the confining nature of the QCD vacuum. The simplest choice satisfying all of these requirements is a Gaussian function of the Euclidean four-momentum:

\[ f(p) = \exp(-p^2/\Lambda^2), \]

where \( \Lambda \) is our cut-off parameter describing the range of the nonlocality. We use this form factor in the numerical calculations presented here. Other choices which also possess these features are likely to lead to qualitatively similar results.

With this choice of form factor, the model does exhibit confinement of the quarks for large enough values of the dynamical mass. This arises because the equation \( p^2 = M(p)^2 = M(0)^2 \exp[2p^2/\Lambda^2] \) in Minkowski space has no real solutions. Hence the quark propagator has no real poles and quarks do not appear as asymptotic states. The quark propagator does have a pair of poles with complex masses, corresponding to quarks which have a finite lifetime as isolated particles. This is similar to the situation in other models for confinement based on approximate Schwinger-Dyson equations [8,7].

The model has three adjustable parameters: \( G, \Lambda \) and \( m_c \). We fix two of these using the pion mass and decay constant. The remaining free parameter can then be chosen to give reasonable values for the dynamical quark mass and decay constant.

III. BETHE-SALPETER EQUATION

The pion is constructed, as in the NJL model, by solving the corresponding Bethe-Salpeter equation. For a separable interaction this equation is just a geometric series. For example, the \( T \)-matrix in the pseudoscalar isovector channel can be written in the form
\[ T(p_1, p_2, p_3, p_4) = f(p_1)f(p_2)f(p_3)f(p_4)[G + G^2 J_{PP}(p) + G^3 J_{PP}(p)^2 + \cdots] \quad (5) \]
\[ = f(p_1)f(p_2)f(p_3)f(p_4) \frac{G}{1 - G J_{PP}(p)}, \]

where \( p = p_1 + p_2 = p_3 + p_4 \) and \( J_{PP} \) is given by a quark loop with two nonlocal pseudoscalar insertions:

\[ J_{PP}(p) = \text{tr} \int \frac{d^4k}{(2\pi)^4} \left[ i \gamma^5 \frac{f^2(k + \frac{1}{2}p)}{k^2 + \frac{1}{2}p} + M(k + \frac{1}{2}p) i \gamma^5 \frac{f^2(k - \frac{1}{2}p)}{k^2 - \frac{1}{2}p} + M(k - \frac{1}{2}p) \right]. \quad (6) \]

The pion mass and wave function renormalisation can be determined from the position and residue of the pole in the propagator in this channel. For momenta in the vicinity of the pion pole, \( p^2 = -m_\pi^2 \), we write

\[ \frac{G}{1 - G J_{PP}(p)} = \frac{Z_\pi}{p^2 + m_\pi^2} + \cdots. \quad (7) \]

Assuming that the symmetry-breaking current quark mass is much smaller than the dynamical mass, we now expand \( J_{PP} \) about the chiral limit. To first order in the current quark mass and \( p^2 \) it can be written in the simple form

\[ J_{PP}(p) \simeq \frac{1}{G} - \frac{m_c}{M_0(0)} \langle \bar{\psi} \psi \rangle - \frac{p^2}{Z_\pi}, \quad (8) \]

where \( M_0(p) \) is the dynamical mass at zero current quark mass. In obtaining this we have made use of the gap equation and the expression for the quark condensate,

\[ \langle \bar{\psi} \psi \rangle = \nu \int \frac{d^4k}{(2\pi)^4} \frac{M_0(k)}{k^2 + M_0(k)^2}, \quad (9) \]

in the chiral limit. To leading order in \( m_c \), the pion mass is given by

\[ m_\pi^2 = -\frac{Z_\pi}{M_0(0)^2} m_c \langle \bar{\psi} \psi \rangle, \quad (10) \]

and the wave function renormalisation by

\[ Z_\pi^{-1} = \nu \int \frac{d^4k}{(2\pi)^4} \frac{M(k)^2 - M'(k)M(k)k^2 + (M'(k))^2k^4}{(k^2 + M_0(k)^2)^2}, \quad (11) \]

where the prime denotes differentiation with respect to \( k^2 \).
IV. PION DECAY CONSTANT

In order to calculate the pion decay constant $F_\pi$ for this model we need to construct an axial current which is conserved in the chiral limit. The usual local current,

$$j_{5}^{\mu\alpha} = \frac{1}{2} \overline{\psi} \gamma^\mu \gamma^5 \tau^\alpha \psi,$$

is not conserved in the presence of the nonlocal interaction of Eq. (1) and so it cannot be the symmetry current of this model. This can be seen from the results of Diakonov and Petrov [6] where the pion decay constant calculated from this current fails to satisfy the Gell-Mann–Oakes–Renner (GOR) relation [9]. In addition, the correlator for this current is not transverse, another signal that it is not the appropriate symmetry current.

