The Euclidean Hopf algebra $U_q(e^N)$ and its fundamental Hilbert space representations

Gaetano Fiore

Sektion Physik der Universität München,
Theresienstrasse 37, D-80333 München, Germany

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Abstract

We construct the Euclidean Hopf algebra $U_q(e^N)$ dual of $Fun_q(e^N)$ by realizing it as a subalgebra of the differential algebra $Diff_q(N)$ on the quantum Euclidean space $e^N$. In fact, we extend our previous realization [11] of $U_q(\mathfrak{so}(N))$ within $Diff_q(N)$ through the introduction of q-derivatives as generators of q-translations. The fundamental Hilbert space representations of $U_q(e^N)$ turn out to be of highest weight type and rather simple “lattice-regularized” versions of the classical ones. The vectors of a basis of the singlet (i.e., zero-spin) irrep can be realized as normalizable functions on $e^N$, going to distributions in the limit $q \to 1$.


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Introduction

One of the most appealing facts explaining the present interest for quantum groups [5] is perhaps the idea that they can be used to generalize the ordinary notion of space(time) symmetry. This generalization is tightly coupled to a radical modification of the ordinary notion of space(time) itself. From this viewpoint inhomogeneous group symmetries such as Poincaré’s and the Euclidean one yield physically relevant candidates for quantum group generalizations: Minkowski space $M^4$ and Euclidean $\mathbb{R}^N$ one are then the corresponding space(time) manifolds.

A major physical motivations for such generalizations is the desire to discretize space(time) (or momentum space) in a “wise” way for QFT regularization purposes. As known, standard lattices used in regularizing QFT do not carry representations of discretized versions (in the form of discrete subgroups) of the associated inhomogeneous groups; actually, the notion of a group is too tight for this scope. For instance, the Euclidean cubic lattice is invariant only under a discretized version of the translation subgroup of the Euclidean group, but not of the rotation one. On the contrary, the notion of symmetry provided by quantum groups is broad enough to allow the existence of lattices whose points are mapped into each other under the action of inhomogeneous $q$-groups, as we will explicitly see in the case of the Euclidean symmetry.

One way to develop the inhomogeneous quantum-group program is to introduce a pair consisting of a “homogeneous” (e.g. Lorentz’s, resp. $SO(N)$) quantum group and the associated quantum space [7,20] comodule algebra; the quantum group coacts homogeneously on the quantum space. The inhomogeneous quantum group [26] can be constructed by doing the braided semi-direct product [17] of the homogeneous one with the corresponding quantum space itself, thought as braid group [16] of translations. The latter is the proper generalization of the ordinary group of translations.

This approach is probably closest to most physicists’ way of thinking of and dealing with space(time) and its inhomogeneous symmetry; it turns out to be a rather viable one for $q$-deformations, too. Here we adopt it in considering the $N$-dimensional ($N \geq 3$) Euclidean space $\mathbb{R}^N$ introduced in [7] and its $\mathbb{R}^{N \times SO(N)}$ symmetry [16,26].

In absence of deformations, the generators of Euclidean Lie algebra $e^N$ can be realized as differential operators acting on $Fun(^N_\phi)$, more precisely the generators of $so(N)$-rotations can be realized as the angular momentum components and the generators of translations as pure derivatives respectively. In other words the algebra $Fun(^N_\phi)$ of functions on $\mathbb{R}^N$ is the base space of (reducible) representations of $so(N)$, of the abelian group $N$ of translations, and of $e^N = \mathbb{R}^{N \times so(N)}$.

Since the structure of $Fun(SO_\phi(N))$ is intimately related [7] to the structures of $Fun(^N_\phi)$ and of the $SO_\phi(N)$-covariant differential calculus [2] which can be built on it, it is interesting to ask whether analogues of these facts occur in the $q$-deformed case; in proper language, whether $Fun(^N_\phi)$ can be considered as a left (or right) module of the universal enveloping algebra $U_\phi(so(N))$, the latter being realized as some subalgebra $U_\phi^N$ of the algebra of differential operators $Diff(^N_\phi)$ on $\mathbb{R}^N$, and of some suitable $q$-deformed version of the abelian algebra of infinitesimal translation; altogether, whether $Fun(^N_\phi)$ can be considered as a left (or right) module of some suitable $q$-deformed version of the euclidean algebra of u.e.a. type.

In this work we give positive answers to these questions. Actually, we construct (chapter 2) a $q$-deformed version $U_\phi(e^N)$ of the Euclidean algebra of u.e.a. type by requiring that the previous conditions are satisfied. Moreover, we study the fundamental Hilbert space representations of $U_\phi(e^N)$ (chapter 3). Chapter 1 provides the reader with the necessary preliminaries, so that the whole paper is basically self-contained. For more details see Ref. [12].
The construction is based on our work in Ref. [11], of which this is the natural development. There we realized the Hopf algebra $U_{q^{-1}}(so(N))$ as a subalgebra of $Diff_q^{N}$ of $Diff_q^{N}$; more precisely, we found $Diff_q^{N}$-realizations of the Drinfeld-Jimbo [5] generators of $U_{q^{-1}}(so(N))$ and showed how the Hopf structure of $U_{q^{-1}}(so(N))$ can be derived in a natural way from the simple commutation relations between these generators and the coordinates $x^i$ of $N$.

Legitimated by these positive results, in the present work we extend $U_{q^{-1}}$ into a new (closed) algebra $u_{q}(e^N)$ by adding to its Drinfeld-Jimbo generators the $q$-derivatives $\partial^i$ as generators of translations (section 2.1). To endow $u_{q}(e^N)$ with a Hopf structure we proceed as in the case of $U_{q}$; we realize that we need to enlarge $u_{q}(e^N)$ further by introducing one more generator $A \in Diff_q^{N}$, generating dilatations. We call $U_{q}(e^N)$ the resulting Hopf algebra. In section 2.2 we see that $U_{q}(e^N)$ can be considered as the dual of the Hopf algebras $Fun(E_{q}^{N})$, $Fun(E_{q}^{N})$, where $E_{q}^{N}, E_{q}^{N}$ are the two versions of $N \times SO_q(N)$ (corresponding to the two possible braidings, see section 1.2.), where the structure of $Diff_q^{N}$ induces (only at the algebra level) a real structure for $U_{q}(e^N)$. The $*$-structure however is not compatible with the Hopf algebra, due to the presence of $\Lambda$; this is the dual situation of what happens for $Fun(E_{q}^{N})$, $Fun(E_{q}^{N})$, where the $*$-structure is incompatible with the algebra. $^1$ In section 2.3 we define a new set of generators of the $q$-Euclidean algebra, in view of the study of representations, in section 2.4 we give the Casimirs of $u_{q}(e^N)$.

The whole construction is a very economic one, since it relies on the use of only $2N$ objects $\{x^i, \partial_j\}$ (the coordinates and derivatives, i.e. the generators of $Diff_q^{N}$) with already fixed commutation relations, and $*$-relations, giving all the generators of $U_{q}(e^N)$, the corresponding algebra, Hopf algebra and $*$-algebra structures, together with a bunch of representations (the ones contained in the reducible representation $Fun_q(N)$, i.e. having zero "spin"). Actually, the algebra relations of $U_{q}(e^N)$ follow from the algebra relations of $Diff_q^{N}$, and $U_{q}(e^N)$ gets a bialgebra by deriving the natural coalgebra structure associated to it when we think of its elements as differential operators on $Fun_q(N)$; the compatibility of the coalgebra with the algebra follows from the associativity of $Diff_q^{N}$ and $Fun_q(N)$; finally, the antipode is found by consistency. Once the structure of $U_{q}(e^N)$ is determined in this way, we abstractly postulate it, so as to allow for all representations, i.e. representations with arbitrary $U_{q}(so(N))$ highest weight.

$U_{q}(e^N)$ is in many aspects the Euclidean analogue of the $q$-deformed Poincare$^*$ Hopf algebra of Ref. [29]. In both cases the inhomogeneous Hopf algebra contains the homogeneous one as a Hopf subalgebra which can be obtained by setting $q^{-1} = 0$, $\Lambda = 1$. In fact, all the commutation relations in $U_{q}(e^N)$ are homogeneous in $p$, contrary to what happens for inhomogeneous Hopf algebras obtained through contractions [4, 14, 15].

Chapter 3 is devoted to the study of fundamental (i.e. irreducible one-particle) representations of $U_{q}(e^N)$; since we are interested in potential applications to quantum physics, we look only for Hilbert space ones. We will call them "irreps" in the sequel. Only the $*$-algebraic structure $u_{q}(e^N)$ of $U_{q}(e^N)$ is involved (the Hopf structure is irrelevant at level of irreducible representations), so no problem of compatibility of $*$ with the coalgebra arises.

In section 3.1 we do an abstract study. We choose a Cartan subalgebra (i.e. a complete set of commuting observables) consisting basically of two parts, $[\frac{N+1}{2}]$ squared momentum components and $[\frac{N}{2}]$ angular momentum components. In subsection 3.1.1, 3.1.2 we do their spectral analysis separately; this is possible by means of the $L$ generators introduced in section 2.3. The points of the spectra make up a $q$-lattice. One important fact is that the irreps turn out to be of highest

$^1$This is an already known problem [29], common to many approaches to inhomogeneous quantum groups. We think that the way out should be sought in a non-standard way to realize Hilbert space tensor products of elementary systems into composite ones, and hope to report progress on this point elsewhere.
weight type, and they can be obtained from tensor products of a single one (the singlet, i.e., the one describing a particle with zero total spin) with some representation of $U_q(\mathfrak{so}(N))$. The spectra of all observables are discrete, in particular the spectra of squared momentum components, as expected. The corresponding eigenvectors are normalizable and make up an orthogonal basis of the Hilbert space of each irrep. In other words we have “lattice-regularized” the irrep, in the sense mentioned before. In subsection 3.1.3 we concentrate on the structure of the singlet representation. Modelling out singular vectors amounts to strengthening the Serre relations of the $L$-roots; a cumbersome “kinematical parity asymmetry” appears in the structure of the spectra of the angular momentum observables, which is an unusual feature for a “lattice” theory [22]; of course, it disappears in the limit $q \rightarrow 1$.

In section 3.2 we find a nonstandard configuration-space realization of the abstract singlet representation, actually based on a pair (the unbarred and the barred) realizations of the abstract vectors as functions on $\mathbb{R}^N_\theta$; the two realizations are put together in the definition of the scalar product as a $\mathbb{R}^N_\theta$-space integral. For instance, in the $N = 2n + 1$ the highest weight vector is represented by a $q$-plane wave directed in the $x^0$-direction. A suggestive final comment to sections 3.1, 3.2 singles out that ultimately the $x^0$-coordinate description of the space of physical states (heuristically, the Euclidean space itself) arises naturally from the existence of its Hilbert-space structure.

In section 3.3 we briefly analyze the limit $q \rightarrow 1$ of the representations.

We can think of the irreps studied in chapter 3 as describing the (time-independent) dynamics of a free nonrelativistic particle with arbitrary “generalized” $U_q(\mathfrak{so}(N))$-spin on $\mathbb{R}^N_\theta$. The subalgebra $\hat{u}_q(e^N) = u_q(e^N)/\{\Lambda = 1\}$ can be considered as the quantum group symmetry of the Hamiltonian

$$H := \frac{(p \cdot p)}{2M}.$$  \hspace{1cm} (1)

of the system; therefore all states with a given energy should be obtained from each other by the action of $\hat{u}_q(e^N)$, as in the classical case, different eigenspaces of the energy should be obtained from each other by the action of the dilatation operators $\Lambda^{\pm 1}$. More interestingly the study of chapter 3 can be considered as a warm-up before the construction of positive mass $q$-Wick-rotated versions of the representations [25] of the $q$-Poincaré algebra of Ref. [29, 21], in view of a $q$-Euclidean formulation of QFT. In fact, the $q$-deformed version of Wick-rotation in $3 + 1$ dimensions [19] rotates our subalgebra $\hat{u}_q(e^N)$ into the $q$-deformed Poincaré algebra of that Ref. For such (so-called virtual) representations one has to impose a condition equivalent to the positivity of the energy in Minkowski spacetime[6]. We hope to report on this point elsewhere.

As anticipated in Chapter 2, there are some alternative approaches to inhomogeneous quantum groups. A different (and chronologically preceding) approach is based on contractions of homogeneous quantum groups of higher rank, see for instance Ref. [4, 14, 15]; another one based on projections of bicovariant differential calculi on homogeneous quantum groups of higher rank has been recently proposed in Ref. [3].

Some notational remarks are necessary before the beginning. We will assume that $q$ is generic; in chapter 3 we have to consider $q \in \mathbb{R}^+$, and we will assume that $0 < q \leq 1$; the case $q > 1$ can be treated in an analogous way, and in doing so one essentially will interchange the roles of annihilation and creation operators. We set $\hbar = \hbar(N) = \frac{\Theta - 1}{2} \; N = 2n + 1$ to allow a compact way of writing relations valid both for even and odd $N$. In our notation a space index $i$ takes the values $i = -n, -n + 1, \ldots, -1, 0, 1, \ldots n$ if $N = 2n + 1$, and $i = -n, -n + 1, \ldots, -1, 1, \ldots n$
if \( N = 2n \). When \( N = 2n \) there is a complete symmetry of \( \text{Diff}^N \) under the exchange 
\[ x^{-1}, \theta^{-1} \rightarrow x^1, \theta^1, \] 
and this implies a complete invariance of the validity of all the results of 
this chapter and of the following ones under the exchange of indices 
\(-1 \rightarrow 1\), so that we will 
normally omit writing down explicitly the results that can be obtaind 
by such an exchange. We 
will often use the shorthand notation 
\[ [A, B]_a := AB - aBA \quad (\Rightarrow [\cdot, \cdot]_1 = [\cdot, \cdot]). \]
Finally, in our 
conventions := \{0, 1, 2, \ldots\}.

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1 Preliminaries

In this chapter we recollect some basic definitions and relations characterizing:

1) the algebra \( \text{Fun}(R_q^N) \) (\( O_q^N(C) \) in the notation of [7]) of functions on the quantum euclidean space \( R_q^N \), \( N \geq 3 \), and the algebra \( \text{Diff}(R_q^N) \) of differential operators on \( R_q^N \);

2) the quantum group \( \text{Fun}(SO_q(N)) \) [7] and its inhomogeneous extension [26] \( \text{Fun}(E_q^N) \), \( E_q^N := R_q^N \# SO_q(N) \);

3) the exterior algebra on \( q \)-variables, modulo the commutation relations (the symmetric and antisymmetric, namely the \( q \)-deformed versions of the corresponding corepresentation in verse:

\( \text{Notice that} \ \hat{R} = R \). The \( q \)-deformed metric matrix \( C := ||C_{ij}|| \) is explicitly given by

\[
C_{ij} := q^{-\delta_i^j} \delta_{i,j},
\]

where

\[
(\rho_i) := \begin{cases} (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2} - n) & \text{if } N = 2n + 1 \\ (n - 1, n - 2, \ldots, 0, 1 - n) & \text{if } N = 2n. \end{cases}
\]

Notice that \( N = 2 - 2\rho_0 \) both for even and odd \( N \). \( C \) is not symmetric and coincides with its inverse: \( C^{-1} = C \). Indices are raised and lowered through the metric matrix \( C \), for instance

\[
a_i = C_{ij} a^j, \quad a^i = C^{ij} a_j.
\]

Both \( C \) and \( \hat{R} \) depend on \( q \) and are real for \( q \in \mathbb{R} \). \( \hat{R} \) admits the very useful decomposition

\[
\hat{R} = qP_S - q^{-1}P_A + q^{1-N}P_1.
\]

\( P_S, P_A, P_1 \) are the projection operators onto the three eigenspaces of \( \hat{R} \) (the latter have respectively dimensions \( \frac{N(N+1)}{2}-1, \frac{N(N-1)}{2}, 1 \)): they project the tensor product \( x \otimes x \) of the fundamental corepresentation \( x \) of \( SO_q(N) \) into the corresponding irreducible corepresentations (the symmetric modulo trace, antisymmetric and trace, namely the \( q \)-deformed versions of the corresponding ones of \( SO(N) \)). The projector \( P_1 \) is related to the metric matrix \( C \) by \( P_1 ij = \frac{c_{ij} c_{j1}}{Q_N} \); the factor \( Q_N \) is defined by \( Q_N := C^{ij} C_{ij} \). \( \hat{R}^{\pm1}, C \) satisfy the relations

\[
[f(\hat{R}), P \cdot (C \otimes C)] = 0, \quad f(\hat{R}_{12}) \hat{R}^{\pm1}_{23} \hat{R}^{\pm1}_{12} = \hat{R}^{\pm1}_{23} \hat{R}^{\pm1}_{12} f(\hat{R}_{23})
\]

(\( P \) is the permutator: \( P_{ij}^{kl} := \delta_i^k \delta_j^l \) and \( f \) is any rational function); in particular this holds for \( f(\hat{R}) = \hat{R}^{\pm1}, P_A, P_S, P_1 \).

Let us recall that the unital algebra \( \text{Diff}(R_q^N) \) of differential operators on the real quantum Euclidean plane \( R_q^2 \) is essentially defined as the space of formal series in the (ordered) powers of the \( \{x^i\}, \{\partial_i\} \) variables, modulo the commutation relations

\[
P_A^{ij} \partial_i x^j = 0 \quad P_A^{ij} \partial_i \partial^j = 0
\]
and the derivation relations

$$\partial_j x^j = \partial_i^j + q R_{ij}^k x^k \partial_k$$

(actually we will enlarge it a bit more by allowing also arbitrary integer powers of the dilatation operator $\Lambda$, which is defined below). The subalgebra $Fun(R_q^N)$ of "functions" on $R_q^N$ is generated by \{x^i\} only. Below we will give the explicit form of these relations.

