Self-dual solutions of 2+1 Einstein gravity with a negative cosmological constant

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Abstract

All the causally regular geometries obtained from (2+1)-anti-de Sitter space by identifications by isometries of the form $P \rightarrow (\exp 2\pi \xi) P$, where $\xi$ is a self-dual Killing vector of $so(2, 2)$, are explicitly constructed. Their remarkable symmetry properties (Killing vectors, Killing spinors) are listed. These solutions of Einstein gravity with negative cosmological constant are also shown to be invariant under the string duality transformation applied to the angular translational symmetry $\phi \rightarrow \phi + a$. The analysis is made particularly convenient through the construction of global coordinates adapted to the identifications.

1 Introduction.

It has been discovered recently [1] that 2+1 Einstein gravity with a negative cosmological constant, even though devoid of local degrees of freedom, allows for black hole solutions with features quite similar to those occurring in the standard four-dimensional theory. The geometry of these black hole solutions have been investigated in depth in [2], and the remarkable supersymmetry properties of the extreme black hole with mass equal to angular momentum, as well as of the massless black hole ground state, have been studied in [3]. Further analysis of the 2+1 black hole may be found in [4].

The talk presented by one of us (M.H.) at the Santiago meeting was devoted to a survey of these results. Since these can be found in the existing literature, we shall not repeat them here. Rather, we shall analyse a question that has come across in the study of the geometry of the (2+1)-black hole [2], namely, that of determining all “self-dual” solutions of Einstein gravity with a negative cosmological constant. [In what sense these solutions are self-dual is defined precisely in the next section]. Although these solutions do not belong to the black hole family, they exhibit quite interesting geometric features that make them worth of study. Namely, they have four Killing vectors, and possess in addition two Killing spinors. Furthermore, they are invariant under the string duality transformations applied to the angular translational symmetry $\phi \rightarrow \phi + a$. 
2 Self-dual metrics: definition.

As shown in [2], the (2+1)-black hole may be obtained from anti-de Sitter space by identifications. These identifications are made as follows. Any Killing vectors $\xi$ of the anti-de Sitter metric defines a one-parameter subgroup of isometries:

$$P \rightarrow \exp(t\xi)P$$ (1)

We shall consider the case where the orbits (1) are isomorphic with the real line. The mappings (1) for which $t$ is an integer multiple of a “basic” step, taken conventionally as $2\pi$,

$$P \rightarrow \exp(t\xi)P, \quad t = 0, \pm 2\pi, \pm 4\pi, ...$$ (2)

define a discrete subgroup isomorphic to $Z$.

Since the transformations (2) are isometries, the quotient space obtained by identifying points that belong to a given orbit, inherits from anti-de Sitter space a well-defined metric which has the same constant negative curvature and which is thus also a solution of the Einstein equations. The identification process leads therefore to new solutions of the Einstein equations which differ from anti-de Sitter space in their global properties. These solutions are completely characterized by the group (2), which we shall call the identification group. The identification group, in turn, is completely characterized by the Killing vector $\xi$ that generates it.

The Killing vector $\xi$ belongs to the Lie algebra $so(2, 2)$. Two Killing vectors $\xi$ and $\xi'$ in the same $SO(2, 2)$ conjugacy class yield isomorphic quotient spaces. Thus, the quotient spaces may be classified by the conjugacy classes of $SO(2, 2)$ in $so(2, 2)$.

The Lie algebra $so(2, 2)$ is the direct sum of two copies of $sl(2, \mathbb{R})$,

$$so(2, 2) = sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$$ (3)

Let $J_{ab}$ be the Killing vectors of anti-de Sitter metric. In terms of the embedding

$$-u^2 - v^2 + x^2 + y^2 = -l^2$$ (4)
of anti-de Sitter space in a four-dimensional flat space of signature
\[ ds^2 = -du^2 - dv^2 + dx^2 + dy^2, \]  
one has
\[ J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b} \]
with \( x^a \equiv (u, v, x, y) \).