Various techniques are available for construction of a conserved current in the presence of nonlocal interactions. A rather elegant one is to make the action locally invariant by coupling the quarks to right- and left-handed gauge fields, $A_\mu^{R,L} [10]$. Some of these fields can be identified with those of the electromagnetic or weak interactions, but in general they need not be physical. The corresponding currents are obtained by differentiating the action with respect to the gauge fields and then setting those fields equal to zero.

The locally invariant action is constructed by replacing the ordinary derivative by the covariant one in the kinetic term. The nonlocal interaction can be made invariant by introducing parallel transport operators,

$$W_{R,L}(x, y) = \text{Pexp} \left[ \int_0^1 d\lambda \frac{dz^\mu}{d\lambda} A^\mu(z)(1 \pm \gamma_5) \right],$$

where the integral follows a path $z(\lambda)$ between the appropriate right- or left-handed quark fields at space-time points $x$ and $y$. This is analogous to the gauge-invariant point splitting used in deriving anomalies [11]. Although this method is convenient in a version of the model (1) with only one quark flavour, it becomes rather cumbersome for the in a model with two or more flavours because of the non-abelian nature of the chiral symmetry.

Alternatively one can use a Noether-like method of construction, evaluating the divergence of the local current with the aid of the equations of motion obtained from the action.
(1) (with \( m_e = 0 \)). For the axial isospin current, which we are interested in here, this gives

\[
\partial_\mu \left( \frac{1}{2} \bar{\psi} \gamma^\mu \gamma^5 \tau^a \psi \right) = - \int d^4x_1 \ldots d^4x_4 \alpha(x_1, x_2, x_3, x_4) \bar{\psi}(x_1) \tau_a i\gamma^5 \psi(x_2) \bar{\psi}(x_3) \psi(x_4) \times [\delta(x_1 - x) - \delta(x_2 - x) + \delta(x_3 - x) - \delta(x_4 - x)] + \frac{1}{2} \int d^4x_1 \ldots d^4x_4 \alpha(x_1, x_2, x_3, x_4) \bar{\psi}(x_1) [\tau_a, \tau_b] \psi(x_3) \bar{\psi}(x_2) \tau_b i\gamma^5 \psi(x_4) \times [\delta(x_1 - x) - \delta(x_3 - x)],
\]

where we have assumed that \( \alpha \) is symmetric under permutations of its arguments. The r.h.s. of this equation, which arises from the variation of the nonlocal interaction under local axial isospin transformations, must be re-expressed as a divergence if we are to identify a conserved current. Ball and Ripka [4] do this by making a derivative expansion of the nonlocality.

However such an expansion is not essential: a trick suggested by the previous method is to write the differences of delta functions as line integrals, for example,

\[
\delta(x_1 - x) - \delta(x_4 - x) = \int_0^1 d\lambda \frac{d\alpha}{d\lambda} \partial_\mu \partial^{\mu} \delta(z - x),
\]

where the path \( z(\lambda) \) runs from \( x_1 \) to \( x_4 \). Interchanging the order of differentiation with respect to \( x \) and the various integrals leaves the r.h.s. of (14) in the form of a total divergence from which the the nonlocal contribution to the symmetry current can be identified. If we choose, for convenience, straight line paths such as

\[
z(\lambda) = (1 - \lambda)x_1 + \lambda x_4,
\]

then the nonlocal current is

\[
j_{5[\mu]}^{\nu a} = \int d^4x_1 \ldots d^4x_4 \int_0^1 d\lambda \alpha(x_1, x_2, x_3, x_4)
\]

\[
\times \left\{ \bar{\psi}(x_1) \tau_a i\gamma^5 \psi(x_3) \bar{\psi}(x_2) \psi(x_4) \left[ (x_4^\nu - x_1^\nu) \delta(\lambda x_1 + (1 - \lambda)x_4 - x) - (x_3^\mu - x_2^\mu) \delta(\lambda x_2 + (1 - \lambda)x_3 - x) \right] \right.
\]

\[
- i\epsilon^{abc} \bar{\psi}(x_1) \tau_c \psi(x_3) \bar{\psi}(x_2) \tau_b i\gamma^5 \psi(x_4) (x_3^\mu - x_1^\mu) \delta(\lambda x_1 + (1 - \lambda)x_3 - x) \right\}.
\]