For any function $f(x) \in Fun(R_q^N)$, $\partial f$ can be expressed in the form

$$\partial f = f_i + f_j^i \partial_j, \quad f_i, f_j^i \in Fun(R_q^N)$$

(with $f_j^i$ uniquely determined) upon using the derivation relations () to move step by step the derivatives to the right of each $x^i$ variable of each term of the power expansion of $f$, as far as the extreme right. We denote $f_j$ by $\partial f$. This defines the action of $\partial_i$ as a differential operator $\partial_i : f \in Fun(R_q^N) \mapsto \partial_i f \in Fun(R_q^N)$; we will say that $\partial_i f$ is the "evaluation" of $\partial_i$ on $f$. For instance:

$$\partial_i 1 = 0, \quad \partial_i x^j = C^{ij}, \quad \partial_i x^i x^j = C^{ij} x^j + q R^{-1} \bar{\partial}_j x^k C^{kj}$$

By its very definition, $\partial_i$ satisfies the generalized Leibnitz rule:

$$\partial_i (fg) = \partial_i fg + (\partial_i f)g, \quad f, g \in Fun(R_q^N),$$

$(\partial_i f) = f_j^i$. In a similar way one can define [11] the evaluation of differential operators corresponding to the angular momentum components. One of the results of this paper will be that we will be able to write explicitly the linear operators $\partial_i^q$ as differential operators, $\partial_i^q \in Diff(R_q^N)$.

If $q \in R$ one can introduce an antilinear involutive antihomomorphism $*$:

$$s^* = i \hat{id} \quad \text{and} \quad (AB)^* = B^* A^*$$

on $Diff(R_q^N)$. On the basic variables $x^i$ $*$ is defined by

$$(x^i)^* = x^j C_{ji}$$

whereas the complex conjugates of the derivatives $\partial^i$ are not combinations of the derivatives themselves. It is useful to introduce barred derivatives $\bar{\partial}$ through

$$\bar{\partial}^* = -q^{-N} \partial C_{ji}.$$  

They satisfy relation (7) and the analogue of (8) with $q, \hat{R}$ replaced by $q^{-1}, \hat{R}^{-1}$. These $\bar{\partial}$ derivatives can be expressed as functions of $x, \partial$ [23], see formula (22).

Let us define

$$(a \cdot b)_j := \sum_{i=1}^j a^{-1} b_{-i} + \left\{ \begin{array}{ll} a^{1+}\delta_{ij} & \text{if} \quad N = 2n + 1 \\ 0 & \text{if} \quad N = 2n, \end{array} \right. \quad \text{and} \quad 0 < j \leq n$$

(when this causes no confusion we will also use the notation $a \cdot b := (a \cdot b)_n$).

Relations (7), (8) defining $Diff(R_q^N)$ amount respectively to

$$\eta^i \eta^j = q \eta^j \eta^i, \quad i < j, \quad \sum_{i=j}^j \eta_i \eta_i = (1 + q^{-2j}) (\eta \cdot \eta_j) \quad j = 1, 2, \ldots, n; \quad \eta^i = x^i, \partial^i,$$
and
\[
\begin{align*}
\partial_k x^i &= q x^i \partial_k - (q^2 -1) q^{-\delta_{jk} \delta_{kl}} x^k \partial_{-l}, & j < -k, j \neq k \\
\partial_k x^i &= q x^i \partial_k, & j > -k, j \neq k \\
\partial_{-k} x^i &= x^i \partial_{-k}, & k \neq 0 \\
\partial_k x^i &= 1 + q^2 x^i \partial_k + (q^2 -1) \sum_{j \geq 1} x^j \partial_j, & i > 0 \\
\partial_k x^i &= 1 + q^2 x^i \partial_k + (q^2 -1) \sum_{j \geq 1} x^j \partial_j - q^{-2y}(q^2 -1)x^{-i} \partial_{-i}, & i < 0 \\
\partial_{-k} x^i &= 1 + q x^0 \partial_0 + (q^2 -1) \sum_{j \geq 0} x^j \partial_j, & \text{(only for } N \text{ odd)).}
\end{align*}
\]

Here are some useful formulae for the sequel (sum over } l \text{ is understood):}
\[
\partial (x \cdot x)_l = q^2 x^i \partial_i + q^2 (x \cdot x)_l \partial (\partial \cdot \partial)_{l} (\partial \cdot \partial)_{l} + q^2 x^i (\partial \cdot \partial)_{l} \partial (x \cdot x)_l = \partial + q^2 x^i (\partial \cdot \partial)_{l} (\partial \cdot \partial)_{l} (\partial \cdot \partial)_{l} \partial + (1-q^2) x^i (\partial \cdot \partial)_{l} \partial + (1-q^2) x^i (\partial \cdot \partial)_{l} \partial + (1-q^2) x^i (\partial \cdot \partial)_{l} \partial.
\]

The dilatation operator } \Lambda \text{ is given by}
\[
\Lambda^2 := 1 + (q^2 -1) x^i \partial_i + q^{N-2}(q^2 -1)^2 (x \cdot x) (\partial \cdot \partial).
\]

(note the change of convention } \Lambda \rightarrow \Lambda^2 \text{ w.r.t. Ref. } [11, 23]); it fulfils the relations
\[
\Lambda x^i = q x^i \Lambda, \quad \Lambda \partial^i = q^{-1} \partial \Lambda.
\]

Then one can prove [23] that
\[
\partial^i = \Lambda^{-2} [\partial^i + q^{N-2}(q^2 -1)^2 x^i (\partial \cdot \partial)]
\]

The operator } B \text{ is defined by
\[
B := \Lambda^{-1} [1 + q^{N-2}(q^2 -1)] (x \cdot \partial) =: \Lambda^{-1} B
\]

and satisfies the relations
\[
[B, (x \cdot x)] = 0 = [B, (\partial \cdot \partial)].
\]

Under complex conjugation
\[
\Lambda^* = q^{-N} \Lambda^{-1}, \quad B^* = B.
\]

By definition a scalar } I(x, \partial) \in \text{Diff}(R_q^N) \text{ transforms trivially under the coaction } \phi_L \text{ (defined in subsection 1.2) associated to the quantum group of symmetry } SO_q(N, R) [7]: \phi_L(I) = 1 \otimes I. \text{ In Ref. [8] we showed that any scalar } I(x, \partial) \in \text{Diff}(R_q^N) \text{ can be written as a function of } x \cdot x, \partial \cdot \partial. \text{ Using relations (20), (23) it is straightforward to verify that the product } (x \cdot x)(\partial \cdot \partial) \text{ can be written as a linear combination of } \Lambda^2, \text{ BA, } 1. \text{ Therefore any scalar } I \text{ with natural dimension zero can also be written as a function } I = I(B, \Lambda) \text{ only.}

1.2 The Euclidean quantum groups } E_q^N, \tilde{E}_q^N

\(E_q^N, \tilde{E}_q^N\) can be introduced as braided semidirect products \(SO_q(N)\) [16] with the two possible covariant braiding; in this way the braided Hopf algebra \(SO_q(N)\) is embedded into an ordinary one, a process which was called “bosonization” [16]. The result is the one which was found in Ref [26] and which we briefly review here.
The Euclidean Hopf algebra $\Fun(E_q^N)$ := $\Fun(N \times SO_q(N))$ as an algebra is unital and is generated by elements $w^\pm 1, T_j^i, y^i$ satisfying relations
\[ P^i_{jk} T^k_j T^l_m = T^l_m T^k_j P^i_{jk}, \quad T^i_j C^i_j = T^i_j C^i_j T^i_j, \]  
and cross relations
\[ w y^i = q^{-1} y^i w, \quad [w, T^i_j] = 0, \quad y^i T^i_j = T^i_j y^i m; \]  
$1_{E_q^N}$ denotes the unit of the algebra. The coproduct, counit and antipode $\phi^E, \varepsilon^E, S^E$ on the generators $w^\pm 1, T^i_j$ are defined by
\[ \phi^E(T^i_j) = T^i_j \otimes T^i_j, \quad \varepsilon(T^i_j) = 1, \quad S^E(T^i_j) = C^i_j T^m_m C^i_j, \]  
$\phi^E(w) = w \otimes w, \quad \varepsilon(w) = 1, \quad S^E(w) = w^{-1}$
\[ \phi^E(y^i) = w^{-1} T^i_j \otimes y^i + y^i \otimes 1, \quad \varepsilon(y^i) = 0, \quad S^E(y^i) = -w S(T^i_j) y^i = -w(C T^i_j y^i). \]  
$\phi^E, \varepsilon^E, S^E$ are extended as algebra (anti)homomorphisms as usual. $\Fun(SO_q(N))$ is the Hopf subalgebra generated by $T^i_j$ only.

Similarly we define the Euclidean Hopf algebra $\Fun(\tilde{E}_q^N)$ := $\Fun(N \times SOq(N))$ (corresponding to the second choice of the braiding) by replacing $q, R, w \rightarrow q^{-1}, R^{-1}, w^{-1}$ in formulae (28)-(31); the corresponding generators of translations will be denoted by $\tilde{y}$ (instead of $y$). Note a change of notation w.r.t to ref. [16]: our $y^i, T^i_j$ are $\tilde{x}^i, M^i_j$ in the notation of that reference.

When $q \in \mathbb{C}$ we can define the complex conjugation $\ast$ as an antilinear involutive antihomomorphism mapping $\Fun(E_q^N) \rightarrow \Fun(E_q^N)$ by defining it on the generators through the formulae
\[ (T^i_j) \ast = C^i_j T^m_m C^i_j, \quad \tilde{y}^i = (y^i)^* C^i_j, \quad w^\ast = w^{-1}. \]  
One can immediately verify that we cannot impose the reality condition $y^i = (y^i)^* C^i_j$ which we adopted for the quantum space $q$ (formula (13)), otherwise we would spoil the algebra relations (28).b.

Finally $\Fun(N)$ can be equipped with a left (or right, as well) $E_q^N$- and $\tilde{E}_q^N$-comodule structure introducing respectively the left coactions defined on the basic generators by
\[ \phi^L_L(x^i) := w^{-1} T^i_j \otimes x^j + y^i \otimes 1 \quad \text{or} \quad \phi^R_R(x^i) := w T^i_j \otimes x^j + \tilde{y}^i \otimes 1, \]  
and is extended as an algebra homomorphism; the $SO_q(N)$ left coaction $\phi_L$ can be obtained from $\phi^E_L$ by setting $w = 1, y = 0$ in the previous formula.

1.3 $q$-Epsilon tensor and Hodge duality

The exterior algebra $\Lambda(R_q^N)$ over $R_q^N$ is defined as the algebra generated by “ 1-forms ” $\xi^i$ satisfying the relations
\[ P^i_{jk} \xi^j = 0 = P^i_{jk} \xi^k; \]  
it is equipped with the same comodule structure as $\Fun(R_q^N)$. In Ref. [9] we showed that, as in the classical case, $\Lambda(R_q^N)$ admits only subspaces of forms of degree $m \leq N$, and that there exists a unique $N$-form (the “ volume ” form $dV$), which transforms as a scalar (since it is mapped by the coaction to itself times the $q$-determinant, and the latter is essentially equal to one).
The existence of such a $N$-form implies the existence of a $q$-deformed completely antisymmetric tensor with $N$ indices $\varepsilon^{i_1i_2\cdots i_N}$ with the same fundamental properties as in the classical case, by

$$\xi^{i_1}\xi^{i_2}\cdots\xi^{i_N} = dV \varepsilon^{i_1i_2\cdots i_N} \quad \text{and} \quad dV := \xi^{-N}\xi^{1-N}\cdots\xi^{N}$$  \hfill (35)

We give the explicit expression for the tensor $\varepsilon_q$ in the case $N = 3,4$:

$$
\begin{array}{cccc}
\varepsilon_q^{11} = 1 & \varepsilon_q^{10} = -q & \varepsilon_q^{01} = -q & \varepsilon_q^{1-1} = q \\
\varepsilon_q^{2-1} = -q^2 & \varepsilon_q^{00} = -q(q^2 - q^{-2}) & \varepsilon_q^{10} = 0 & \text{otherwise}
\end{array}
$$

Exactly as in the classical case we can define a Hodge duality operation $*: \Lambda(R^N_q) \rightarrow \Lambda(R^N_q)$ by

$$*\xi^{i_1}\cdots\xi^{i_N} := A_m\varepsilon^{i_1\cdots i_m}\xi^{i_{m+1}}\cdots\xi^{i_N}$$ \hfill (38)

the normalization constant $A_m$ is such that $(*)^2 = \pm 1$. It is easy to verify that $(\xi^{i_1}\cdots\xi^{i_m})^* (\xi^{i_1}\cdots\xi^{i_m})$ transforms under the coaction as a scalar.

### 1.4 The $*$-Hopf algebra $U^N_q \simeq U_{q^{-1}}(so(N))$

**Theorem 1** ([11]) The Hopf algebra of $u.e.a.$ type $U_{q^{-1}}(so(N))$ can be realized as “the algebra of the angular momentum on $R^N_q$”, i.e., the subalgebra $U^N_q$ of $\text{Diff}(R^N_q)$ whose elements commute with any scalar $I(x,\partial) \in \text{Diff}(R^N_q)$.

A Poincaré-Birkhoff-Witt basis of $U^N_q$ is provided by the generators $\{B,l^{ij}\}$, where $B$ was defined in formula (23) and

$$l^{ij} := \Lambda^{-1} \mathcal{P}_{A \beta \delta} \xi^i \xi^j \partial^{\beta} x^\delta = -q^{-2} \Lambda^{-1} \mathcal{P}_{A \beta \delta} \partial^{\beta} x^\delta$$ \hfill (39)

(note the change of convention $L \rightarrow l$ w.r.t. Ref. [11]). Of course the $l^{ij}$ are not all algebraically independent and have rather involved commutation relations.

There exist closed commutation relations between the generators $l^{ij}, B$ and $v^i (v^i = x^i, \partial, \bar{\partial}$ etc.). We derive here only the commutation relations with $B$:

**Proposition 1** [12]

$$v^i B = q^{-1} \frac{q^N + 1}{1 + q^{N-2}} B v^i + \frac{(q^{-1} - q)(q^4 - 1)}{(1 + q^{N-2})(1 + q^{2-N})} l^{ij} v^i$$ \hfill (40)

**Proof.** Using formula (5) the definitions (15),(39), and equations (23) it is easy to prove the following relation:

$$x^i (\partial \cdot \partial) = x^i \partial_i + \frac{(q^2 + 1)q^{2-N}}{1 + q^{-N}} L^{ij} \partial_j$$ \hfill (41)
Then,

\[ B\partial = q^{-2}\partial B + \frac{1 - q^{-2}}{1 + q^{-N}} \left[ q^{-N} \partial + (q^2 - 1) x^i (\partial \cdot \partial) \right] \]

\[ B\partial = q^{-2}\partial B + \frac{1 - q^{-2}}{1 + q^{-N}} \left[ q^{-N} B\partial + \frac{(1 - q^{-2})(q^2 - 1)}{(1 + q^{-N})(1 + q^{2N})} L^i \partial_j \right]. \tag{42} \]

We find relation (40) if we collect terms in \( B\partial \) together, multiply both sides of this identity by \( q^2 \Lambda^{-1} \), use relation (21) and the definition \( B = B^{\Lambda^{-1}} \).