The non-vanishing brackets of the \( so(2, 2) \)-algebra read explicitly:
\[
\begin{align*}
[J_{01}, J_{02}] &= -J_{12} & [J_{02}, J_{03}] &= -J_{01} \\
[J_{01}, J_{03}] &= -J_{13} & [J_{02}, J_{12}] &= J_{01} \\
[J_{01}, J_{12}] &= J_{02} & [J_{02}, J_{23}] &= -J_{03} \\
[J_{01}, J_{13}] &= J_{03} & [J_{03}, J_{13}] &= J_{01} \\
[J_{03}, J_{23}] &= J_{02} & [J_{12}, J_{13}] &= -J_{23} \\
[J_{12}, J_{23}] &= -J_{13} & [J_{13}, J_{23}] &= J_{12}
\end{align*}
\]

One may take as generators of the first copy \( sl(2, \mathbb{R}) \) (”self-dual” generators)
\[
\begin{align*}
\xi^{(0)} &= \frac{1}{2}(J_{01} + J_{23}) \\
\xi^{(1)} &= \frac{1}{2}(J_{02} + J_{13}) \\
\xi^{(2)} &= \frac{1}{2}(J_{03} - J_{12})
\end{align*}
\]

and the generators of the second copy \( sl(2, \mathbb{R}) \) (”anti-self dual” generators)
\[
\begin{align*}
\eta^{(0)} &= \frac{1}{2}(J_{01} - J_{23}) \\
\eta^{(1)} &= \frac{1}{2}(J_{02} - J_{13}) \\
\eta^{(2)} &= -\frac{1}{2}(J_{03} + J_{12})
\end{align*}
\]

The commutation relations between the \( \xi \)'s and the \( \eta \)'s are:
\[
\begin{align*}
[\xi^{(0)}, \xi^{(1)}] &= \xi^{(2)}, & [\xi^{(0)}, \xi^{(2)}] &= -\xi^{(1)}, & [\xi^{(1)}, \xi^{(2)}] &= -\xi^{(0)} \\
[\eta^{(0)}, \eta^{(1)}] &= \eta^{(2)}, & [\eta^{(0)}, \eta^{(2)}] &= -\eta^{(1)}, & [\eta^{(1)}, \eta^{(2)}] &= -\eta^{(0)}
\end{align*}
\]
Of course, one has also:

$$[\xi_\alpha(\beta), \eta_\nu] = 0. \quad (12)$$

As shown in [2], the Killing vectors used in making the identifications for the black hole solutions are neither self-dual nor anti-self-dual (for any value of the mass and angular momentum). Nevertheless, identifications by self-dual (or anti-self-dual) Killing vectors lead to interesting geometries, which we shall call self-dual (respectively anti-self-dual).

Since the anti-self-dual case can be obtained from the self-dual one by the parity transformation $x^3 \rightarrow -x^3$, which is an isometry, it is enough to consider the self-dual case.

There are clearly three subcases to be considered, since there are only three inequivalent types of elements in $\mathfrak{sl}(2, \mathbb{R})$: those that are spacelike (type A), those are timelike (type B) and those that are lightlike (type C) (with the Killing-Lorentz metric of the $\mathfrak{sl}(2, \mathbb{R})$ algebra). By redefinitions, one may always assume:

Type A: $\xi = a \xi_{(1)}$

Type B: $\xi = a \xi_{(0)}$

Type C: $\xi = \xi_{(0)} + \xi_{(1)}$

The norms of the Killing vectors (as tangent vectors to the anti-de Sitter space, with the anti-de Sitter metric) are given by:

Type A: $\xi \cdot \xi = \frac{a^2}{4} r^2$

Type B: $\xi \cdot \xi = -\frac{a^2}{4} r^2$

Type C: $\xi \cdot \xi = 0$

and are also constant. Thus, the orbits of the Killing vectors are geodesic. This is quite remarkable since the anti-de Sitter norm of a generic Killing vector is in general position dependent and grows as $r$ as one recedes from the origin.