The full current is the sum of this and the local piece (12).
Note that none of these procedures determines the current uniquely. The ambiguity shows up most clearly in the arbitrary choice of path for the line integrals but it affects all methods for constructing a conserved current from a nonlocal action. It arises because the requirement that the divergence of the current vanish only fixes the longitudinal part of the current; the transverse part remains undetermined. In the derivative expansion of [4] it corresponds to the fact that one can always add to the current combinations of field derivatives whose divergence vanishes by construction. This problem is well known in nuclear physics: the longitudinal components of exchange currents can be related to phenomenological nucleon-nucleon forces, while the transverse currents require a specific model for the underlying meson exchanges.\textsuperscript{3} For our present purpose, calculation of $F_\pi$, only the longitudinal current is needed and so our results are not affected by this ambiguity.

The pion decay constant, $-iF_\pi p^\mu = \langle 0 | j_5^\mu (0) | \pi^- (p) \rangle$, can be found by extracting the pion pole from the correlator of two axial currents. This correlator can be written

\[ J^\mu_{PA}(p) \frac{G}{1 - GJ_{PP}(p)} J^\rho_{PA}(p) = \frac{4F_\pi^2 p^\mu p^\rho}{p^2 + m_\pi^2} + \cdots , \]

(18)

where $J_{PP}(p)$ is given by the quark loop described in the previous section and $J^\mu_{PA}(p)$ is given by a similar loop with one pseudoscalar and one axial current insertion. Equating the residues at the poles on both sides, one finds

\[ 4F_\pi^2 p^\mu p^\rho = Z_\pi J^\mu_{PA}(p) J^\rho_{PA}(p) , \]

(19)

where $p^2 = -m_\pi^2$.

The contribution of the local piece of the current to $J^\mu_{PA}(p)$ is straightforward and has the form:

\[ J^\mu_{PA\{loc\}}(p) = \text{tr} \int \frac{d^4k}{(2\pi)^4} \left[ \gamma^\mu \gamma^5 \frac{1}{\slashed{k} + \frac{1}{2} \slashed{p} + M(k + \frac{1}{2}p)} \gamma^5 \frac{f(k + \frac{1}{2}p) f(k - \frac{1}{2}p)}{\slashed{k} - \frac{1}{2} \slashed{p} + M(k - \frac{1}{2}p)} \right] . \]

(20)

\textsuperscript{3}See, for example, [12].
The nonlocal contribution is more involved and so we do not present the details of its evaluation here. Its main features can be seen from the form of the the nonlocal current \( (17) \) in momentum space,

\[
\mathcal{J}_5^{\mu}(p) = G \frac{p^\mu}{p^2} \int \frac{d^4k_1 \cdots d^4k_4}{(2\pi)^8} \overline{\psi}(k_1) i\gamma_5 \tau^a \psi(k_3) \overline{\psi}(k_2) \psi(k_4) \delta(k_1 + k_2 - k_3 - k_4 + p)
\]

\[
\times \left[ f(k_1 + p)f(k_2)f(k_3)f(k_4) - f(k_1)f(k_2 + p)f(k_3)f(k_4) + f(k_1)f(k_2)f(k_3 + p)f(k_4) - f(k_1 + p)f(k_2)f(k_3)f(k_4 + p) \right] + \cdots,
\]

where the terms involving commutators of isospin matrices have not been written out since they do not contribute to \( F_\pi \). Each of the terms shown in Eq. (21) involves the product of pseudoscalar isovector and scalar isoscalar densities. The vacuum to one-pion matrix element of the nonlocal current consists of similar terms, each of which is a product of two integrals: one similar in its Dirac structure to \( J_{PP} \), the other like the self-energy in Eq. (3).