A more manageable Poincaré-Birkhoff-Witt basis of \( U_q^{\mathfrak{g}} \) is provided by the set \( \{ L^{i,j}, k^i \} \) (\( i < j, \neq -j; n \geq l > 1 \)), which was constructed in Ref. [11]; \( k^i \)'s generate a Cartan subalgebra. In particular the elements \( L^{-i,j}, L^{-i,-j}, k^i(k^i)^{-1} i = h, h + 1, \ldots, n \) are " Chevalley generators " of \( U_q^{\mathfrak{g}} \) coinciding with the Drinfeld-Jimbo generators of \( U_q^{\mathfrak{g}} \) up to rescaling of the roots \( L \) by suitable functions of \( k^i \). All the other roots \( L^{i,j} \) can be constructed starting from them as follows:

\[ [L^{-i,j}, L^{-l,k}]_q = q^{l-j} L^{-i,j} \quad [L^{-i,j}, L^{-l,j}]_q = q^{l-j+1} L^{-i,j}, \quad n \geq k > l \geq j \geq \begin{cases} 0 & \text{if } N = 2n + 1 \\ -1 & \text{if } N = 2n \end{cases} \tag{43} \]

\[ [L^{-i,j}, L^{-l,j}]_{q^\pm 1} = q^{l-j} L^{i,j} \quad [L^{-i,j}, L^{-l,j}]_{q^\pm 1} = q^{l-j} L^{-i,j}, \quad 2 \leq l < k \leq n \tag{44} \]

\[ [L^{1,0}, L^{2,0}] = q^{-1} L^{1,2} \quad [L^{-1,0}, L^{-2,0}] = L^{-1,2}, \quad 1 < k \leq n \quad \text{if } N = 2n + 1. \tag{45} \]

Then the commutation relations satisfied by the Chevalley generators can be summarized in the following way. Commutation relations between the generators of the Cartan subalgebra and the simple roots:

\[ [k^i, L^{(1-k),\pm k}]_a = 0 \quad a = \begin{cases} q^k & \text{if } i = k \leq n \\ q^{-k} & \text{if } i = k - 1 \\ 1 & \text{otherwise} \end{cases} \quad [k^i, k^j] = 0; \tag{46} \]

commutation relations between positive and negative simple roots:

\[ [L^{1-m,m}, L^{-k,k}]_a = 0 \quad a = \begin{cases} q^{-1} & m \pm 1 = k \\ 1 & \text{if } k \neq m, m \pm 1 \\ m, k \geq h(N) + 1, \end{cases} \tag{47} \]

\[ [L^{12}, L^{-21}] = 0 \quad [L^{-12}, L^{-21}] = 0 \quad \text{if } N = 2n, \tag{48} \]

\[ \begin{cases} [L^{1-m,m}, L^{-m,m}]_q = q^{-1} + q^{1-k} & \text{if } N = 2n + 1; \\ [L^{0,1}, L^{-1,0}]_q = q^{1-k} & \text{if } N = 2n + 1; \end{cases} \tag{49} \]

Serre relations:

\[ [L^{1-m,m}, L^{1-k,k}] = 0 \quad [L^{-m,m-1}, L^{-k,k}] = 0 \quad m, k > 0, \quad |m - k| > 1 \tag{50} \]

\[ [L^{1+j,m-m-j}, L^{2+m,m}]_a = 0 \quad [L^{m,m-2}, L^{j+m-m-j}]_a = a = \begin{cases} q & \text{if } j = 0 \\ q^{-1} & \text{if } j = 1 \end{cases} \quad m \geq 3 \tag{51} \]

\[ \begin{cases} [L^{0,1}, L^{12}]_{q^{-1}} = 0 & [L^{-21}, L^{-1,0}]_{q^{-1}} = 0 \quad \text{if } N = 2n + 1; \\ [L^{-12}, L^{0,2}]_q = 0 & [L^{-2,0}, L^{-2,1}]_q = 0 \quad \text{if } N = 2n + 1. \tag{52} \end{cases} \]

When \( q \in \mathbb{R} \) the complex conjugation (13),(14) acts on the Chevalley generators in the following way:

\[ (k^i)^* = k^i, \quad (L^{1-k,k})^* = q^{-2} L^{1-k,k-1} \quad k \geq 2, \quad \begin{cases} (L^{11})^* = q^{-1} L^{10} & \text{if } N = 2n + 1 \\ (L^{12})^* = q^{-2} L^{2,1} & \text{if } N = 2n \tag{53} \end{cases} \]
There exist closed commutation relations between the generators of $U_q^N$ and the coordinates $x^i$. Let $m \geq h(N) + 1$. Then

$$[k^0, x^i]_{\omega_x} = 0, \quad \hbar = 1, 2, \ldots, n;$$

$$[L^{\hbar}, x^i] = -q^{-1}x^i$$

and in all the remaining cases

$$[L^{1-m,m}, x^i]_{\omega_x} = q^{m} \sum_{i} (\delta_{i,m} - \delta_{i,-1}) x^{i+1}$$

$$[L^{-m,m-1}, x^i]_{\omega_x} = q^{m} \sum_{i} (\delta_{i,m} - \delta_{i,-1}) x^{i-1},$$

where

$$a_{m,i} := q^{2(m-\delta_{i,m})}, \quad b_{m,i} := (a_{m-1,i}) (a_{m,i})^{-1}.$$  

The commutation relations of $k^0$, $L^\hbar$'s with $\partial_x$, $\partial_i$ are the same, since $\partial_x \propto [\partial_x, x^i]_{\omega_x}$, $\partial_i \propto [\partial_x, x^i]_{\omega_x}$, and the $k$, $L$'s commute with scalars. Considering $L^\hbar$'s, $k^0$'s as differential operators on $Fun(R_q^N)$ (see formula (11)) one gets the Hopf algebra structure of $U_q$ in a natural [11]. In particular the Leibnitz rule of these differential operators determines its coassociative coproduct $\phi : U_q^N \rightarrow U_q^N \otimes U_q^N$, which on the generators $k^i$, $L^\hbar$ takes the form

$$\phi(k^i) = k^i \otimes k^i.$$  

$$\phi(L^{1-m,m}) = L^{1-m,m} \otimes 1 + (k^{m-1}(k^{m}-1)) \otimes L^{1-m,m},$$

$$\phi(L^{-m,m-1}) = L^{-m,m-1} \otimes 1 + (k^{m-1}(k^{m}-1)) \otimes L^{-m,m-1}$$

where

$$m \geq \begin{cases} 1 & \text{if } N = 2n + 1, \\ 2 & \text{if } N = 2n. \end{cases}$$

(1' here denotes the unit of $Diff(R_q^N)$, which acts as the identity when considered as an operator on $Fun(R_q^N)$, and $k^0 \equiv 1'$), and is extended to all of $U_q^N$ as an homomorphism. The counit $\epsilon$ is an homomorphism defined by (see formulae (9),(11)):

$$\epsilon(u) = u1_{Fun(R_q^N)}; \quad u \in U_q^N, \quad u \equiv \text{differential operator on } Fun(R_q^N).$$

which on the generators takes the form

$$\epsilon(L^j) = L^j 1_{Fun(R_q^N)} = 0, \quad \epsilon(k^i) = k^{i} 1_{Fun(R_q^N)} = 1.$$  

$\epsilon, \phi$ are matched so as to form a bialgebra. For instance, relation (58) is a consequence of the Leibnitz rule

$$k^i (f \cdot g) = (k^i f) \cdot (k^i g), \quad f, g \in Fun(R_q^N),$$

which is a consequence of commutation relations (54) and definition (60). The validity of the axioms of a bialgebra follow from the properties of $Diff(R_q^N)$, in particular the coassociativity of $\phi$ follows from the associativity of the Leibnitz rule, which in turn is a consequence of the associativity of $Fun(R_q^N)$, and the fact that $\phi, \epsilon$ are algebra homomorphisms follows from the associativity of the whole $Diff(R_q^N)$. Finally the antipode $\sigma$ is found by consistency with the coalgebra; it is an antihomomorphism which on the generators takes the form

$$\sigma(k^i) = (k^i)^{-1}.$$  

$$\sigma(L^{1-m,m}) = -(k^{m}(k^{m}-1)) \otimes L^{1-m,m}$$

$$\sigma(L^{-m,m-1}) = -(k^{m}(k^{m}-1)) \otimes L^{-m,m-1}$$

where

$$m \geq \begin{cases} 1 & \text{if } N = 2n + 1, \\ 2 & \text{if } N = 2n. \end{cases}$$

The Hopf structure of the inhomogeneous extension $U_q(\nu_N)$ of $U_q^N$ will be determined by exactly the same procedure.
2 Euclidean Hopf algebra $U_q(e^N)$ of u.e.a. type

2.1 Construction of $U_q(e^N)$

In this section we are going to introduce a $q$-deformed Euclidean Hopf algebra $U_q(e^N)$, of u.e.a. type in $N$ dimensions as an extension of $U_q^N = U_q^{so(N)}$, more precisely by adding to the generators of the latter "infinitesimal" generators $p'$ of translations and a generator $\Lambda$ of dilatations. $U_q(e^N)$ will be dual of both the Hopf algebras $Fun(E^N_q)$, $Fun(E_q^N)$ (see Chapter 2) through two different parings.

We first introduce the algebraic structure of the Hopf algebra $U_q(e^N)$.

The dual of a copy $U_q$ of the Euclidean quantum space (thought as the braid group of finite translations) is another copy $U_q$ of this quantum space, with generators $p'$ such that $\langle p_i, x^j \rangle = \delta^i_j$, and $Fun (SO_q (N))$ coacts on it in the same way (see formula (33)), provided we use the contravariant components $p' = C_{ij} p_i$.

The condition that the $p'$ generate $Fun(U^{N}_q)$ of course means that they satisfy the commutation relations

$$\mathcal{P}_A \vec{i}, \vec{j} = \Lambda = 0.$$  \hfill (65)

The dual version of the statement that $Fun (SO_q (N))$ coacts on $Fun(U^{N}_q)$ in the same way as on $Fun(U^{N}_q)$ is that the commutation relations between the generators of $U^{N}_q$ and $p'$ must be the same as those with $x^i$ (this is no surprise since we already saw that also the derivatives $\partial_i, \tilde{\partial}$ satisfy commutation relations (54)-(56)). We rewrite them here for convenience. Let $m \geq h(N) + 1$. Then:

$$\begin{align*}
\{ [L^{1-m,m}, p^{m-1}]_q &= - q^{\delta_m^m} p^m \\
[L^{m,m-1}, p^m]_q &= - q^{\delta_m^m} p^{m-1} \\
[L^{m}, p^m] &= - q^{-1} p^m \\
[L^{-1,0}, p^m] &= p^{-1} i f N = 2n + 1
\end{align*}$$

\hfill (66)

We see that the algebra generated by $L, k, p$ is closed.

To find a coproduct we will need to introduce one more generator, the generator $\Lambda$ of dilatation, such that:

$$[\Lambda, p_{q-1}] = 0 \quad [\Lambda, k] = 0 \quad [\Lambda, L] = 0;$$

we will see that it is the dual of $w$ of section 1.2.

Note that all the commutation relations involving $p$, (65)-(68), are homogeneous in these generators.

**Definition** In the sequel $u_q(e^N)$ will denote the algebra generated by $L, k, p, u_q(e^N)$ the algebra generated by $L, k, p, \Lambda^{\pm 1}$; the generators $L, k$ satisfy relations (46)-(52).

$U_q^N \subset u_q(e^N) \subset u_q(e^N)$ as subalgebras; more precisely these subalgebras can be projected into each other by setting $\Lambda = 1$ and $p = 0$ respectively in formulae (65)-(68).

**Remark** Note that there exists a natural embedding $u_q(e^N) \hookrightarrow u_q(e^{N+2})$ obtained by setting equal to zero all the generators of $p'$, $L^2, k^2$ of $u_q(e^{N+2})$ where either $i$ or $j$ takes the values $\pm (n+1)$.

Actually as a set of algebraically independent generators of $u_q(e^N)$ one should take the Chevalley generators of $U_q^N$, $\Lambda^{\pm 1}$ and one particular momentum component (e.g. $p_0$), since equations (66) can be used to define the others; then relations (67) will be linear in $p$ and...
of degree \(d \geq 1\) in the Chevalley generators and will play a role analogous to Serre relations (50-52). For instance, with the above choice \(p_n = \partial_n\) the new “Serre” relations to be added to the \(U_q^N\) ones would read

\[
[\Lambda, \partial_n]_{q^{-1}} = 0 \quad [\Lambda, k] = 0 \quad [\Lambda, L] = 0; \quad \quad \quad \quad (69)
\]

\[
k^i \partial_n = q^{-2k^i} \partial_n k^i \quad [L^{-m,n-1}, \partial_n] = [L^{1-m,n}, \partial_n] \quad h(N) + 1 \leq m < n \quad \quad \quad \quad (70)
\]

\[
[L^{-m,n-1}, \partial_n]_{q^{-1}} = 0 \quad [L^{1-m,n}, [L^{1-m,n}, \partial_n]]_{q^{-1}} = 0.
\]

The commutation relations among \(L, k, \partial, \Lambda\) have been determined by realizing these generators as differential operators on \(\text{Fun}_{q}(N)\); now we can think of them as abstract generators of an algebra, i.e., no more as elements of \(\text{Diff}_{q}(N)\). At the representation-theoretic level this will be necessary since we are interested in finding all representations of \(u_q(e^N)\) and not only those with zero \(U_q(sl(N))\)-spin.

We try now to endow \(u_q(e^N)\) with some Hopf structures compatible with its algebraic structure.

We first look for the coalgebra structure. Again, as we did in section 1.4 with \(U_q^N\), our postulates will be suggested by the explicit realization of the generators \(p^i\) as differential operators on \(\text{Fun}_{q}(N)\).

A first possibility is to realize \(p^i\) by \(p^i = \partial^i\), a second, by \(p^i = \overline{\partial}^i\). Using relations (18),(22),(39) one can easily show that more generally \(p^i \equiv f(\Lambda)(\alpha \partial^i + \beta \overline{\partial})\) is a realization of the \(p^i\)'s as differential operators on \(e^N\). The following theorem shows that there are no more alternatives.

**Theorem 2** The only way to realize the generators \(p^i\) satisfying the algebra (65)-(68) as elements of \(\text{Diff}_{q}(N)\) is by a combination

\[
p^i \equiv f(\Lambda)(\alpha \partial^i + \beta \overline{\partial}), \quad \quad \quad \quad \alpha, \beta \in \mathbb{C}, \quad f(t) \in \mathbb{D} (71)
\]

**Proof.** The commutation relations (66)-(67) impose for \(p^i\) the general form \(p^i = S_1 \partial + S_2 x\), where \(S_1, S_2 \in \text{Diff}_{q}(N)\) are scalars. The commutation relation (68) implies that the natural dimensions \(d\) of \(S_1, S_2\) are given by \(d(S_1) = 0, d(S_2) = 2\). Hence, using the results mentioned at the end of section 1.1, we are led to an improved ansatz \(p^i = S_1(B, \Lambda) \partial^i + S_2(B, \Lambda) x^i \partial \cdot \partial\) (recall that \(B \in U_q^N\)). Now we replace this general form for \(p^i\) into the commutation relations (65). We move \(\partial \cdot \partial\) to the right of \(S_1, S_2, x^i, \partial\) by using relations (18),(21),(23). The three coefficients of the powers 0,1,2 of \(\partial \cdot \partial\) must vanish independently:

\[
\begin{align*}
\mathcal{P}^A_{B, k} S_2(B, \Lambda) \partial^i S_1(B, \Lambda) \partial^k &= 0 \\
\mathcal{P}^A_{B, k} [S_1(B, \Lambda) \partial^i S_2(B, \Lambda) x^k + S_2(B, \Lambda) x^i S_1(B, \Lambda q^2) \partial^k + q^{2-N} S_2 x^k S_2(B, \Lambda q^2) \partial^i] &= 0 \\
\mathcal{P}^A_{B, k} S_1(S_1, \Lambda) x^k S_2(B, \Lambda) x^i &= 0.
\end{align*}
\]

We expand \(S_i\) in a power series \(S_i = \sum_{n=0}^{\infty} B^n S_i(B^n)\) and use the commutation relations (40) to move the \(\partial\) (resp. the \(x\)) to the right of all the \(B^n\) in the first (resp. third) equation; we get an expansion in powers of the independent generators \(B, \Lambda\). Setting all their coefficients equal to zero implies \(S^n_{ij} \equiv 0\) \(n \geq 1\) (only the coefficient of the constant term vanishes automatically because of the relation (7)). In other words \(S_i = S_i(\Lambda)\) only. This allows to rewrite the second equation as

\[
[-q^2 S_1(S_2(\Lambda q) + S_2 S_1(\Lambda q) + S_2 S_1(\Lambda q^2)q^{-N})]^{i,j} \partial \cdot \partial = 0.
\]
This implies that the term in square brackets must vanish. A nontrivial solution of the equations (72) is therefore $S_0 = 0$, $S_1 = S_1(\Lambda)$, yielding the solution with $\beta = 0$ given in the claim. If $S_0 \neq 0$ we can set $s(\Lambda) := \frac{S_1(\Lambda)}{S_1'(\Lambda)}$, and the preceding equation is equivalent to the new one

$$q^{2-N} - q^2 s(\Lambda) + s(q\Lambda) = 0. \quad (74)$$

Expanding $s(\Lambda)$ in power series one immediately finds that the general solution of the latter equation is $s = \frac{q^{2-N}}{q^2 - 1} + a\Lambda^2$, $a \in \mathcal{L}$, which yields the remaining solutions of the claim (after use of eq. (72)).

**Remark** Note that $\partial' \in u_\Phi(e^\Lambda)$ for all realizations (71) of $\partial'$; for, if e.g. we had picked $\partial'$ as original realizations of $\partial'$, it is easy to check using formulae (19),(22),(23) that

$$\partial' = \frac{1 + q^{-N}}{q^2 - 1} \Lambda^{-1} [B, \partial'] + \Lambda^2 \partial' = \Lambda^{-1} [B, \partial'], \quad (75)$$

Now we can use theorem 2 to derive the natural coalgebra structure associated to the realization $u_\Phi(e^\Lambda) \subset \text{Diff}(\Lambda)$, and then convert it into a Hopf one by finding the antipode. We follow the same approach used in section 1.4 with $U_q^\Lambda$, i.e. the counit derives from evaluation of differential operators on the unit $1 \in \text{Fun}(\Lambda)$, the coproduct from their Leibnitz rules, and the antipode from consistency with the coproduct, according to the axioms of a Hopf algebra.

Because of the remark following theorem 1, it is sufficient to show explicitly the existence and the form of coproduct and counit $\phi, \epsilon$ only on the generators $\partial' = \partial'$; even better, only on one particular “momentum” component, say $\partial_0$. In fact, we can determine the action of $\phi, \epsilon$ on the other components $\partial'$ iteratively through commutators (66) of $\partial_0$ with the Chevalley generators of $U_q^\Lambda$, and on the other realizations (71) by using relation (75). In both steps, we use the basic property that $\phi, \epsilon$ are algebra homomorphisms.