For comparison with the classification of the $so(2, 2)$ elements given in the appendix A of [2], we observe that type A corresponds to type $I_b$ with $\lambda_2 = -\lambda_1 = a/2$, type B corresponds to type $I_c$ with $b_1 = b_2 = a/2$ and type C corresponds to type $II_b$ with $b = 0$. [Note that there is a misprint in table I of the appendix of [2], where the Killing vector.
for type $II_b$ should read $(b - 1)J_{01} + (b + 1)J_{23} + J_{02} - J_{13}$ instead of the incorrect $(b - 1)J_{01} + (b - 1)J_{23} + J_{02} - J_{13}$. 2

\section{(2+1)-anti-de Sitter space as a group manifold.}

One may understand the above properties of the Killing vectors by recalling that (2+1)-anti-de Sitter space is the group manifold of $SL(2,\mathbb{R})$. This is most easily seen by defining new variables

$$
\alpha = \frac{u + x}{l}, \quad \delta = \frac{u - x}{l}, \quad \beta = \frac{v - y}{l}, \quad \gamma = \frac{-v + y}{l}
$$

in terms of which the equation $u^2 + v^2 - x^2 - y^2 = l^2$ becomes the unit determinant condition

$$
\alpha \delta - \beta \gamma = 1
$$

for the matrix

$$
A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
$$

Furthermore, one verifies straightforwardly that the anti-de Sitter metric coincides with the Killing metric $l^2 tr(A^{-1} dA)^2$ on the group $SL(2,\mathbb{R})$.

Any group manifold $G$ (with the Killing metric) is invariant under the isometry group $G_L \times G_R$, where $G_L$ and $G_R$ are respectively the groups of left and right translations (of course, $G_L$, $G_R$ are isomorphic to $G$). The isometry group is a direct product because left and right translations commute. In addition, the infinitesimal generators of $G_L$ (respectively $G_R$) have constant norm because, as the Killing metric, they are invariant under right (respectively left) translations, which are transitive on $G$. In our case, $(SL(2,\mathbb{R}))_L$ is generated by the self-dual Killing vectors, while $(SL(2,\mathbb{R}))_R$ is generated by the anti-self-dual Killing vectors.

\footnote{There is another misprint on page 1521 of the same paper, where the Casimir invariants $I_1$ and $I_2$ are incorrectly expressed in terms of $\omega^a_{ab}$ and $\omega^a_{ab}$. The correct expression are $I_1 + I_2 = \omega^a_{ab} \omega^{+ab}$ and $I_1 - I_2 = \omega^a_{ab} \omega^{-ab}$.}
4 Self dual metrics: construction.

In order to construct the self-dual metrics of type $A$, we shall introduce a *global* parametrization of anti-de Sitter space adapted to the Killing vector used in making the identification. More precisely, we shall introduce new coordinates covering the full anti-de Sitter manifold, in which the Killing vector $a\xi_{(1)}$ is just $\frac{\partial}{\partial \phi}$.

We shall actually do more, namely, we shall also arrange so that the anti-self-dual Killing vector $2\eta_{(0)}$ is $\frac{\partial}{\partial t}$. [We choose $2\eta_{(0)}$ and not any other Killing vector for two reasons: (i) $\eta_{(0)}$ commutes with $\xi_{(1)}$; (ii) it is everywhere linearly independent from $\xi_{(1)}$ on anti-de Sitter space. The factor 2 is a matter of normalization convention]. To simplify the analysis, we start with the case $a = 2$. We consider next the case of an arbitrary $a$.

Along the orbit of $-2\xi_{(1)}$, one has

$$\begin{align*}
\frac{dv}{d\phi} &= y, \quad \frac{dy}{d\phi} = v, \quad \frac{du}{d\phi} = x, \quad \frac{dx}{d\phi} = u
\end{align*}$$

(16)

where $\phi$ is a parameter that distinguishes the points on the same orbit. Similarly, along the orbit of $2\eta_{(0)}$, one has

$$\begin{align*}
\frac{dv}{dt} &= u, \quad \frac{du}{dt} = -v, \quad \frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x
\end{align*}$$

(17)

We thus introduce the following parametrization of adS

$$\begin{align*}
u &= l(\cosh r \cosh \phi \cos t + \sinh r \sinh \phi \sin t) \\
v &= l(\cosh r \cosh \phi \sin t - \sinh r \sinh \phi \cos t) \\
x &= l(\cosh r \sinh \phi \cos t + \sinh r \cosh \phi \sin t) \\
y &= l(\cosh r \sinh \phi \sin t - \sinh r \cosh \phi \cos t)
\end{align*}$$