After a certain amount of effort, the total \( J_{PA}^\mu(p) \) can be expressed in terms of the same integral as in \( Z_\pi^{-1} \) above, and so can be written as

\[
J_{PA}^\mu(p) = -2ip^\mu M_0(0)Z_\pi^{-1}
\]

to lowest order in \( m_\pi \). Using this in Eq. (19), we see that \( F_\pi \) can be related to the wave function renormalisation by

\[
F_\pi^2 = M_0(0)^2 Z_\pi^{-1}.
\]

The GOR relation follows at once from this and Eq. (10) for the pion mass:

\[
F_\pi^2 m_\pi^2 = -m_\pi \langle \overline{\psi} \gamma^5 \psi \rangle.
\]

**V. RESULTS AND DISCUSSION**

We fix the current quark mass \( m_\pi \) and one combination of \( G \) and \( \Lambda \) to give a pion mass of 140 MeV and a pion decay constant of 93 MeV. Results for the dynamical quark mass
and the quark condensate are shown in the table. The variation of these quantities with the cut-off $\Lambda$ is qualitatively similar to that obtained using the local NJL model with the usual regularisation schemes [2,3].

Typically the dynamical quark mass in quark models is taken to be in the region 300–400 MeV. The average current mass of the up and down quarks is believed to be about 5–10 MeV at the momentum scale relevant to hadronic models [13]. From the GOR relation, this corresponds to a quark condensate in the range $\langle \bar{q}q \rangle \equiv \frac{1}{2} \langle \bar{\psi} \psi \rangle \simeq -(210 \text{ MeV})^3$ to $-(260 \text{ MeV})^3$. From the table we see that parameter sets with cut-offs of around 1 GeV yield reasonable results for quark masses and the condensate.

For large enough values of the dynamical mass, the quark propagator ceases to have poles at real $p^2$ in Minkowski space, and quarks become confined. In the chiral limit, this happens for

$$\frac{M_0(0)}{\Lambda} > e^{-\frac{1}{2}}.$$

(25)

Neglecting the effect of the small current quark mass, we see that our quarks are confined for those parameter sets which give $M_0(0)$ above about 300 MeV. The critical point for confinement is thus in the middle of the region which yields reasonable values for the condensate.

Other related models have been studied by various authors. In particular Diakonov and Petrov [5] have used a nonlocal interaction with a separable form suggested by an instanton approach to the QCD vacuum. The expression for $F_\pi$ given by Eqs. (23, 11) agrees with theirs [6]. However those authors never explicitly calculate $F_\pi$ from the conserved axial current. Instead they take the pion wave function renormalisation, and assuming that PCAC holds, determine $F_\pi$ from it using Eq. (23) (the equivalent is Eq. (30) of their paper). When they attempt to calculate $F_\pi$ using the local piece of the current alone, they find an expression which was originally obtained by Pagels and Stokar [14]. The incompleteness of this expression of Pagels and Stokar and the fact that the full symmetry current requires a nonlocal contribution have also been pointed out by Holdom and coworkers in the context of a constituent quark model with a nonlocal self-energy [15]. In fact the model of [15] has
many features in common with the one we study here, although it is not derived from an underlying quark-quark interaction. Ball and Ripka [4] discuss in detail the construction of conserved currents in such models.

Buballa and Krewald [16] have also studied a model of the form (1), with a different choice of form factor. They too find quark confinement but their form factor is chosen to avoid the appearance of complex poles in the quark propagator. The propagator in their model does however contain some unusual analytic properties, in the form of additional cuts arising from the square roots in the form factor. The appearance of extra singularities in the quark propagator seems unavoidable in models for confinement based on Schwinger-Dyson equations [8,7]. Moreover the model of Ref. [16] still requires an explicit cut-off since the form factor fails to regulate the momentum integrals in loops.

Another related model has been studied by the Rostock group [17]. It is based on an instantaneous interaction with a separable dependence on the three-momenta. That model is of course not covariant and so, for example, there will be different pion decay constants associated with the space and time components of the axial current. Schmidt et al. have calculated $F_\pi$ from the time component; since their interaction is instantaneous, that component of the current is purely local and the extra terms described here do not appear.

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\begin{table}
\begin{center}
\begin{tabular}{ccccc}
\hline
$M_0(0)$ & $\Lambda$ & $-\langle \bar{q}q \rangle_0^1$ & $m_c$ \\
\hline
200 & 2222 & 330 & 2.4 \\
250 & 1414 & 257 & 5.0 \\
300 & 1075 & 222 & 7.7 \\
350 & 887 & 200 & 10.5 \\
400 & 768 & 185 & 13.3 \\
450 & 684 & 173 & 16.2 \\
500 & 623 & 165 & 18.9 \\
\hline
\end{tabular}
\end{center}
\end{table}

\textbf{TABLE I.} Results for dynamical quark mass and quark condensate, with $F_\pi = 93$ MeV and $m_\pi = 140$ MeV. All quantities are in MeV.