From

$$[\Lambda, x]_q = 0 \quad (76)$$

and

$$\partial_n x^n = 1 + q^2 x^n \partial_n, \quad \partial_n x^{-n} = x^{-n} \partial_n, \quad \partial_n x^l = q x^l \partial_n, \quad |l| < n \quad (77)$$

we infer

$$\phi(\Lambda) := \Lambda \otimes \Lambda \quad \epsilon(\Lambda) = 1 \quad \sigma(\Lambda) = \Lambda^{-1} \quad (78)$$

and

$$\phi(\partial_0) := \partial_0 \otimes 1' + \Lambda(k^0)^{1/2} \otimes \partial_0 \quad \epsilon(\partial_0) = 0 \quad \sigma(\partial_0) = -\Lambda^{-1}(k^0)^{1/2} \partial_0. \quad (79)$$

$\phi, \epsilon$ are extended as algebra homomorphisms, $\sigma$ as an algebra antihomomorphism, to the rest of $u_\Phi(e^\Lambda)$. Then $u_\Phi(e^\Lambda)$ is turned into a Hopf algebra, what we call $U_q^\Lambda$, since the basic axioms are satisfied on the generators. In particular we readily find when $i \geq h$, for instance,

$$\phi(\partial_i) = \partial_i \otimes 1' + \Lambda(k^i)^{1/2} \otimes \partial_i + (1 - q^2) \sum_{j > i} q^{-h} \Lambda L^{-i,j}(k^i)^{1/2} \otimes \partial_j \quad \epsilon(\partial_i) = 0 \quad (80)$$

$$\sigma(\partial_i) = -\Lambda^{-1}(k^i)^{1/2} \partial_i + (q^2 - 1) \sum_{j > i} q^{-h} \Lambda^{-1}(k^i)^{1/2} \sigma(L^{-i,j}) \partial_j. \quad (81)$$

As for the choice of “conjugated ” translation generators $\partial' \equiv \partial'$, in principle the actions of $\phi, \epsilon, \sigma$ can be determined from formula (75) in the remark, but practically it is more convenient
to find them using the same procedure as for $p^i = \partial^i$. Only, we pick now as initial component $p_{-n} \equiv \tilde{\partial}_{-n}$. From
\[ \tilde{\partial}_{-n} x^{-n} = 1 + q^{-2} x^{-n} \tilde{\partial}_{-n}, \quad \tilde{\partial}_{-n} x^n = x^n \tilde{\partial}_{-n}, \quad q \tilde{\partial}_{-n} x^l = x^l \tilde{\partial}_{-n}, \quad |l| < n \] (82)
we find
\[ \tilde{\partial}(\tilde{\partial}_{-n}) = \tilde{\partial}_{-n} \otimes 1 + \Lambda^{-1} (k^n)^{\frac{1}{2}} \otimes \tilde{\partial}_{-n}, \quad \tilde{\tau}(\tilde{\partial}_{-n}) = 0 = \tilde{\tau}(\tilde{\partial}), \quad \tilde{\tau}(\tilde{\partial}_{-n}) = -\Lambda (k^n)^{\frac{1}{2}} \tilde{\partial}_{-n}. \] (83)

So far we have not considered the $*$-structure. Of course, we would be finally interested in a $*$-Hopf algebra, i.e., in a Hopf algebra endowed with a $*$ operation which is compatible with the Hopf axioms. Whenever $q \in \mathbb{C}$, the $*$-structure of $Diff_q(\mathbb{C})$ induces at the algebra level a consistent $*$-structure for $U_q(e^N)$, see formulae (13), (14), (53): in other words, $U_q(e^N)$ by itself is a $*$-algebra. We can select a particular realization (71) of the momenta $p^i$ by imposing the same reality condition valid for the $x^i$,
\[ (p^i)^* = p^j C_{ji}. \] (84)

**Corollary 1** The only way to realize the generators $p^i$ satisfying the algebra (65)-(68) and the $*$-relations (84) as elements of $Diff_q(\mathbb{C})$ is (up to a global real factor) through
\[ p^i \equiv -i(\partial q^j + \tilde{\partial}). \] (85)

However, this is not of much help, since

**Proposition 2** The above $*$-structure is incompatible with the Hopf one.

**Proof.** That $*$ and $\phi$ do not commute can be immediately checked on $\Lambda$:
\[ \left[ (\ast \otimes \ast) \circ \phi \right](\Lambda) = q^{-2N} \Lambda^{-1} \otimes \Lambda^{-1} \neq q^{-N} \Lambda^{-1} \otimes \Lambda^{-1} = \phi(\Lambda^*). \] (86)

In other words, $U_q(e^N)$ is not a $*$-Hopf algebra. This is a major problem for physical applications, since it prevents the standard construction of Hilbert space representations of composite physical systems through tensor product of Hilbert space representations of fundamental ones, and is common to many inhomogeneous $q$-deformed Hopf algebras.

We think that the way out of this problem should be sought in a non-standard way to perform tensor products of Hilbert spaces.

The $*$-algebra structure is all what we will use to find the fundamental (i.e. one-particle) Hilbert space representations of $U_q(e^N)$ (see Chapter 3). On $p^i$ we will then impose the $*$-relations (84) at an abstract level. However, to find a realization of the singlet (i.e. spin-zero) representation on $\mathbb{N}_0 \times \mathbb{N}_0$ configuration space $\tilde{\mathcal{V}}$, i.e., realize the carrier space of this representation as a subspace of $Fun_{\mathbb{Z}_2}(\mathcal{V})$, we will use both $\partial$ and $\tilde{\partial}$ in a non-standard way (section 3.2).

Summing up, our definition of the quantum Euclidean Hopf algebra (of u.e.a. type) $U_q(e^N)$ can be taken as follows

**Definition:** $U_q(e^N)$ as an algebra is generated by elements $\Lambda^{\pm 1} k^i, L^{\pm (i+1)}, L^{\pm (i-1)}$ ($i = h, h + 1, \ldots, n$) and $p_n$ (in the $N = 2n$ case one should also add $L^{\pm 2}, L^{\pm 3}, \ldots$). They satisfy the commutation relations (46)-(54), (65)-(68). The Hopf structure is given by (58)-(61), (63), (64), (78), (79).
2.2 The dual Hopf algebras of $U_q(e^N)$

In this section we recognize that $U_q(e^N)$ is the dual Hopf algebra of both versions $\text{Fun}(E^N_{q-1})$, $\text{Fun}(E^N_q)$ of the Euclidean Hopf algebra of functions-on-the-group type introduced in Ref. [26], and reviewed in Chapter 2. This means, incidentally, that suitable completions of $\text{Fun}(E^N_q)$, $\text{Fun}(E^N_q)$ will coincide.

To prove that statement one has to find a pairing $\langle , \rangle$ between $U_q(e^N)$ and $\text{Fun}(E^N_q)$ (resp. $\text{Fun}(E^N_q)$) satisfying linearity and the two postulates

\begin{equation}
\langle ab, A \rangle = \langle a \otimes b, \phi^F(A) \rangle \quad \langle a, AB \rangle = \langle \phi(a), A \otimes B \rangle
\end{equation}

\begin{equation}
\langle a \otimes b, A \otimes B \rangle = \langle a, A \otimes b, B \rangle.
\end{equation}

Since $\text{Fun}(E^N_q)$, $\text{Fun}(E^N_q)$ are not dual quasi-triangular Hopf algebras (implying that the basic commutation relations between all its generators cannot be put in the form (26), with a suitable $R$ matrix), the pairing cannot be introduced through a big algebras, since then it can be extended in a consistent way from the generators to the rests of the algebras.

As a set of generators of $U_q(e^N)$ we choose the Chevalley generators $L, k$ of $U^N$ together with $\partial_y, \Lambda^{\pm 1}$ (resp. $\partial_y, \Lambda^{\pm 1}$). As a set of generators of $\text{Fun}(E^N_q)$ (resp. $\text{Fun}(E^N_q)$) we choose the generators $w^{\pm 1}, T_j^{\pm}$ together with $y^j$ (resp. $\tilde{y}^j$) (see section 2.4). Thus, our problem is reduced to exhibiting a pairing such that the $U^N_q$ algebra together with relations (69)-(70) (resp. the analogue for $\partial_y$) is compatible with the coalgebra structure of $\text{Fun}(E^N_q)$ (resp. $\text{Fun}(E^N_q)$), and the algebra relations (26)-(28) of $\text{Fun}(E^N_q)$ (resp. $\text{Fun}(E^N_q)$) are compatible with the coalgebra of $U_q(e^N)$.

Proposition 3 $U_q(e^N)$ id dual of $\text{Fun}(E^N_q)$, $\text{Fun}(E^N_q)$ respectively through the natural pairings

\begin{equation}
\begin{cases}
\langle \Lambda, w^{\pm 1} \rangle = q^{\pm 1}, & \langle \Lambda, T_j^{\pm} \rangle = \delta^j_i, & \langle \Lambda, y^j \rangle = 0 \\
\langle \partial_y, w^{\pm 1} \rangle = 0 & \langle \partial_y, y^j \rangle = \delta^j_i, & \langle \partial_y, T_j^{\pm} \rangle = 0 \\
\langle L^i, w \rangle = 0 & \langle L^i, y^j \rangle = 0 & \langle k^j, w^{\pm 1} \rangle = 1 & \langle k^j, y^j \rangle = 0 \\
\langle U_q^N, \text{Fun}(SO_q^{(1)}(N)) \rangle: \text{the pairing of Ref. [7] with } q \rightarrow q^{-1}
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
\langle \Lambda, w^{\pm 1} \rangle = q^{\pm 1}, & \langle \Lambda, T_j^{\pm} \rangle = \delta^j_i, & \langle \Lambda, y^j \rangle = 0 \\
\langle \partial_y, w^{\pm 1} \rangle = 0 & \langle \partial_y, y^j \rangle = \delta^j_i, & \langle \partial_y, T_j^{\pm} \rangle = 0 \\
\langle L^i, w^{\pm 1} \rangle = 0 & \langle L^i, y^j \rangle = 0 & \langle k^j, w \rangle = 1 & \langle k^j, y^j \rangle = 0 \\
\langle U_q^N, \text{Fun}(SO_q^{(1)}(N)) \rangle: \text{the pairing of Ref. [7] with } q \rightarrow q^{-1}
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
\langle \Lambda, w^{\pm 1} \rangle = q^{\pm 1}, & \langle \Lambda, T_j^{\pm} \rangle = \delta^j_i, & \langle \Lambda, y^j \rangle = 0 \\
\langle \partial_y, w^{\pm 1} \rangle = 0 & \langle \partial_y, y^j \rangle = \delta^j_i, & \langle \partial_y, T_j^{\pm} \rangle = 0 \\
\langle L^i, w^{\pm 1} \rangle = 0 & \langle L^i, y^j \rangle = 0 & \langle k^j, w \rangle = 1 & \langle k^j, y^j \rangle = 0 \\
\langle U_q^N, \text{Fun}(SO_q^{(1)}(N)) \rangle: \text{the pairing of Ref. [7] with } q \rightarrow q^{-1}
\end{cases}
\end{equation}

Of course the pairing of the generators with the units is given by $\langle 1', A \rangle = \varepsilon(A)$,

$\langle a, 1 \rangle = \varepsilon(a) = 1, a, 1' \in U_q(e^N)$, $A, 1 \in \text{Fun}(E^N_q)$ and $\varepsilon$ is the counit in $\text{Fun}(E^N_q)$.

Proof: See the Appendix.
As a consequence of the above definition, it is easy to show that the pairing between $U^N$ and the subalgebra $P$ of translation generators extends in a trivial way, $<P, U^N> = 0$, whereas the pairing between the latter and the generators $y, \bar{y}$ of finite translations extends first as

$$<\partial_i, y^j> = \delta_i^j = <\partial_i, \bar{y}^j>, \quad (91)$$

and then as

$$<\partial_{i_1} \partial_{i_2} \ldots \partial_{i_l}, y^{j_1} y^{j_2} \ldots y^{j_m}> = \left\{ \begin{array}{ll}
0 & \text{if } m \neq l \\
(1; \bar{R}[2; \bar{R}][m; \bar{R}])^{i_1 j_2 \ldots i_l j_m} & \text{if } m = l,
\end{array} \right. \quad (92)$$

for the unbarred objects and for the barred ones by replacing $\bar{R} \rightarrow \bar{R}^{-1}$. Here, (with a notation suggested by, but slightly different from that of Ref. [18])

$$[h, M] := 1 + M_{\lambda-1, \lambda} + M_{\lambda, \lambda-1} + \ldots + M_{m-1, m} + M_{m-2, m-1} + \ldots + M_{2, 1}.$$

Note that formulae (92) alone can be obtained also by using the fact that the braid group $N_\sigma^N$ (resp. $N_\sigma^g_\delta$) of momenta and the braid group $N_\sigma^N$ of finite translations $x$ are "braided" paired, in the sense that their pairing is generated from the basic formulae (28), (29), provided $\phi$ is replaced by the coaddition $\Delta$ (3.49) and $\phi$ is replaced by a braided one $\Delta'$, $\Delta'(p^i) := p^i \otimes 1 + 1 \otimes p^i$.

From the viewpoint of duality the difficulty one meets in finding $s$-structures compatible with the Hopf ones in $Fun(E^N_q, U_q(e^N))_{q, \sigma, \delta}^N$ (resp. $Fun(\bar{E}^N_q, U_q(\bar{e}^N))_{q, \sigma, \delta}^N$) are all related and are a general feature of inhomogeneous quantum groups of BWM type [1].

2.3 New $L$ generators of the Euclidean algebra $u_q(e^N)$

In Chapter 3 we will construct the fundamental Hilbert space representations of the Euclidean algebra. We will show (Proposition 7) that for such representations either $(p \cdot p)_i \equiv 0$ identically, $\forall i \geq h$, or all $(p \cdot p)_i$ are strictly positive definite. In the former case the algebra reduces to the homogeneous one $U^N_q$, in the latter case, which we here consider, it follows that we can define the inverse of $(p \cdot p)_i$.

Let us consider the relations

$$[L^{-m_1}, (p \cdot p)_i] = 0 \quad \text{if } l \neq m; \quad (94)$$

$$[L^{-m, m+1}, (p \cdot p)_m] = -q^{m+1}p^m p^{-m+1}, \quad (\Rightarrow [L^{-m, m+1}, \frac{1}{(p \cdot p)_m}] = \frac{q^{m+1}}{(p \cdot p)_m^2} p^{-m} p^{m+1}], \quad (95)$$

which can be easily drawn from equations (66), (67). As a consequence of the second formula, $L$ would not map eigenvectors of $(p \cdot p)_m$ into each-other. However, one can define improved generators $L$ which actually do. From (67) we find

$$\left\{ \begin{array}{ll}
[L^{-m, m+1}, \frac{1}{(p \cdot p)_m}] = \frac{q^m}{(p \cdot p)_m^2} p^{-m} p^{m+1}, \\
[L^{1, 2}, \frac{1}{(p \cdot p)_m}] = \frac{1}{(p \cdot p)_m} p^2, \quad & \text{if } N = 2m;
\end{array} \right. \quad (96)$$

therefore we define

$$\left\{ \begin{array}{ll}
L^{-m, m+1} := L^{-m, m+1} + \frac{q^m}{(p \cdot p)_m} p^{-m} p^{m+1}, \\
L^{-m, m-1} := L^{-m, m-1} + \frac{q^m}{(p \cdot p)_m} p^{-m} p^{m-1}
\end{array} \right. \quad (97)$$
(similarly for $L^{13}$). Note that this redefinition is possible only when $q \neq 1$. The basic property of the new generators is the fact that if $i > 0$ then (see definition (57))

$$[L^{-m,m+1}, p^i]_{-m} = 0 \quad [L^{-m-1,m}, p^i]_{-m} = 0,$$

implying

$$[L^{-m,m+1}, (p \cdot p)] = 0 = [L^{-m-1,m}, (p \cdot p)] \quad \forall i, m.$$  

Moreover, it is easy to see that the $L^i$ satisfy the same $*$-conjugation relations as the $L^i$.

Let us find out now the commutation relations satisfied by the $L^i$. We can define other roots $L$ starting from simple ones, just in the same way as we did with the $L^i$, using relations (43)-(45) (with the replacement $L \to L$). $L$ roots can be divided into positive and negative ones according to the same convention used for the $L^i$.

**Proposition 4** Let $k \geq h + 1$. The commutation relations between positive and negative simple roots are

$$[L^{1-m,m}, L^{-k,m-1}] = 0 \quad a = \begin{cases} q^{-1} & m \pm 1 = k \\ 1 & k \neq m, m \pm 1, \end{cases}$$

$$[L^{12}, L^{-5,1}] = \frac{(p \cdot p)^2 p^i}{(1 - q^2)(p \cdot p^i)^2}, \quad [L^{-12}, L^{-5,1}] = \frac{(p \cdot p)^2 p^{-1} p^{-1}}{(1 - q^2)(p \cdot p^i)^2}, \quad \text{if} \quad N = 2n,$$

$$\begin{cases} [L^{1-m,m}, L^{-m,m}] = q^{1+2m-1} + C_m & 2 \leq m \leq n \\ [L^{1-m,m}, L^{-m,m}] = q^{1+2m-1} + C_1 & \text{if} \quad N = 2n + 1 \end{cases}$$

where

$$C_1 := \frac{q^2}{1-q^2} \left[ 1 + q \frac{(p \cdot p^i)}{(p \cdot p^i)^2} \right] \quad \text{if} \quad N = 2n + 1,$$

$$C_m+1 := \frac{q^{2m}}{1-q^2} \left[ 1 - \frac{(p \cdot p^i)_{m-1}(p \cdot p^i)_{m+1}}{(p \cdot p^i)^2} \right], \quad m \geq 1.$$  

**Proof.** Use equations (47)-(48),(96),(99) perform explicit computations. \(\triangleleft\)

The list of commutation relations involving the $L^i$ is completed by the following proposition:

**Proposition 5** The $[[k, l]$ relations, the Serre relations and the $s$-relations for the $L$ generators are the same as those of the $L^i$ generators.

**Proof:** explicit computations. \(\triangleleft\)

Summing up, the commutations relations among the $L^i$ are the same as those among the $L^i$, if we add some “central charges” ($C_m$).

**Remark** Note that the embedding $\hat{u}_q(e^N) \hookrightarrow \hat{u}_q(e^{N+2})$ mentioned after formula (68) is obtained now by setting equal to zero all the generators $p^i, L^j, k^i$ of $\hat{u}_q(e^{N+2})$ having $i = \pm (n+1)$ for some space index $i$.