(18)

or equivalently, in terms of the light-like variables $\alpha = \frac{u+v}{\sqrt{2}}$, $\delta = \frac{v+u}{\sqrt{2}}$, $\beta = \frac{v-u}{\sqrt{2}}$, $\gamma = -\frac{v-u}{\sqrt{2}}$,

$$\begin{align*}
\alpha &= e^{\phi}(\cosh r \cos t + \sinh r \sin t) \\
\beta &= e^{-\phi}(\cosh r \sin t + \sinh r \cos t) \\
\gamma &= -e^{\phi}(\cosh r \sin t - \sinh r \cos t) \\
\delta &= e^{-\phi}(\cosh r \cos t - \sinh r \sin t)
\end{align*}$$

(19)

That this is a good parametrization of anti-de Sitter space can be seen as follows. First, any $(\alpha, \beta, \gamma, \delta)$ given by (19) clearly solves the
equation $\alpha \delta - \beta \gamma = 1$. Conversely, let $(\alpha, \beta, \gamma, \delta)$ be a solution of $\alpha \delta - \beta \gamma = 1$. Define $\phi$ and $r$ through

$$e^{4\phi} = \frac{\alpha^2 + \gamma^2}{\beta^2 + \delta^2}$$

and

$$\sinh 2r = \alpha \beta + \gamma \delta$$

Because $\alpha \delta - \beta \gamma = 1$, this last relation implies

$$\cosh^2 2r = (\alpha^2 + \gamma^2)(\beta^2 + \delta^2).$$

It remains to determine $t$. Since $(\alpha, \gamma)$ belongs to the circle of radius $e^{\frac{1}{2}}2r (\alpha^2 + \gamma^2 = e^{2\phi} \cosh 2r$ by (20) and (22)), there is a unique $t'$ defined by

$$\alpha = e^\phi (\cosh r \cos t' + \sinh r \sin t')$$
$$\gamma = -e^\phi (\cosh r \sin t' - \sinh r \cos t')$$

Similarly, there is a unique $t''$ such that

$$\beta = e^{-\delta} (\cosh r \sin t'' + \sinh r \cos t'')$$
$$\gamma = e^{-\delta} (\cosh r \cos t'' - \sinh r \sin t'')$$

The relations (21) and $\alpha \delta - \beta \gamma = 1$ imply $\sin(t'' - t') + \cos(t'' - t') \sinh 2r = \sinh 2r$ and $\cos(t'' - t') - \sin(t'' - t') \sinh 2r = 1$, i.e. $t'' = t'$. We define $t$ by $t = t' = t''$, which shows that given $(\alpha, \beta, \gamma, \delta)$ solution of $\alpha \delta - \beta \gamma = 1$, there are unique $(t, r, \phi)$ in terms of which $\alpha, \beta, \gamma$ and $\delta$ can be written as in (18).

In the coordinates $(t, r, \phi)$, the anti-de Sitter metric reads

$$ds^2 = -dt^2 + d\phi^2 + 2 \sinh (2r) dt d\phi + d\alpha^2$$

and the Killing vectors $2\xi_{(1)}$ and $2\eta_{(0)}$ are given by

$$2\xi_{(1)} = \frac{\partial}{\partial \phi}, 2\eta_{(0)} = \frac{\partial}{\partial t}$$

The coordinates $\phi$ and $r$ range over the entire line. This is also true for $t$ if one passes to the universal covering of $\text{adS}$, as we shall do from now on,

$$-\infty < t < +\infty, \hspace{0.5cm} -\infty < \phi < +\infty, \hspace{0.5cm} -\infty < r < +\infty$$
If instead of $2\xi_{(1)}$, one takes as identification vector $a\xi_{(2)}$, one needs to make the rescaling $\phi \rightarrow \frac{a}{2}\phi$, in terms of which the metric becomes

$$ds^2 = \tilde{\ell}(-dt^2 + \frac{a^2}{4}d\phi^2 + a \sinh 2r dt d\phi + dr^2)$$

(28)

and $a\xi_{(1)} = \frac{\partial}{\partial t}$, $2\eta_{(0)} = \frac{\partial}{\partial r}$.

The self-dual metric is obtained by making $\phi$ periodic. Thus, it is also given by (28),

$$ds^2 = \tilde{\ell}(-dt^2 + \frac{a^2}{4}d\phi^2 - a \sinh 2r dt d\phi + dr^2)$$

(29)

but now, $\phi$ is an angle, $-\infty < t < +\infty$, $0 < \phi < 2\pi$, $-\infty < r < +\infty$.

From now on, we shall set $l = 1$. Note that the self-dual metrics of type $B$ are obtained by keeping $\phi$ unperiodic and by making $t$ periodic. Similarly, it is easy to verify that the self-dual metric of type $C$ is given by

$$ds^2 = 2r^2 dt d\phi + \frac{1}{r^2} dr^2$$

(30)

For both types $B$ and $C$, there are closed causal curves. This is not the case for type $A$, as we now show.

5 Absence of closed causal curves.

The self-dual metric (28) is causally regular in the sense that there is no closed causal curves. This is easy to verify because the coordinates $(t, \phi, r)$ provide a global covering of the manifold. A closed causal curve must fulfill

$$\left(\frac{dt}{d\lambda}\right)^2 - \frac{a^2}{4} \left(\frac{d\phi}{d\lambda}\right)^2 - \frac{a}{2} \sinh 2r \frac{dt}{d\lambda} \frac{d\phi}{d\lambda} - \left(\frac{dr}{d\lambda}\right)^2 \geq 0$$

(31)

If it closed, $t$ must come back to its original value. Thus, there must be a point on the curve where $\frac{dt}{d\lambda} = 0$. But this contradicts the above inequality (and the fact that the tangent vector is never zero since $\lambda$ is a good parametrization of the curve).

The self-dual space is also geodesically complete (as is anti-de Sitter) and singularity-free. Hence, it is a perfectly acceptable solution of the Einstein equation. This stationary, axially symmetric solution was missed in [1, 2] because it was assumed from the very beginning that the rotational Killing vector $\frac{\partial}{\partial \phi}$ had non constant norm.
6 Holonomies

The self-dual metric may also be described in terms of Chern-Simons holonomies. It is well known that (2+1)-gravity with a negative cosmological constant is equivalent to the Chern-Simons theory with the $SO(2, 2)$ gauge group [5]. The self-dual solution (28) corresponds to the following $SO(2, 2)$ flat connection,

$$A = a[\cosh r(J_{02} + J_{13}) - \sinh r(J_{23} + J_{01})]d\phi$$
$$[-\sinh r(J_{13} - J_{02}) + \cosh r(J_{01} - J_{23})]dt$$
$$-J_{03}dr$$

(32)

which, in turn, can be transformed under the $SO(2, 2)$ gauge-transformation

$$U = \exp -t(J_{01} - J_{23}) \circ \exp rJ_{03}$$

to the $SO(2, 2)$ flat connection

$$A = a(J_{02} + J_{13})$$

(33)

for which the holonomy is manifestly equal to

$$\exp \oint [a(J_{02} + J_{13})d\phi] = \exp 2\pi a(J_{02} + J_{13})$$

(34)

7 Killing vectors.

The self-dual metric has by construction four Killing vectors, namely $\xi_{(1)} = \frac{\partial}{\partial \phi}$ belonging to the self-dual $sl(2, \mathbb{R})$, and the tree vectors $\eta_{(0)}, \eta_{(1)}, \eta_{(2)}$ belonging to the anti-self-dual $sl(2, \mathbb{R})$ and commuting therefore with $\xi_{(2)}$. In the coordinates $(t, \phi, r)$, these vectors are given explicitly by

$$2\xi_{(1)} = \frac{\partial}{\partial \phi}$$

(35)

$$2\eta_{(0)} = \frac{\partial}{\partial t}$$

(36)

$$\eta_{(1)} = \frac{1}{2} \tanh 2r \cos 2t \frac{\partial}{\partial t} + \frac{\cos 2t}{2 \cosh 2r} \frac{\partial}{\partial \phi} + \frac{1}{2} \sin 2t \frac{\partial}{\partial r}$$

(37)

$$\eta_{(2)} = -\frac{1}{2} \tanh 2r \sin 2t \frac{\partial}{\partial t} - \frac{\sin 2t}{2 \cosh 2r} \frac{\partial}{\partial \phi} + \frac{1}{2} \cos 2t \frac{\partial}{\partial r}$$

(38)
One may ask whether there are any other independent Killing vectors besides (35)-(36). It is easy to see that the answer to this question is negative. Indeed, as shown in [2], the problem amounts to determining all $SO(2, 2)$ Killing vectors $\eta$ of anti-de Sitter space that commute with the $SO(2, 2)$ matrix $\exp 2\pi \xi_{(1)}$,

$$\left[ \exp 2\pi \xi_{(1)}, \eta \right] = 0$$

(39)