**Definition** We denote by $u_q^{+N}$ the subalgebra of $u_q(e^N)$ generated by the positive roots $L^i$ (resp. by the negative roots $L^i$) and, in the case $N = 2n$ only, by $p^{\pm 1}$.
2.4 Casimirs of $\hat{u}_q(e^N)$

As in the classical case, $\hat{u}_q(e^N)$ has $n+1-h$ casimirs; their definition mimics the classical one. They can be built in the most compact way as follows. Define the "Pauli-Lubanski" $(2l+1)$-form

\[ \omega_{2l+1} := (\omega)^l p \]  

\[ \omega := l^i \xi_l \xi_l, \quad p := p^l \xi_l \]  

\[ l = 0, 1, \ldots, n-1. \]  

(105)

**Theorem 3** The $n+1-h$ casimirs $\Omega^i$ of $\hat{u}_q(e^N)$ are defined by

\[ \Omega^i \ dV := \omega_{2l+1} \land^N \omega_{2l+1}. \]  

\[ \Omega^m \ dV := \omega_{2m+1} \quad \text{if} \quad N = 2n+1, \]  

(106)

where $dV$ denotes the volume $N$-form, and $\land$ denotes the Hodge duality operation introduced in section 2.4.1.

**Proof.** The general proof will be given in a separate paper [13]; it involves the use of the general $[l^i, p^j]$ commutation relations and of specific properties of the $q$-deformed completely antisymmetric projector with $m \leq N$ indices. The theorem will be verified in the cases $N = 3, 4$ using the $L$ generators in next subsection.

The irreps of $\hat{u}_q(e^N)$ are characterized by the values of the casimirs.

In particular, when $l = 0$ we find the square momentum casimir

\[ \Omega^0 \equiv (p \cdot p)_0; \]  

(108)

when $N = 4$, $\Omega^i$ is the $q$-deformed analogue of the Wick-rotated casimir which is constructed from the Pauli-Lubanski vector $w^i$ defined by $w^i \xi_l := * \omega^l$ and gives the intrinsic spin of each irrep. Using the properties of the $\epsilon^q_l$ tensor, one can show that $w^i$ really generate a quantum Euclidean space, i.e. satisfy

\[ \mathcal{P}^i_{A \beta} w^\beta w^i = 0. \]  

(109)

Of course, the casimirs $\Omega^i, l \geq 1$, are zero in the singlet representation, i.e. that characterized by the trivial weight, since then both $p^l$ and $l^i$ can be expressed as differential operators on $N_q$, namely $l^i = \mathcal{P}^i_{A \beta} \partial^\beta A^{-1}$, $p^l = \partial^l A^2$, and $\mathcal{P}^j_{A \beta} \partial^\beta \partial^i = 0$.

2.4.1 The casimirs of $\hat{u}_q(e^N)$ in the cases $N = 3, 4$ in terms of the $L, k, p$ generators

**Proposition 6** When $N = 3, 4$, the Casimirs $\Omega^i$ (107), (106), in terms of $p, L, k$ generators take respectively the form

\[ \Omega_1 = p^1(k^1)^{-1} = q(q + 1)\frac{(p \cdot p)_1}{p^2}(k^1)^{1} + q^2(1 - q)(1 - q^2)L^{-1,0}L^{0,1}(k^1)^{1} \]  

(110)

and

\[ \Omega_1 = (L^{-2,1}L^{1,2})(L^{2,1}L^{-1,2})k^2(p \cdot p)_1 + \frac{q^2}{(q^2 - 1)^2}(p \cdot p)_1[k^1(L^{-2,1}L^{1,2}) + (k^1)^{-1}(L^{2,1}L^{-1,2})] \]  

\[ + \frac{q^2}{(q^2 - 1)^2} \left[ 1 - q^2 k^2(p \cdot p)_2 \right] - \frac{q^2}{(1 - q^2)^2}(p \cdot p)_1[p^{-1}L^{-2,1}L^{1,2} + p^1L^{2,1}L^{-1,2})k^2. \]  

(111)
Proof. We prove that $\Omega_1$ as defined by equation (107), (106) take the forms (110), (111); then it is straightforward to verify that they are casimirs of $u_q(e^N)$ using the commutation relations of the preceding subsection.

Case $N = 3$. Using the definition of $\mathcal{P}_A$ one verifies that there are only three independent $q^j, l^{-1,0}, l^{0,1}, l^{1,-1}$, say. $l^{0,1} = (k^{1})^\dagger L^{0,1}, l^{-1,0} = (k^{1})^\dagger L^{-1,0}, l^{1,-1} = -\frac{(k^{1})^\dagger L^{0,1}}{q^j}$ see ref. [11]. $B$ can be easily expressed as a linear function of the quadratic casimir $C_{\Omega_1}$ of $U_q^2$, which in terms of $L, k$ generators reads

$$C_{\Omega_1} = q^\frac{1}{2}L^{-1,0}L^{0,1}(k^{1})\frac{1}{2} + \frac{[(k^{1})^\dagger L^{0,1} - (k^{1})^\dagger L^{-1,0}]}{(q-q^{-1})(q^\frac{1}{2} - q^{-\frac{1}{2}})};$$  \hspace{1cm} (112)

one finds

$$B(1+q) = (k^{1})^\dagger + k^{-\frac{1}{2}} + q^2(q - q^{-1})(q^\frac{1}{2} - q^{-\frac{1}{2}})L^{-1,0}L^{0,1}(k^{1})\frac{1}{2}; \hspace{1cm} (113)$$

This allows an expression for $\Omega_1$ involving only $L, k$ generators. Finally, one replaces $L$'s as functions of $k^{1}, L^{0,1}, L^{-1,0}$ and finds the expression (110).

The case $N = 4$ can be proved in a similar way [13]. $\diamond$

3 The fundamental Hilbert space representations of $u_q(e^N)$

3.1 Construction

As noticed before, when $q \in \mathbb{C}$, $u_q(e^N)$ is a $*$ algebra, with $*$-relations (14), (25), (53). We remind that a $*$-representation $\Gamma [27]$ of $*$-algebra $A$ on a Hilbert $\mathcal{H}$ space (briefly: Hilbert space representation) is essentially a representation of $A$ such that $\Gamma(a^*) = \Gamma(a)^\dagger$ ( $T^\dagger$ is the adjoint of $T$) at least on a dense subset of the Hilbert space. In this section we will determine the fundamental representations of $u_q(e^N)$. We will find a basis of $\mathcal{H}$ and show how the generators of $u_q(e^N)$ are to be represented as operators on the elements of the basis. We will not deal with questions regarding domains of definition of the operators; this is premature at this stage, and is out of the scope of this work. The positivity of the scalar product

$$\begin{cases}
(u, u) \geq 0, \\
(u, u) = 0 \Rightarrow u = 0,
\end{cases} \hspace{1cm} \forall u \in \mathcal{H} \hspace{1cm} (114)$$

will be imposed $apriori$ at each step of our construction, and of course will be essential in determining the structure of the representations.

3.1.1 Spectra and eigenspaces of the squared momentum observables; the action of $p, k, L$ on the points of the “$q$-lattice”

Contrary to the classical case, the momenta $p$ don't commute with each-other, therefore cannot be chosen as (part of) a set of commuting observables to study the Hilbert spaces of the irreps. Among the commuting observables of a complete set characterizing an irrep we can always take (see sections 1.4, 2.1)

$$p_0, (p \cdot p)_1, ..., (p \cdot p)_{n-1}, (p \cdot p)_n; k^1, ..., k^n \hspace{1cm} (p_0 \equiv 0 \hspace{1cm} if \hspace{1cm} N = 2n)$$  \hspace{1cm} (115)

(in fact we will see that they actually make up a complete set for the “singlet ” irrep). Notice that if we had taken $(p \cdot p)_0$ instead of $p_0$ we could not distinguish between positive and negative
eigenvalues of $p_l$. It is easy to realize from the commutation relations of $u_q(e^N)$ that the sign of $p_l$ will be the same within each irrep.

We make an ansatz, assuming existence of eigenspaces of the first $n + h$ observables consisting only of normalizable eigenvectors; then we find that $\mathcal{H}$ consists of eigenspaces of normalizable eigenvectors, too.

Let $\mathcal{H}$ be the Hilbert base space of the considered irrep of $u_q(e^N)$. Given an eigenspace $\mathcal{H} \subset \mathcal{H}$ of $(p \cdot p)_n$ with eigenvalue $M^2$, $\mathcal{H}_{\tau_0} := \Lambda^{2 \tau + 2} \mathcal{H}$ $(\pi_n \in \mathcal{H})$ will be an eigenspace as well, with eigenvalue $M^2 q^{2(l+2)}$. Here $M^2$ is a nonnegative constant characterizing the irrep; it has the dimension of a mass squared and is determined up to integer powers of $q^2$. Each $\mathcal{H}_{\tau_n}$ will be an irrep of $u_q(e^N)$. Finally, the irrep of $u_q(e^N)$ characterized by $M = 0$, i.e. $(p \cdot p)_n \equiv 0$, is actually an irrep of $U_q^N$.

Therefore our problem is thus reduced to the study of the irreps of $u_q(e^N)$.

**Proposition 7** There are only the following two alternatives in $\mathcal{H}$:

\[
\begin{cases}
1) (p \cdot p) \equiv 0 & \text{identically} \\
2) (p \cdot p) > 0 & \text{strictly}
\end{cases}
\forall i = h, h + 1, ..., n; \quad (116)
\]

**Proof.** As a trivial observation, $(p \cdot p) \geq 0$, since $p^l p_j = q^{2l_1} (p^j)^* p_l$. Then $(p \cdot p)_n \equiv 0$ implies $(p \cdot p) \equiv 0$.

Suppose now that $(p \cdot p)_m > 0$ (starting from $m = n$), and assume (per absurdum) that there exists $|\phi \rangle \in \mathcal{H}$ such that $(p \cdot p)_{m-1} \langle \phi | = 0$. Formulae (15),(16) then imply $p^{\pm (m-1)} | \phi \rangle = 0$, $p^m | \phi \rangle \neq 0$. Because of relation (66), this in turn implies

\[
p^{-m} L^{1-m,m} | \phi \rangle \propto | L^{1-m,m}, p^{m-1} | \phi \rangle \propto p^{-m} | \phi \rangle \neq 0,
\]

and, by taking the norm and using relations (66),(67),(16), the contradiction:

\[
0 \neq | \phi | L^{-m,m-1} \cdot p^{-m} p^{m-1} L^{-m,m} | \phi \rangle =
\]

\[
< \phi | L^{-m,m-1} (L^{1-m,m} p^{m-1} p^{m-1} + q^{2l} p^{m} p^{m-1}) | \phi \rangle = 0. \quad \diamond
\]

In the sequel $M^2 > 0$. From (16) we find

\[
(p \cdot p)_l = p^l p^{-l} q^{2l} + q^{-2l} (p \cdot p)_{l-1} = p^{-l} q^{2l} (p \cdot p)_{l-1}
\]

Each $p^l p^{-l}$ is positive definite ($\Rightarrow (p \cdot p) > (p \cdot p)_{l-1} > 0 \quad \forall l$).

Assume that $|\psi \rangle \in \mathcal{H}$ is an eigenvector of all $(p \cdot p)_k$. According to eq. (16), $|\psi_{l,r} \rangle := (p^k)^* | \psi \rangle (r \in \mathcal{H})$ will also be an eigenvector of all of them; the eigenvalues of $|\psi \rangle$, $|\psi_{l,r} \rangle$ will differ by an integer power of $q$. Let $a_i$ be the eigenvalues of $(p \cdot p)_i$ on $| \psi \rangle$. The norm of $|\psi_{l,r} \rangle$ will be given by

\[
< \psi_{l,r} | \psi_{l,r} > = < \psi_{l-1,r} | \psi_{l-1,r} > + a_l - q^{2l-2} a_{l-1}
\]

If $a_{l-1} \neq 0$, there must exist a $r$ such that $(p^{-1})^{r+1} | \psi \rangle = 0$, otherwise the above norm would get negative for large $r$. In other words there must exist a state which is annihilated by $p^{-1}$.

Let us start by setting $l = n$ in the previous argument; then we iterate it by setting $l = n - 1, ..., h + 1$. We infer the existence of a nonempty subspace $\mathcal{H}_{\pi_n} \subset \mathcal{H}_{\tau_0}$ such that $(p \cdot p)_l = M^2 q^{2(l+2)}$, $p^{-l} \mathcal{H}_{\pi_{n+1}} = 0 \quad \forall l$. Let $\pi := (\pi_n, \pi_{n+1}, ..., \pi_0) \in \mathcal{N}_h$. We define $\mathcal{H}_l := (p^l)^{\tau_n} \cdot (p^{l+1})^{\tau_{n+1}} \mathcal{H}_{\pi_{n+1}}$. Clearly the maps $p^k : \mathcal{H}_l \rightarrow \mathcal{H}_{\pi_{n+1}}$ are invertible ($p^{-1}$ only in the orthogonal complement of its kernel), the inverse being $[p^k]^{-1} = [p^{-k}]^{\pi_{n+1}} p^{-k}$, and $[p^k]^{-1} = [p^{-k}]^{\pi_{n+1}} p^{-k}$ respectively, as one can easily check using equations (119). Therefore we arrive at the preposition.
Proposition 8 \( \mathcal{H} \) can be decomposed into the direct sum

\[
\mathcal{H} = \bigoplus_{p \in \mathbb{G}^{n-1} \times \mathbb{H}^n} \mathcal{H}_p, \quad \mathcal{H}_p := \Lambda^p((p^p)^{\tau_n}, \ldots (p^{p+1})^{\tau_n}) \mathcal{H}_\circ, \tag{121}
\]

of orthogonal eigenspaces \( \mathcal{H}_p \) of the observables \( (p \cdot p) \).

\[
(p \cdot p) \mathcal{H}_p = M^2 q^l, \quad \mathcal{H}_p \quad l = h, h + 1, \ldots, n. \tag{122}
\]

In the case \( N = 2n + 1 \) we will attach superscripts \( \pm \) when we want to specify that we are dealing with irreps characterized by positive (resp. negative) eigenvalues of \( p_0 \). Then:

\[
p_0 \mathcal{H}_\circ = \pm M |1 + q^{-1}|^{1/2} q^{i+1} \mathcal{H}_\circ^{\pm}. \tag{123}
\]

Remarks. As expected the spectra of \( (p \cdot p) \) are discrete; they are particularly simple, since they consist only of \( q \)-powers. Note that none of them contains the zero eigenvalue (but the latter is an accumulation point of the spectra); in particular \( (p \cdot p)_n \) is zero always, i.e. there is no state in which the nonrelativistic quantum particle is at rest. Furthermore, note that fixing the values of the observables \( (p \cdot p)_l \) selects a \( n \)-Torus within momentum space, in other words \( \mathcal{H}_p \) will consist of states with a “support” confined in \( n \)-Torus \( T_n \) in momentum space. As we will see, \( h_k := \log q \) \( (i = 1, 2, \ldots, n) \) will play the role of “action variable” observables conjugated to the \( n \) “angle” variables of the torus.

Assume that \( |\phi > \in \mathcal{H} \) is an eigenvector of \( k^l \); \( k^l |\phi > = \lambda_l |\phi >, \forall i = 1, \ldots, n \). It is easy to realize from eqs. (46), (67), that application of all generators of \( u_0(e^N) \) to \( |\phi > \) will yield eigenvectors of \( k^l \) with eigenvalues \( \lambda q^j \), with fixed \( \lambda_i \) and \( j := (j_1, \ldots, j_n) \in \mathbb{Z}^n \). We can assume without loss of generality that \( 1 \geq \lambda_i > q^2 \). We will see in the sequel which further restrictions there are on the values of \( \lambda_i, j_i \). Summing up, our Hilbert space will be spanned by orthonormal vectors \( |\tilde{x}, \tilde{j}; \alpha > \) such that

\[
(p \cdot p) |\tilde{x}, \tilde{j}; \alpha > = M^2 q^j, \quad |\tilde{x}, \tilde{j}; \alpha > = \lambda q^j |\tilde{x}, \tilde{j}; \alpha >. \tag{124}
\]

\( \alpha \) stands for possible further labels necessary to completely identify the vectors of a basis of \( \mathcal{H} \). They occur if the set of commuting observables (115) is not complete on \( \mathcal{H} \); they label the eigenvalues of the commuting observables which are to be added to the ones reported in formula (115), to get a complete set.