Now, the solutions of (39) form a subalgebra of $so(2, 2)$, which contains $\mathbb{R} \oplus sl(2, \mathbb{R})$, where $\mathbb{R}$ is the subalgebra of the "self-dual" $so(2, 1)$ generated by $\xi_{(1)}$. There are only two subalgebras containing $\mathbb{R} \oplus sl(2, \mathbb{R})$, namely, $\mathbb{R} \oplus sl(2, \mathbb{R})$ itself or the full $so(2, 2)$. Since this latter case is excluded (there exist elements of $so(2, 2)$ that are not invariant by $\exp 2\pi \xi_{(1)}$), we conclude that the isometry algebra of the quotient space is $\mathbb{R} \oplus sl(2, \mathbb{R})$. There are thus only four independent Killing vectors.

8 Killing spinors.

The metric (25) can be viewed not just as a solution of Einstein equation, but also as a solution of adS supergravity with zero gravitini. As such, it possesses exact supersymmetries.

Exact supersymmetries are by definition supersymmetry transformations leaving the metric (25) (with zero gravitini) invariant. The spinor parameters of these transformations solve the "Killing spinor equation"

$$D_{\lambda} \psi = \frac{\epsilon}{2i} \gamma_{\lambda} \psi,$$

(40)

where $\epsilon = 1$ or $-1$ depending on the representation of the $\gamma$ matrices.

As is well known, there are two inequivalent two-dimensional irreducible representations of the $\gamma$ matrices in three spacetime dimensions. One may be taken to be $\gamma(0) = i\sigma^2, \gamma^{(1)} = \sigma^1$ and $\gamma^{(2)} = \sigma^3$, where the $\sigma^k$ are the Pauli matrices. The other is given by $\gamma^{(\lambda)} = -\gamma^{(\lambda)}$. We shall consider here the simplest supergravity model with negative cosmological constant involving both representations, namely, $(1, 1)$ adS supergravity [6].

The anti-de Sitter metric $ds_{ads}^2 = -dt^2 + d\phi^2 + 2 \sinh 2rdtd\phi + dr^2$ ($-\infty < \phi < +\infty$) possesses four Killing spinors, two for each
inequivalent representation of the $\gamma$ matrices. In the tetrad frame

$$
\begin{align*}
  h_{(0)} &= \cosh rdt - \sinh r d\phi \\
  h_{(1)} &= \sinh rdt + \cosh r d\phi \\
  h_{(2)} &= dr
\end{align*}
$$

the Killing spinors are given by

$$
\psi = \left[ ((\epsilon - 1)(A \cosh \phi + B \sinh \phi) e^{-r/2} + (\epsilon + 1) \sin t + K e^{r/2} \right] \\
\times (1 + \gamma^{(2)}) \\
+ \left[ ((\epsilon - 1)(A \sinh \phi + B \cosh \phi) e^{r/2} + (\epsilon + 1) \cos t + K e^{-r/2} \right] \\
\times (1 - \gamma^{(2)}) E
$$

where $E$ is a constant spinor and $A, B, K$ are constant.

Since the self-dual metric can be obtained from $ds_{ads}^2$ by making appropriate identifications, it possesses locally as many Killing spinors as anti-de Sitter space. However, only a subset of these Killing spinors are, in general, compatible with the identifications, i.e., invariant under the transformations of the discrete group used in the identifications. So, whereas all the local integrability conditions for the Killing equations are fulfilled [3], there may be no Killing spinor at all because of global reasons.

Since the self-dual metric is obtained by making the identifications $\phi \sim \phi + a \pi$, the only Killing spinors of the self-dual metric are those that are periodic or anti-periodic in $\phi$. By inspection of (42), one finds that these are given by:

$$
[\sin(t + K)e^{r/2}(1 + \gamma^{(2)}) + \cos(t + K)e^{-r/2}(1 - \gamma^{(2)})]E
$$

Thus, whereas anti-de Sitter space has four Killing spinors, the self-dual geometry has only two Killing spinors (two for one representation of the $\gamma$ matrices and zero for the other).
9 String duality.