In the case \( N = 2n + 1 \) we will attach superscripts \( \pm \) when we want to specify that we are dealing with irreps characterized by positive (resp. negative) eigenvalues of \( p_0 \). Then:

\[
p_0 \mathcal{H}_\circ = \pm M |1 + q^{-1}|^{1/2} q^{i+1} \mathcal{H}_\circ^{\pm}. \tag{125}
\]

In the sequel we look for a definition of the action of all the generators of \( u_0(e^N) \) on these vectors which is consistent with all the algebra relations; maximal extension of the domain of these operators (in particular of the symmetric ones, in order to get self-adjoint operators in \( \mathcal{H} \)) is out of the scope of this work. To analyze the action of the other generators of \( u_0(e^N) \) it is convenient to separate the action of changing \( \tilde{x} \) from that of changing \( \tilde{j} \) by using the operators \( L \) instead of the operators \( L \); in fact, relation (99) implies \( L : \mathcal{H}_\circ \rightarrow \mathcal{H}_\circ \). Then
Theorem 4. On a basis \( \{|\vec{x}; \vec{j}, \alpha>\} \) (\( \vec{x} \in \mathbb{R}^n \times \), \( \vec{j} \in \mathbb{Z}^n \)) of \( \mathcal{H} \)

\[
p'|\vec{x}; \vec{j}, \alpha> = M[1 - q^{(2n-1)+1}]^\frac{1}{2} \sum_{i=1}^{2(1+\tau_2)} |\vec{x} + \vec{e}_{i-1}; \vec{j} + \vec{y}_{i}, \alpha'>
\]

\[
p'^-|\vec{x}; \vec{j}, \alpha'> = M[1 - q^{(2n-1)}]^{-\frac{1}{2}} \sum_{i=1}^{2(1+\tau_2)} |\vec{x} - \vec{e}_{i-1}; \vec{j} - \vec{y}_{i}, \alpha'>.
\]

Here \( l > h, \vec{e}_i \in \mathbb{R}^n, \vec{y}_i \in \mathbb{Z}^n \) with \( \vec{e}_1^n = \vec{e}_1^n \). We have

\[
\begin{align*}
p_0|\vec{x}; \vec{j}, \alpha> &= \pm M[1 + q^{-1}]^\frac{1}{2} q^{\tau_2} |\vec{x}; \vec{j}, \alpha>, & \text{if } N = 2n+1, \ |\vec{x}; \vec{j}, \alpha> \in \mathcal{H}^k; \\
p^\pm|\vec{x}; \vec{j}, \alpha'> &= M q^{\tau_2} |\vec{x}; \vec{j} \pm y_1, \alpha'>, & \text{if } N = 2n
\end{align*}
\]

(we have set all the arbitrary phase factors equal to 1). Moreover

\[
\begin{align*}
L^{-m,m}_\tau|\vec{x}; \vec{j}, \alpha> &= D_m(\vec{x}, \vec{j}, \alpha)|\vec{x}; \vec{j} + y_m - y_{m-1}, \alpha'>, \\
L^{m,-m-1}_\tau|\vec{x}; \vec{j}, \alpha> &= D_{-m}(\vec{x}, \vec{j}, \alpha)|\vec{x}; \vec{j} + y_m - y_{m-1}, \alpha'>, \\
k|\vec{x}; \vec{j}, \alpha> &= \lambda q^{\prime} |\vec{x}; \vec{j}, \alpha>
\end{align*}
\]

The coefficients \( D_m, D'_m \) depend on the particular irrep under consideration; the precise domain of \( \vec{j} \) will be given in formula (152).

Proof. Let us consider for instance the proof of relation (7.10). One starts from the ansatz \( p|\vec{x}; \vec{j}, \alpha> = A|\vec{x} + \vec{e}_{i-1}; \vec{j} + \vec{y}_{i}, \alpha> \), takes the norm of this vector, uses hermitean conjugation and knowledge of the eigenvalues of formula (7.7) to find \( |A| \); the arbitrary phase factor of \( A \) is taken equal to one for convenience. The equality \( \lambda_1 = \lambda_1 \in [1, q^2] \) will be proved in next subsection. \( \diamond \)

Formula (7.15) implies that the range of \( j_1 \) in the case \( N = 2n \) is

Theorem 4 summarizes the essential features of the promised “\( q \)-latticization” in momentum space \( \mathbb{F}_q \). For each vector \( \vec{x} \), the equations \( \nu(p,p) = M^2 q^{|\vec{x}|} \) single out a \( n \)-torus submanifold \( T^n \) within \( \mathbb{F}_q \), on which the states of \( \mathcal{H}_q \) have support. However, the additional specification of a vector \( \vec{j} \) selects in \( \mathcal{H}_q \) state(s) having well-defined angular momentum components \( k \), but no well-defined \( p \)-angles: in other words the support of each state is not concentrated on a point of \( T^n \subset \mathbb{F}_q \). For no choice of a complete set of commuting observables the corresponding eigenvectors would have a point-like support in \( \mathbb{F}_q \), since no such set can include \( N \) functions of the \( (n \)-non-commuting variables) \( p \)'s. The \( q \)-lattice \( \{ (\vec{x}, \vec{j}) \} \) has to be understood in a space where \( n + 1 - h \) dimensions (corresponding to the first \( n + 1 - h \) observables (115)) are of “momentum” type, and the remaining are of “angular momentum” type. The action of the generators \( p, L, \Lambda^{\pm 1} \) on a vector \( |\vec{x}; \vec{j}, \alpha> \) can be visualized as a mapping of the point \( (\vec{x}, \vec{j}) \) of the \( q \)-lattice into one of its nearest neighbour points.

In fig. 1 we draw the \( 1 \)-tori \( T^1 \subset \mathbb{F}_q \) (circles) and the corresponding action of \( \Lambda^{\pm 1}, p^{\pm 1} \) in the case \( N = 3 \).

The number and the form of the extra parameters \( \alpha \), the domain of \( \vec{j} \)'s, the values of \( \lambda_1 \) of formula (124) and the explicit form of \( D_m, D'_m \) can be determined only after a detailed study of the structure of the subspaces \( \mathcal{H}_n \). Due to equations (98),(121), if we knew the structure of any subspace \( \mathcal{H}_n \) and the way the \( L \)'s operators act on it, we would be able to extend this knowledge in a straightforward way to all the other subspaces, through application to \( \mathcal{H}_n \) of powers of the momenta.
3.1.2 Structure of \( \mathcal{H}_q \)

As a particular subspace we take \( \mathcal{H}_q \); next, we are going to investigate its structure. Note that the “central charges” of formulae (103), (104) reduce to

\[
C_m|_{\mathcal{H}_q} = \begin{cases} 
0 & \text{if } m > h + 1 \\
\frac{q^{-1} - q^{1}}{1-q} & \text{if } N = 2n + 1 \text{ and } m = 1 \\
\frac{q^{1} - q^{-1}}{1-q} & \text{if } N = 2n \text{ and } m = 2 
\end{cases}
\]  

(130)

Correspondingly the triples \( T_m := L^{1-m,n}, L^{-m,m-1}, k^m(k^{m-1})^{-1} \) generate \( U_q(\mathfrak{su}(2)) \) subalgebras (since they satisfy the same commutation relations of \( M^{1-m,n}, M^{-m,n-1}, k^m(k^{m-1})^{-1} \) when \( m > h + 1 \), whereas in the remaining cases they generate subalgebras characterized by the following modified relations:

\[
[L^{0,1}, L^{-1,0}] = q^{1} \frac{k^{1} - (k^{1})^{-1}}{1-q^2}
\]  

(131)

if \( N = 2n + 1 \) and

\[
\begin{align*}
[L^{1,1}, L^{-2,1}] &= \frac{q^{1} - q^{-1}}{(1-q^2)(p-p^2)}, \\
[L^{1,1}, L^{-2,1}] &= \frac{q^{1} - q^{-1}}{(1-q^2)(p-p^2)}, \\
[1^{1,1}, L^{-2,1}] &= 0 \\
[1^{1,1}, L^{-2,1}] &= 0,
\end{align*}
\]  

(132)

\[
[L^{\pm 1,2}, L^{-3,1}] = \frac{(k^{1})^{-1}(k^{1})^{1}}{1-q^2},
\]  

(133)

if \( N = 2n \).

As a first task we want to determine the constants \( \lambda \) involved in the definition (124) of the eigenvalues of the \( k^i \).

To each triad \( T_m, m > h + 1 \), we can apply the representation theory of \( U_q(\mathfrak{su}(2)) \). \( \mathcal{H}_q \) can be completely decomposed into the direct sum of the representation spaces of the irreps of each \( U_q(\mathfrak{su}(2)) \) tripe \( T_m \), separately. Therefore well-known results concerning the irreps of \( U_q(\mathfrak{su}(2)) \) imply that there exists \( \lambda \) characterizing each irrep of \( T_m \) such that the eigenvalues of \( log_q([k^m](k^{m-1})^{-n}) \) are \(-1, 1, \ldots, 1\), implying \( log_q(\frac{\lambda}{\lambda - 1}) \in \) in all \( \mathcal{H}_q \). We recall that these well-known results follow from the existence of both an highest and a lowest weight within each irrep of \( U_q(\mathfrak{su}(2)) \).

It remains to evaluate \( \lambda \). We are going to show that this constant is not constrained by the representation theory of the algebra \( (131), (132), (133) \).

\( \mathcal{H}_q \) can be completely decomposed into the direct sum of the representation spaces \( \mathcal{H}_q^s \) of the \( s \)-irreps of these latter algebras. Let us study them.

We start from the first algebra (i.e. with odd \( N \)). It is immediate to realize that, because of the embedding mentioned after Proposition 5, the casimir \( \Omega \) of the algebra \( (131) \) is formally given by the casimir \( \Omega = \Omega|_{\mathcal{H}_q} \) of \( \tilde{u}_q(e^2) \), namely (see formula (110))

\[
\Omega = L^{-10}L^{01}(k^1)^{1} + q^{1} \frac{(k^{1})^{-1} - (k^{1})^{1}}{(q^3 - 1)(q - 1)}.
\]  

(134)

Let \( \Omega \mathcal{H}_q = \omega \mathcal{H}_q^s \), \( \omega \in \) , let \( | \psi \rangle \in \mathcal{H}_q^s \) be an eigenvector of \( k^1, k^1| \psi \rangle = \mu^2| \psi \rangle \), and define \( | \psi_m \rangle := (L^{01})^m| \psi \rangle \). Then

\[
< \psi_{m+1} | \psi_m > = q^{1} < \psi_{m} | L^{-10}L^{01} | \psi_m > = q^{1} \frac{1 - \mu^{-2}q^{-2m}}{(q^3 - 1)(q - 1)}.
\]  

(135)
Assuming $\mu > 0$, we realize that $\langle \psi_{m+1} | \psi_{m+1} \rangle$ would get negative for large $m$ unless it vanishes for some $m$. The latter condition means the existence of a highest weight vector $|\tau_i > \in \mathcal{H}_G$,
\[
L^{n+1} |\tau_i > = 0, \quad k^j |\tau_i > = \tau_i^j |\tau_i > ,
\]
and $\omega^j = q^{\frac{1}{k^j-1}} |\tau_i^j >$. If we repeat the same argument with $|\psi_m > = (L^{-10})^m |\psi >$, we see that its norm keeps positive for large $m$, hence there exists no lowest weight vector and no restriction on the value of $\tau_i$. The initial vector $|\psi >$ can be reconstructed from $|\tau_i >$ through $|\psi > = a(L^{-10})^m |\tau_i >$ (with some $a \in $); in fact each power of $L^{-10}L^{10}$ can be expressed as a diagonal operator which is a combination of $\omega^j(k^1)^{-1}$ and a function of $k^1$, therefore $(L^{-10})^m (L^{10})^m$ is diagonal.

If it were $\mu < 0$ we would find a lowest weight vector and no highest weight, on the contrary; for the sake of brevity, in the sequel we will assume that $\mu > 0$.

Since, if $N = 5, 7, ...$, different subspaces $\mathcal{H}^2 \subset \mathcal{H}_G$ are mapped into each other by the remaining $L$ operators - as well as by the $p^l$ - in such a way that the $k^1$ eigenvalues are only rescaled by even powers of $a$, we infer that the constant $\lambda_i (1 \geq \lambda_i > q^2)$ involved in the $k^1$ eigenvalues is characteristic of the $\hat{u}_q(e^N)$ irrep and unconstrained. Its value is a function of the casimirs of the irrep. Note that no classical analogue of such representations with $1 > \lambda_i > q^2$, $\lambda_i$ is available.

Now we consider the $N = 2n$ case. Since the expression for the casimir of the algebra (132)+133) is not so simple, we prefer to use the Lemma 3 of the appendix to prove the existence of highest weight vectors (in the sense of vectors that are annihilated by $L^{k^1}$). In fact there are an infinite series, for, if $|\phi >$ is one, then $(L^{10})^l |\phi >$ is an independent one $\forall l \in $. For proving that the whole representation space can be reconstructed from the highest weight vectors one needs using the explicit expression for the casimir, and we omit the computations here. Reasoning as in the $N = 2n+1$ case one concludes as before that there is a unique constant $\lambda_i$ (involved in the $k^1$ eigenvalue) characterizing the irrep.

Let $P^+N$ be the subalgebra of $\hat{u}_q(e^N)$ generated by $p^l$, $l > h(N)$, and let
\[
\hat{u}_q^{N} := \hat{u}_q^{-N} \circ P^+N
\]
($u_q^N$ were defined at the end of section 2.3). As a second point comes

Theorem 5 The subspace $\mathcal{H}_G$ of "highest weight vectors", i.e.
\[
\mathcal{H}_G := \{ |\phi > \in \mathcal{H} \mid L|\phi > = 0, \quad p^{-L} |\phi > = 0 \quad \forall L \in u_q^{+N} , l > h \}
\]
is nonempty and $\mathcal{H}_G = \hat{u}_q^{-N} \mathcal{H}_G$. It is one-dimensional in the case $N = 2n$ and infinite dimensional in the case $N = 2n$. In the latter case a basis of $\mathcal{H}_G$ is provided by the vectors \{(\mu^1)^l |\phi > , \quad |\phi > is any nontrivial vector of $\mathcal{H}_G$. Using the results of Proposition 8 we can say that $|\phi >$ is cyclic in $\mathcal{H}$ w.r.t. the subalgebra $u_q^{N}$. (In the sequel by "the highest weight vector" we will mean a particular one of these vectors, in the case $N = 2n$). The eigenvalues $k^i$ of the operators $k^i$ are of the type $k^i = q^{2j} \lambda_i$, $j_i \in $, and the constant $\lambda_i$, $1 \geq \lambda_i > q^2$, is a function of the casimirs characterizing the irrep.

Proof. We note that it is sufficient to prove the theorem in $\mathcal{H}_G$, by proposition 41. The proof for $N = 3, 4$ amounts to the discussion preceding the claim. The general proof will be partially given in the appendix.
Contrary to the case of representation theory of $H_v$, due to the presence of non-vanishing $C_1, C_2$ in the commutation relations (131), (133) when $N$ is respectively odd, even. For this reason the application of $u^{-N}_q$ to a highest weight vector generates an infinite dimensional space (the whole $H_d$ when $N = 2n + 1$, its subspace characterized by odd/even $j_1$ if $N = 2n$).

In the sequel we will stick to irreps characterized by $\lambda = 1, q$ (among which we can find those having classical analogue).

For this class of irreps we can introduce a vector $\vec{w} \in \mathbb{R}$ such that $k^i |\phi> = q^{\vec{w}}|\phi>$. The vector $\vec{w}$ depends on the casimirs and together with the value of $M$ completely characterizes an irrep. We will therefore attach it as a superscript to the symbol $H$ and write $H^{\vec{w}}$ when we want to specify that we are considering the irrep with highest weight $\vec{w}$; correspondingly, we will attach superscripts to the ket symbols: $|\vec{0}, \vec{j}, \ldots, \vec{s}>^\vec{w}$; the highest weight vector itself will be denoted by $|\vec{0}, \vec{w}>^\vec{w}$. So far we have avoided a heavy notation distinguishing the abstract generators $p, L, k, \bar{L}$ of the Euclidean algebra from their realizations as operators belonging to a particular representation of the algebra. From now on we will denote by $\Gamma^{\vec{w}}$ the irrep with highest weight $\vec{w}$, and we will write $\Gamma^{\vec{w}}(p), \Gamma^{\vec{w}}(L), \ldots$ instead of $p, L, \ldots$ when we want to stress that we are dealing with operators on $H^{\vec{w}}$ representing $p, L, \ldots$.

**Definition.** We define the singlet irrep as the one characterized by the highest weight $\vec{w} = 0$. It will play a crucial role in the sequel, as one expects from comparison with the classical case.

It is immediate to verify that on the singlet irreps of $u_q(e^X)$, $\tilde{u}_q(e^X)$ the casimirs $\Omega_i$ take zero values (see formulae (110),(111)), since they annihilate the corresponding highest weight states. This is no surprise, since they vanish in the classical case as well.

Now we are going to determine possible highest weights $\vec{w}$ and construct generic (not necessarily singlet) Irrep of $u_q(e^X)$ from tensor products. It is easy to verify that by making the tensor product of the singlet irrep $\Gamma^{\vec{w}}$, $\tilde{U}^{\vec{w}}(e^X)$ and an irrep $\Gamma^N_{\lambda, \mu}$ of $U_N$ we find a reducible Hilbert space representation of $u_q(e^X)$ characterized by the same mass, as it occurs in the classical case. In fact, let $\tilde{H}^\vec{w} := \tilde{U}^{\vec{w}} \odot \tilde{H}^\vec{w}_{\lambda, \mu}$ and define $\tilde{\Gamma}^{\vec{w}}$ on $\tilde{H}^\vec{w}$ by

$$\tilde{\Gamma}^{\vec{w}} := (\Gamma^{\vec{w}} \otimes \Gamma^N_{\lambda, \mu}) \circ \phi, \quad \Gamma^N_{\lambda, \mu}(p') := 0 =: \Gamma^N_{\lambda, \mu}(p \cdot p), \quad \Gamma^N_{\lambda, \mu}(L) := \Gamma^N_{\lambda, \mu}(L),$$

where $\phi$ is the coproduct of $U_N(e^X)$. Then it is immediate to verify that $\Gamma^{\vec{w}}(p'), \Gamma^N(L), \tilde{\Gamma}^N(k')$ satisfy the commutations of $U_q(e^X)$ and the spectra of the operators $\tilde{\Gamma}^{\vec{w}}((p \cdot p))$ are the same as in the singlet irrep (i.e. the “mass” scale $M$ characterizing the irrep is the same); in fact, for instance, one can immediately verify that

$$[\tilde{\Gamma}^{\vec{w}}(L^{-m, m+1}), \tilde{\Gamma}^{\vec{w}}(L^{-m-1, m})]_{\vec{w}} = q^{1+2m} \left[ \frac{1 - k^m(k+1)}{q - q^{-1}} + C_m \right]$$

where the central charges $C_m$ were defined in formulae (103),(104).