The self-dual geometry (25) may also be viewed as a solution of the low energy string equations in 2+1 dimensions [7]

\[ S = f(-g)^{\frac{1}{2}} e^{-\phi} \left[ \frac{R}{2} + 4(\nabla \phi)^2 - \frac{1}{l^2} H_{\mu \nu \rho \tau} H^{\mu \nu \rho \tau} \right] \]
\[ R_{\mu \nu} + 2 \nabla_{\mu} \nabla_{\nu} \phi - \frac{1}{4} H_{\mu \lambda \sigma} H^{\lambda \sigma} = 0 \]
\[ \nabla^\nu (e^{-\phi} H_{\mu \nu \rho}) = 0 \]
\[ 4 \nabla^2 \phi - 4(\nabla \phi)^2 + \frac{4}{l^2} + R - \frac{1}{l^2} H^2 = 0 \]

with zero dilaton \( \phi = 0 \) and antisymmetric tensor. \( B_{\lambda \mu} \) given by

\[ B_{\phi t} = \frac{\alpha}{2} \sinh 2r, \quad B_{\phi r} = B_{tr} = 0 \]  \hspace{1cm} (44)

(so that \( H_{\mu \nu \rho} = \frac{2}{3} \epsilon_{\mu \nu \rho} (-g)^{\frac{1}{2}} \neq 0 \)). In (44), \( k \) must be taken equal to \( \ell \).

Thus, part of the cosmological constant arises from the antisymmetric tensors and part of it arises from the term \( \frac{4}{l^2} \).

Given a solution of the low energy string equations with a Killing vector, one may construct by duality another solution of the same equations (see references therein [7, 8]). The duality transformation reads \((g_{\mu \nu}, B_{\mu \nu}, \phi) \rightarrow (g'_{\mu \nu}, B'_{\mu \nu}, \phi')\) with

\[ g'_{xx} = 1/g_{xx}, \quad g'_{\alpha \beta} = g_{\alpha \beta} - (g_{x\alpha} g_{x\beta} - B_{x\alpha} B_{x\beta})/g_{xx}, \]
\[ R'_{xx} = g_{xx}/g_{xx}, \quad B'_{\alpha \beta} = B_{\alpha \beta} + 2 g_{x[\alpha} B_{\beta]x}/g_{xx}, \]
\[ \phi' = \phi - \frac{1}{2} \log |g_{xx}|. \]  \hspace{1cm} (45)

where we assume that the solutions \((g_{\mu \nu}, B_{\mu \nu}, \phi)\) does not depend on \(x\) and where \( \alpha \) and \( \beta \) runs over all coordinates but \(x\).

In [7], the transformation (45) was applied to the \( \phi \) translational symmetry of the black hole solutions to generate "black string solutions". The Killing vector \( \frac{\partial}{\partial \phi} \) is singled out by the fact that it has closed orbits. In the case of the black hole, \( \frac{\partial}{\partial \phi} \) is not self-dual and therefore, does not have a constant norm, \( g_{\phi \phi} \neq \) constant. As a result, the transformation (45) generates a different geometry.

For the self-dual geometries, the Killing vector \( \frac{\partial}{\partial \phi} \) with closed orbits has the quite interesting property of being self-dual and is thus
of constant norm. Accordingly, the transformation (45):

\[
\begin{align*}
g'_{\phi\phi} &= \frac{4}{a^2} \quad g'_{\phi t} = \frac{2}{a} \sinh 2r \quad g'_{\phi r} = 0 \\
g'_{tt} &= -1 \quad g'_{tr} = 0 \quad g'_{rr} = 0 \\
B'_{t\phi} &= \frac{1}{a} \sinh 2r \quad B'_{r\phi} = 0 \quad B'_{rr} = 0 \\
\Phi' &= \Phi - \frac{1}{2} \log |a|
\end{align*}
\]

and simply amounts to replacing \( a \) by \( \frac{a}{2} \). It does not modify the geometry.

We can thus conclude that the self-dual geometries have also remarkable duality properties from the string duality point of view.

\section{Conclusions.}

We have analysed in this paper the self-dual geometries determined from (2+1)-anti de Sitter by identifications generated by a self-dual Killing vector. We have shown that these geometries have quite interesting symmetry properties: they have four Killing vectors and two Killing spinors. They are also invariant under the string duality transformation applied to the angular translational symmetry \( \phi \to \phi + \alpha \).

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\section{References}