We would like now to single out the irreps contained in $\tilde{\Gamma}^{\vec{w}}$. In the classical theory this can be done imposing the “wave-equations” on the tensor product space; each kind of wave equation selects the subspace corresponding to an irrep, which consists of the tensor product
of a one-dimensional representation (characterized by a vector $\vec{p}$) of the translation subalgebra and an Irrep of the little subgroup $SO(N-1) \subset SO(N)$ of the direction of $\vec{p}$ in the momentum space. An equivalent approach is to tensor this one-dimensional representation directly to the little group of $\vec{p}$. It seems difficult to apply either approach in the $q$-deformed case, since we don’t know natural embeddings $U_q(so(N-1)) \hookrightarrow U_q(so(N))$, except when $N = 3,4$. When $N = 3$ we have in fact a natural embedding $U_q(so(2)) \approx U(1) \hookrightarrow U_q(so(3))$: $U_q(so(2))$ is the classical subalgebra generated by $k^1$. In the case $N = 4$ we have also a natural embedding, since $U_q(so(3)) \approx U_q(su(2)) \hookrightarrow U_q(su(2)) \otimes U_q(su(2)) \approx U_q(so(4))$

However, there is a natural way to find the irreps contained in $\tilde{\Gamma}^2$ for any $N$, due to the fact that the representation is of highest weight type; it is the usual procedure that e.g. one uses to determine the irrep decomposition of the tensor product of two irreps of the same (classical) Lie algebra. Using orthogonality, one determines all the highest weight vectors contained in $\tilde{\mathcal{H}}^2$.

For the sake of being explicit let us consider the case $N = 2n + 1$. The vector

$$[\vec{0}, \vec{v}] > \varepsilon := [\vec{0}, \vec{v}] > 0 \otimes \vec{v} >$$

is clearly a highest weight vector of $u_q(e^N)$ (in the sense of theorem 1) and when we apply $u_q^{-N}$ to it we generate the Irrep $(\Gamma^d, \tilde{\mathcal{H}}^d)$; in particular $\Gamma^d(I; L^{-1}; 0) |\vec{0}, \vec{v}, \vec{s} \vec{\varepsilon} > \vec{v} > \in \tilde{\mathcal{H}}^d$.

Since the subspace of $\tilde{\mathcal{H}}^d$ with $\vec{j} = \vec{u} - \vec{y}_1$ is two-dimensional (it is spanned by the vectors $[\vec{0}, \vec{v} > 0 \otimes \vec{v} > 0 - \vec{y}_1 >, [\vec{0}, -\vec{y}_1 > 0 \otimes \vec{v} >$), its orthogonal complement is one-dimensional and we can easily verify that it is spanned by

$$[\vec{0}, \vec{u} - \vec{y}_1 > \vec{d} - \vec{g}_1 : = \left[ q^{-\frac{1}{2}}(u_1)_{q^{-1}} \right]^+ |\vec{0}, -\vec{y}_1 > 0 \otimes \vec{v} > - \left[ \frac{q \frac{1}{2}(1 + \lambda_1 q)}{q^2 - 1} \right] |\vec{0}, \vec{v} > 0 \otimes \vec{v} > (142)$$

(as usual, the tensor product scalar product is defined on the vector of a basis as the product of the scalar product of the tensor factors and is extended by linearity).

$[\vec{0}, \vec{u} - \vec{y}_1 > \vec{d} - \vec{g}_1$ clearly is a highest weight vector itself, because $\Gamma^d$ is a $s$-representation, and when we apply $u_q^{-N}$ to it we generate a different Irrep, $(\Gamma^d - \vec{g}_1, \tilde{\mathcal{H}}^d - \vec{g}_1)$. It is evident that if we reiterate this procedure $2n_1$ times we find $2n_1 + 1$ irreps.

The same argument can be applied in the case $N = 2n$ to the highest weight vector of $(\tilde{\mathcal{G}}^d, \tilde{\mathcal{H}}^d)$. We conjecture that all highest weights (for the class of representations with $\lambda_1 = 1, q$) can be obtained in this way. The final result is summarized in the

**Proposition 9** The irreps of $u_q(e^N)$ (characterized by integer $\lambda_1$) are highest weight irreps. Possible highest weights are of the form $\vec{z} \equiv 0, \vec{w} \equiv \vec{u} - \vec{y}_1$, $0 \leq l \leq 2n_1$, if $N = 2n + 1$, $\vec{u} \equiv \vec{w}(l, l') : = \vec{a} - l \cdot \text{sign}(u_2 - u_1)(\vec{y}_1 - \vec{v}_1) l' (\vec{y}_1 - \vec{v}_1)$, $0 \leq l \leq [u_2 - u_1], 0 \leq l' \leq u_2 + u_1$, if $N = 2n$; $\vec{u}$ denote weights of $U_q^N$. In particular, when $N = 3,4$ the sets $\{\vec{w}\}$ of weight satisfy the relations $\{w_1\}, \{\vec{w}\} \subset \odot$ respectively. We have the following tensor product decomposition

$$\tilde{\mathcal{G}}^d = \bigoplus_{l=0}^{2n_1} \Gamma^d - \vec{g}_1, \text{ if } N = 2n + 1$$

$$\bigoplus_{l=0}^{u_2 + u_1 + 1} \Gamma^d(l, l'), \text{ if } N = 2n + 1$$

Highest weight vectors can be easily determined from the above described tensor product construction procedure.

Note that the irreps with $w_1 < 0$ have no classical analogue.
3.1.3 Moding out singular vectors in the singlet irrep

According to theorem 5, \( \mathcal{H}_2 \) is generated by application of the Borel subalgebra \( u_{\gamma}^{-N} \) to its highest weight vector. A Poincaré-Birkhoff-Witt basis for \( u_{\gamma}^{-N} \) is the set of monomials 
\[
(L_{\alpha_i})^{m_i} \ldots (L_{\alpha_1})^{m_1} (p^{i \gamma}(m_i)_{-1})^{b_{i-1}},
\]
where 
\[
s = \begin{cases} 
2n & \text{if } N = 2n + 1, \\
(n(n-1)) & \text{if } N = 2n.
\end{cases}
\]
by \( L_\alpha \), we denote the \( L^{i\alpha} \) corresponding to the negative root \( \alpha_i \) \( (i = 1, 2, \ldots, s) \), and the roots have been ordered according to an admissible total order \( (\alpha_i > \alpha_i \text{ if } i > l) \). Then, to each such monomial there corresponds a vector of \( \mathcal{H}_2 \); not all of these vectors, however, can be considered as independent.

In fact, as in the classical case, the requirement that the representation is of Hilbert space type makes many combinations of the previous vectors singular. We remind that a vector \( \langle \chi \rangle \) is said to be singular if \( \langle \psi|\chi\rangle = 0 \forall \psi \in \mathcal{H}_2 \). The simplest examples of singular vectors can be found in the singlet representation when \( N > 4 \). They are \( L^{-i_{\alpha_i}}\bar{0}, \bar{0} \rangle \), \( i > h + 1 \). Actually they are orthogonal to all vectors with different weights, by the hermicity of the \( k^l \), and have zero norm because of eq (102), (130) and the definition of the singlet irrep. Therefore we have to set
\[
L^{-i_{\alpha_i}}\bar{0}, \bar{0} \rangle = 0
\]
if we want the basic requirement (114) of a Hilbert space to be satisfied. A slightly more complicated example is provided in the same representation and for \( N \geq 5 \) odd by the vectors \( |1\rangle := L^{-1,0} L^{-2,1} L^{-4,0} \bar{0}, \bar{0} \rangle, |2\rangle := L^{-2,1} L^{-4,0} \bar{0}, \bar{0} \rangle, \) since we find, after straightforward use of formula (102), Proposition 5,
\[
|2\rangle := |1\rangle - \frac{1}{q + q^{-1}} \Rightarrow <2|2\rangle = 0 \quad \Rightarrow \quad <\psi|2\rangle = 0 \quad \forall \psi \in \mathcal{H}_2.
\]
When \( N = 2n \) another example of singular vector is
\[
|\tau\rangle := (L^{-2,-1} - q^2 L^{-2,1} \frac{p^{-1}(k_{-1})_{-1}}{(p+q)^2}) \bar{0}, \bar{0} \rangle.
\]
as it can be easily verified by applying commutation relations (100),(102).

An Irrep of our algebra is obtained by moding out all singular vectors which are generated in the previous construction. Now we want to identify them.

Let us stick for the moment to the singlet representation of \( u_{\gamma}^{-N} \).

Theorem 6 In the singlet Irrep the commutation relations
\[
\begin{cases}
  g_1 := L^{-2,-1} - q^2 L^{-2,1} \frac{p^{-1}(k_{-1})_{-1}}{(p+q)^2} = 0 \\
  \bar{g}_1 := L^{-2,-1} - q^2 L^{-2,1} \frac{p^{-1}(k_{-1})_{-1}}{(p+q)^2} = 0 \\
  g_0 := L^{-i_{\alpha_i} - 1} L^{-i_{\alpha_i} - 1} - \frac{(k_{-1})^2}{q} - q^{-1} \frac{1}{q} L^{-i_{\alpha_i} - 1} L^{-i_{\alpha_i} - 1} = 0
\end{cases}
\]
and their hermitian conjugates hold. All singular vectors of the singlet representation can be obtained by application either of any element of \( u_{\gamma}^{-N} \) to a singular vector, or of some \( g_i \) to a nonsingular vector. Therefore if we mode out the singular vectors, an orthonormal basis \( B_\gamma \) of \( \mathcal{H}_2 \) will consist of the vectors
\[
|\bar{0}, \bar{j}\rangle := (L^{-n,1} j_i \ldots (L^{-n,1} j_i - 1)(L^{-2,1})_{j_i} \left\{ (L^{-1,0} j_i) \frac{\bar{0}, \bar{0} \rangle}{(p+q)^2 j_i} \right\} \bar{0}, \bar{0} \rangle \quad \text{if } N = 2n + 1
\]

\[
(149)
\]
where the integers \( J_i \) satisfy the conditions \( J_{i+1} \leq J_i \) if \( i \geq h+1 \). The integers \( \bar{J} \in \mathbb{N} \) specify the eigenvalues of \( \mathcal{K} \) and are given by

\[
j_i = J_i - J_{i+1} \geq 0 \quad (J_{n+1} \equiv 0)
\]

**Proof:** see the appendix.

**Remarks**

- It is easy to show that the above relations are compatible with the Serre relations of Proposition 5, and the commutation relations (98), namely they yield identities \( 0 = 0 \) when plugged into the latter. The validity of commutation relations (147), (148) depends crucially on the fact that \( C_1 \neq 0, C_2 \neq 0 \) in relations (102).

- As a consequence of formula (149), each weight in the singlet irrep has multiplicity 1, in other words we don’t need extra parameters \( \alpha \) to identify the vectors of the basis \( \mathcal{B}_\alpha \). This is a strong indication that it is convenient to introduce a “configuration space realization” of the singlet irrep; actually we will see in section 3.2 how to represent its vectors as functions on the quantum Euclidean space.

By inverting relations (150) one gets the equivalent relation:

\[
J_i = \sum_{\bar{J} \in \mathbb{N}} j_i
\]

Let us consider now a generic representation \((\Gamma^\mathcal{F}, \mathcal{H}^\mathcal{F})\). Its singular vectors can be determined using relations (147),(148) in \( \mathcal{H}^\mathcal{F} \) and the tensor product construction of \( \Gamma^\mathcal{F}, \mathcal{H}^\mathcal{F} \), described in Proposition 9. Of course, in building \( \mathcal{H}^\mathcal{F} \) through application of \( u^\alpha \) to the highest weight vector \( |0, \bar{u} > >, \) one has to impose relations (147),(148) only on the generators \( L \) acting on the first tensor factor (the singlet representation) of the product (139). The existence of singular vectors in \( \mathcal{H}^\mathcal{F} \) is therefore only due to the existence of singular vectors in the representation \( \Gamma^\mathcal{F} \) of \( u^\alpha \).

**3.1.4 Summary: the structure of \( \mathcal{H}^\mathcal{F}, \mathcal{H}^{\mathcal{F}_\alpha} \)**

Finally, let us summarize and comment a little on the structure of the pre-Hilbert spaces \( \mathcal{H}^\mathcal{F} \).

First, we note that the domain of the labels \( \bar{J} \) of the vectors \( | \bar{J}, \bar{J}, \alpha > \) forming an orthogonal basis of \( \mathcal{H}^\mathcal{F} \) is:

\[
\mathcal{J} := \{ \bar{J} \in \mathbb{N} \mid j_i \leq \pi_{i-1} + w_i, \quad i = h+1, h+2, \ldots, n; \quad j_1 \in \mathbb{N} \}
\]

This follows from inequalities (150), the tensor product construction of Proposition 9 and formulae (98).

**Remark.** The explicit form of the highest weight vectors (which can be determined according to the procedure sketched in the proof of Proposition 9) and the knowledge of the singular vectors of the singlet representation allow in principle to fix each the extra parameter \( \alpha \) appearing in formulae (26),(129) and the coefficients \( D_n, D_{n+1} \) appearing in Theorem 4.

We give an intuitive picture of the physical content of the spectra of the observables (115) in the singlet representation. The subspace \( \mathcal{H}^\mathcal{F}_\alpha := \bigoplus_{| \bar{J}, \bar{J}, \alpha >} \mathcal{H}^\mathcal{F}_\alpha \) is the eigenspace of the observable \( p^{\perp} p = (p \cdot p) - (p \cdot p)_{\perp} \) with the minimum eigenvalue compatible with a given eigenvalue of
\[ (p \cdot p)_i = M^2 q \sum_{a=0}^{2(1+n)} (q^2 - 1); \] the latter quantity never vanishes when \( q \neq 1 \). This means that the there is always a "point zero" momentum component available in the plane of the coordinates \( i, -i \). Now let us ask in which "directions" of this plane this point zero momentum component can be pointed.

The admitted eigenvalues of \( m_0(k) \), i.e. of the angular momentum component in the plane, are \( j_1 \leq 0 \) (see (150)) and show that (except when \( N = 2m, i = 1 \)) only a "clockwise" or "radial" orientation are possible. The anticlockwise is excluded! If \( N = 3 \), for instance, minimum \( p^1 p_1 \) means that the momentum is almost pointed in the \( p^0 \) direction: \( j_1 \) represents the \( p^0 \)-direction component of the (orbital) angular momentum: therefore the "orbital helicity" \( j_1 p_0 \) has a definite sign. We find a sort of a purely "kinematical" parity asymmetry of the allowed momentum space, which is a surprising feature for a lattice theory: in fact, at least usual equispaced lattice theories, which are commonly used nowadays for regularization purposes, cannot have a parity asymmetry by a well-known no-go-theorem [22]. In next section we will see in which sense in the classical limit \( q \to 1 \), however, parity symmetry is recovered.

Both for odd and even \( N \) value of \( j_1 \) is not bounded from below; larger and larger absolute values of the angular momentum with a fixed amount of momentum available in the plane \( i, -i \) (in particular, with the minimum amount \( p^{-1} p_{-1} = (p \cdot p)_1 = (p \cdot p)_{1,-1} \) can be intuitively described only if the corresponding states have larger and larger values of the mean distance from the origin of the plane. Thus we can visualize the states of \( \mathcal{H}_q \) with larger and larger \( |j_1| \) as states with larger and larger mean distance from the origin in the \( x^j, x^{-j} \) plane.

### 3.2 Configuration space realization of the singlet irrep

As known, the undeformed Euclidean algebra \( e^N \) can be realized in the singlet representation as the algebra of differential operators acting on suitable subspaces of the algebra of "functions" on \( N \) configuration space (e.g. the subspace of smooth functions, or square integrable, or distributions...). Working in configuration-space realization rather than at an abstract level is very useful in the classical context for many purposes; for instance, questions regarding in concrete cases the domain of definition of some elements of \( U(e^N) \), considered as operators in the singlet representation, e.g. the essential self-adjointness domain of would-be observables, Fourier (anti)transforms performed by integration in configuration (or momentum) space, etc. are best treated in configuration (or, depending on cases, momentum) space realization. The occurrence of distributions is (ultimately) related to the the fact that the spectrum of each position/momentum operator is continuous.

In the case \( q \neq 1 \) the spectrum is discrete, as we have seen, and the corresponding eigenvectors are normalizable. In the preceding sections we have investigated in an abstract way the singlet irrep of \( u_q(e^N) \), we would now be happy to find its corresponding \( q \)-deformed configuration-space counterpart, if any, in view of further developments of the theory (concerning, for instance, domain or Fourier transform questions). The existence of such a counterpart is expected also because theorem 6 shows that the notation with \( L \) generators applied to \( \hat{0}, \hat{0} \) is too heavy to identify a basis of \( \mathcal{H}_q \) once one has moded out the singular vectors. The vectors of the basis \( B_q \) of formula (149) are labeled just by the corresponding powers \( m, j \) of the generators of \( u_q(e^N) \) applied on \( \hat{0}, \hat{0} \); in other words these vectors don’t depend (apart from a factor) on the order in which these generators are applied. This result can be obtained in a natural way in a configuration-space realization.

The standard way to represent momenta operators \( p^j \) satisfying the reality relations (84)
through $\varphi$-derivatives would be to define them as the real combination (85): then, the scalar product of two vectors would be computed formally as in the case of undeformed quantum mechanics by a $\varphi$-integral involving the corresponding two wave-functions. Concrete computations would be performed formally as in the case of undeformed quantum mechanics by $\varphi$-derivatives would be to define them as the real combination (85); then, the scalar product of two vectors would be computed formally as in the case of undeformed quantum mechanics by a $\varphi$-integral involving the corresponding two wave-functions. Concrete computations in this case would be however extremely hard; in fact, the derivation relations of $x = q^N \partial^x + \partial^x$ with $x$ have no simple expression such as (8) in terms of some $\mathcal{R}$ matrix, but involve $\Lambda, i^{1/2}, B$ in the RHS, so that solving the eigenvalue equations gets rather involved. In Ref. [8] we solved a similar problem in studying another quantum mechanical physical system (the harmonic oscillator) on $\mathcal{N}$ by using a non-standard way of realizing the momentum operators $x$ in “configuration space representation” as a pair of realizations such as $\partial, \bar{\partial}$. We adopt the same approach here (see also Ref. [12]). As for the generators $J_k$ of the subalgebra $\mathcal{U}_{q^N} \subset \mathcal{U}_q(e^N)$, the choice of the standard or nonstandard approach is irrelevant, since they are represented by the same differential operators in both of them and therefore their hermitean conjugation will amount to $s$-conjugation in $\mathcal{D}_q^N$.

We can introduce a pair of realizations (what we call the unbarred and the barred) of the algebra of $\mathcal{U}_q(e^N)$, i.e. a pair of algebra homomorphisms $\rho, \bar{\rho}$

$$\begin{align*}
\mathcal{D}_q^N, \quad \text{Fun}(\mathcal{N}) & \xrightarrow{\rho} \mathcal{U}_q(e^N), \quad \mathcal{H} \\
\mathcal{D}_q^N, \quad \text{Fun}(\mathcal{N}) & \xrightarrow{\bar{\rho}} \mathcal{H}
\end{align*}$$

(153)

through

$$\begin{align*}
\rho(\mu^x) & := -i \Lambda^x \partial^x, \\
\rho(\mu^x) & := -i \bar{\partial} \Lambda^{-x} q^{-N(1+\psi)}, \\
\rho(\mu^x) & := \rho(u) \quad \forall u \in \mathcal{U}_q^N, \\
\rho(\Lambda) & := \Lambda := \bar{\rho}(\Lambda)
\end{align*}$$

(154)

$(a \in \mathbb{C})$. It is easy to check that the hermitean conjugation in $\mathcal{U}_q(e^N)$, defined in formulae (4.30),(4.90) (we call it $\dagger$ here, to avoid confusion) satisfies the relation

$$\rho(u^\dagger) = [\rho(u)]^\ast, \quad \bar{\rho}(u^\dagger) = [\bar{\rho}(u)]^\ast,$$

(155)

where $\ast$ denotes the complex conjugation in $\mathcal{D}_q^N$. To be precise we define the homomorphisms $\rho, \bar{\rho}$ first on the vectors of the basis $\mathcal{B}_q$ (see theorem 6), and then we extend them linearly to all of $\mathcal{H}$.

The $\rho, \bar{\rho}$-images of the vectors of the basis $\mathcal{B}_q$ are determined by the requirement that they satisfy $\varphi$-differential equations which are respectively the $\rho$- and $\bar{\rho}$-image of the equations satisfied by $\varphi^x$. Actually we can limit ourselves to the search of $\rho$- and $\bar{\rho}$-images of the highest weight vector(s), since they are cyclic in the representations spaces. If we find solutions to these differential equations we obtain (two) representations of $\mathcal{U}_q(e^N)$; then we have to see whether they are $s$ representations and, if so, identify them with some irreps studied in the previous section.

This is the program before us. We will see that it can carried through, and that we obtain a unique configuration-space realization of the singlet irrep based on the use of a pair of non-$s$-representations (the unbarred and the barred).

All choices of $a$ in equation (154) are essentially equivalent. The choice $a = -1$ will be particularly convenient for representing scalar products in $\mathcal{H}^0$ as integrals of functions on $\mathcal{N}_q$, and we adopt it here.

Then we can easily prove the important
Lemma 1 If the vector $|\phi\rangle$ satisfies the equations

$$ (A + A^\dagger p_i + u)|\phi\rangle = 0, \quad A \in \mathbb{R}, \quad (p \cdot p)|\phi\rangle = M^2|\phi\rangle $$

and $\varphi := \rho(|\phi\rangle) \in \text{Fun}(\mathcal{H}_q)$ is the function that represents $|\phi\rangle$ in the unbarred realization with $a = -1$, then its barred partner $\bar{\varphi} := \bar{\rho}(|\phi\rangle)$ is given by

$$ \bar{\varphi}(x) = e_q[-m^2 x \cdot x] \varphi(q^{-1} x), \quad m^2 := (q^{-2} - 1)q^{-2}h^{-1}M^2 $$

Proof. After a shift $x \to q^{-1}x$, $\partial \to q\partial$ the hypothesis reads

$$ (A - iA^{-1}q\partial_i + u)\varphi(q^{-1}x) = 0, \quad a \in \mathbb{R}, \quad (p \cdot p)\varphi(q^{-1}x) = M^2\varphi(q^{-1}x); $$

then the thesis can be verified by a straightforward calculation, using the expression (22) of $\partial$ in terms of $\partial, x$ and formulae (18), (19). \top

Let us define

$$ e_q[Z] := \sum_{n=0}^{\infty} \frac{Z^n}{(n)_q!}, \quad \varphi'(Z) := \sum_{n=0}^{\infty} \frac{[-Z]^n}{(n)_q!}, \quad (n)_q := \frac{q^n - 1}{q - 1}. $$

Now comes the

Proposition 10 One can solve the highest weight vector conditions (138) in the $\rho, \bar{\rho}$ realization. The resulting representations of $\bar{\rho}(e^n)$ on $\text{Fun}(\mathcal{H}_q)$ coincide with the singlet representation $(1^0, \mathcal{H}_q^0)$. The cyclic highest weight vector $|0, 0\rangle$ is represented in the unbarred realization by

$$ \rho(|0, 0\rangle) := \varphi_0 = \begin{cases} 
\rho_{-1}[iM_x x^n], & M_0 = \pm (1 + q^{-1})q^{n+1} \quad \text{if} \quad N = 2n + 1 \\
\sum_{n=0}^{\infty} \frac{[-(x^{-1})^n M]}{(n)_q!}, & M_0 = q^{n+1}M \quad \text{if} \quad N = 2n.
\end{cases} $$

The subspace $\rho(\mathcal{H}_q^0) \subset \rho(\mathcal{H}_q)$ is spanned in the unbarred representation by $\rho(B_q)$, whose elements are the functions

$$ \rho(|0, 0\rangle) \propto (x^{-n})^{j_1} \cdots (x^{-n})^{j_i}, \begin{cases} 
(x^{-1})^j e_q^{-1}[iM_x x^n], & N = 2n + 1 \\
(x^{-1})^j e_q^{-1}(x^{-1}) M^n, & N = 2n
\end{cases} \quad \text{if} \quad i \geq h, $$

where $J_1$ was defined in formula (151).

Proof. The most general expression for the function representing the cyclic vector is

$$ \varphi_0 = \sum_{|\alpha| = 0}^{\infty} \sum_{i=0}^{N} A_{i, \alpha \cdots \alpha \cdots} (x^{-n})^{j_{i+1}} \cdots (x^{-n})^{j_i}.$$

The requirement that it is annihilated by $\rho(p^{-i}), i > h$, implies that there can be no dependence on $x^i$ (take in the order $i = n, n - 1, \ldots$ and perform the derivations). Similarly, since on $\mathcal{H}_q$ $L^{-i,i} = L^{-i,i} + \text{cost} \cdot p_i p^{-i}$, the requirement that it is annihilated by $L^{-i,i}$ implies that there can be no dependence on $x^{-i}$. This is straightforward to check when $i > h + 1$, and a little more lengthy when $i = h + 1$. In the case $N = 2n + 1$, for instance, it is easy to check that

$$ L^{-1,1} \varphi_0 = 0 \Rightarrow \begin{cases} 
A_{i-1,i+1} \alpha_{\cdots} \alpha_{\cdots} \propto A_{i-1,i+1} \alpha_{\cdots} \alpha_{\cdots} \quad \text{when} \quad i_{\perp} \geq 0 \\
A_{i-1,i+1} \alpha_{\cdots} \alpha_{\cdots} \propto A_{i-1,i+1} \alpha_{\cdots} \alpha_{\cdots} \quad \text{when} \quad i_{\perp} \geq 1
\end{cases} $$

$$ \Rightarrow \text{which is the desired result.} \quad \Box$$
the first (resp. second) condition comes from setting the coefficient of $(x^{-1})^i \cdot (x^0)^{s-1} x^i$ (resp. $(x^{-1})^i \cdot x^i \cdot (x^0)^{s+1}$) equal to zero. They are incompatible, therefore $A_{s-1} = 0$ if $s > 0$. Finally the requirement that $\phi_0$ is an eigenvector of $\rho(p^3)$ in the case $N = 2n + 1$ or $\rho((p \cdot p_1) k^1)$ in the case $N = 2n$ with the prescribed eigenvalues (following from the conditions $(p \cdot p)_n = M^2$, $p^{-1} | \phi_0 > = 0$) yields the expression (160).

A direct application of the commutation relations (55), (57) yields the expression in formula (161) as the basis of $\rho(\mathcal{H}_5^q)$; the commutation relations (147), (148) can be easily checked applying them to the elements of this basis.

Thus we have obtained two representations of $u_q(e^N)$. They coincide with the singlet representation of the previous section since the eigenvalues of $k^i$ on the highest weight vectors coincide with those of the singlet representation both in the unbarred and in the barred case.

It remains to realize the scalar product yielding the hermiticity structure described in the abstract context in section 7.1.

The relevance of the double realization manifests itself in the

**Theorem 7** The scalar product in $\mathcal{H}_{\overline{5}}$ can be realized in configuration-space by

$$< \phi_1 | \phi_2 > = \int d_\gamma V [\tilde{\phi}_1]^* \phi_2,$$

where $\int d_\gamma V$ is the integration first defined in [8], with a suitable normalization.

**Proof.** Let $< \phi_1 | \phi_2 >' := RH S(7.55)$. Because of the lemma,

$$< \phi_1 | \phi_2 >' = \int d_\gamma V \ e_{\rho} [-m^2 x \cdot x] [\phi_1 (q^{-1} x)]^* \phi_2 (x)$$

$$m^2 := (1 - q^2) q^{-2} e^{-3} M^2; \quad (165)$$

$< , >'$ is a well defined inner product in $\mathcal{H}_{\overline{5}}$; in fact, the integral (7.56) is well defined according to the definition given in ref. [8], due to the presence of the the $q^2$-gaussian damping factor $e_{\rho} [-m^2 x \cdot x]$, which can be taken as "reference function of the integration": in the frame of that definition it suffices to expand $[\phi_1 (q^{-1} x)]^* \phi_2 (x)$ in powers of $x$, perform the integrations and resum the terms to obtain the above integral. From the practical point of view it is convenient, however, to choose directly $[\tilde{\phi}_1]^* \phi_2$, where $| \phi_0 > := |0, 0 >$ stands for the cyclic vector, as (non-scalar) reference function of the integration (in the sense mentioned in ref. [8]), since, as we will see below, all integrals of the form (165) can be evaluated from this basic integral.

Because the integration satisfies Stoke's theorem with derivatives $\partial, \bar{\partial}$ [8], i.e. "boundary terms" vanish, and formula (155) holds, the complex conjugation $*$ followed by an exchange of the two realizations acts as hermitean conjugation of all differential operators of $u_q(e^N)$ with respect to the scalar product of $\mathcal{H}_{\overline{5}}$. This ensures that the inner product $< , >'$ preserves the orthogonality relations between different vectors of the basis $\mathcal{B}_q$ of $\mathcal{H}_{\overline{5}}$ since they have different eigenvalues of the observables (115).

If $| \phi_i > = D_i | \phi_0 >, \ D_i \in u_q(e^N)$ ($u_q(e^N)$ was defined in formula (165)), $i = 1, 2$, then because of formula (155)

$$\int d_\gamma V \ [\tilde{\phi}_1]^* \phi_2 = \int d_\gamma V \ [\tilde{\phi}_2]^* \rho(D_i D_i) [\phi_0 >). \quad (166)$$

Only the $\rho$ image of the nonzero $| \phi_0 >$-component of $D_i D_i | \phi_0 >$, if any, will contribute to the above integral. This explains why the evaluation of all integrals of the form (165) is reduced to the basic integral $\int d_\gamma V [\tilde{\phi}_0]^* \phi_0$, as claimed. Finally, we normalize the integration so that $\int d_\gamma V [\tilde{\phi}_0]^* \phi_0 = 1$. This concludes the proof that $< , >' = < , >$. \hfill \Box
Remark 1 To recognize that the inner product $\langle \cdot, \cdot \rangle$ is sesquilinear it is convenient to consider its more symmetric but equivalent form

$$\langle \phi_1|\phi_2 \rangle := \int d\varphi V \left( [\varphi_1]^* \varphi_2 + [\varphi_1]^* \varphi_2 \right)$$

(167)

(alternatively one could include in the RHS only the second term). In fact, the sesquilinearity of the scalar product is immediate in this form. That this form is equivalent to the former can consider its more symmetric but equivalent form

alternatively one could include in the RHS only the second term. In fact, the sesquilinearity of the classical one. Only when $q \neq 1$ the damping factor $e^{i [-\frac{1}{2} x^2 + \mathcal{F}] \cdot x] \neq 1$ in the integral (165) makes the norms of eigenvectors of the observables (115) finite.

Remark 2. Formula (165) allows to explain from the point of view of configuration space the regularizing effect of taking the $q$-deformed version of the Euclidean Hopf-algebra instead of the classical one. Only when $q \neq 1$ the damping factor $e^{i [-\frac{1}{2} x^2 + \mathcal{F}] \cdot x] \neq 1$ in the integral (165) makes the norms of eigenvectors of the observables (115) finite.

Remark 3 The final lesson we learn from interpreting the results of sections 3.1,3.2 is the following. Requirement (114), of nonnegativity of the scalar product makes the representations of $u_q(e^N)$ of highest weight type. If we stick for simplicity to the highest weight singlet representation of $u_q(e^N)$, then the fact that power series in the coordinates $x^i$ change the component to subspaces $\rho(H_q^i), \rho(H_q^f) \subset Fun_q(e^N)$ which we have constructed) arise as a natural basis to identify elements of the singlet irrep of $u_q(e^N)$ (instead of the vectors obtained by applying combinations of the elements of a Poincare'-Birkhoff-Bott basis of $u_q(e^N)$ to the highest weight state) ultimately is traced back to the requirement (114) that the scalar product is nondegenerate. In fact, equation (114) eliminates a huge amount of singular vectors to appear within the space of such combinations; having moled the latter out, we have shown that the abovementioned power series in $Fun_q(e^N)$ are sufficient to identify the remaining vectors.

3.3 Classical limit of the singlet irrep

In this section we just briefly sketch what we are allowed to mean by “$\lim_{q \to 1} \Gamma_q = \Gamma_{0 \to 1}$”, i.e. by saying that an irrep of $u_q(e^N)$ goes to an Irrep $U(e^N)$ in the limit $q \to 1$. If we compare the behaviour of the representations $\Gamma_q$ of $u_q(e^N)$ under this limit with that of the representations of $U_q(e^N)$ (which is the “compact” subalgebra of $u_q(e^N)$), we find important differences. For simplicity we stick to the singlet Irrep of $u_q(e^N)$; the other Irreps of $u_q(e^N)$ can be obtained by the tensor product(139), and the properties of the Irreps of $U_q(e^N)$ are essentially known.

The commuting observables

$$p_0, (p \cdot p)_{h_1}, \ldots, (p \cdot p)_{h_{n-1}}, (p \cdot p)_{h_0}; h_1, \ldots, h_n$$

$$p_0 \equiv 0 \quad \text{if} \quad N = 2n + 1.$$  

(168)

($h_j := \log^2(k^i)$) make up a complete set both when $q \neq 1$ and $q = 1$. We have chosen $h_j$ instead of $k^i$ because it is the set of generators $\{E^j, h_k, p^i\}$ which has classical commutation relations in the limit $q \to 1$. The eigenvalues of the observables (115) label the vectors of an orthonormal basis $B_q$ (149) (of eigenvectors) of the singlet Irrep for all $q \in (0,1)$: when $q = 1$ the vectors of this basis are distributions (see the configuration space realization of the preceding section), i.e. $H_{q \to 1}$ is no more a Hilbert space but the space of functionals on some space of smooth functions on $N$, e.g. $\mathcal{S}(N)$.  

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For each fixed eigenvector \( |\vec{x}, j\rangle \) the eigenvalues \( j_i \) of \( h_i := \alpha q_{2i}(\vec{k}) \) don’t depend on \( q \) and are integers; whereas the eigenvalues of \((p \cdot p)_n\) (non-uniformly) “collapse” to \( M^2\):

\[
\lim_{q \to 1} c_\ell(q, \vec{x}) = M^2 \quad \text{where} \quad (p \cdot p)|\vec{x}, j\rangle =: c_j(q, \vec{x})|\vec{x}, j\rangle.
\]  

(169)

Therefore all vectors \( |\vec{x}, j\rangle \) of \( B_q \) with fixed \( j \) would have separately the same limit when \( q \to 1 \), and the latter would coincide with a vector of \( B_{q=1} \); consequently, \( \{\lim_{q \to 1} |\vec{x}, j\rangle, |\vec{x}, j\rangle \in B_q\} \neq B_{q=1} \). Therefore the limit \( \lim_{q \to 1} \Gamma_q = \Gamma_{q=1} \) cannot be given a literal sense.

However, we can give a weaker sense to the above limit, as we are going to explain. Assume that \( q = 1 \) and the vector \( \vec{r} := (r_1, \ldots, r_{n-1}, r_n) \) consists of components \( r_n \in \mathfrak{s} \), \( 0 \leq r_i \leq 1 \), \( h \leq i \leq n-1 \), \( \vec{r} \in \pi \); let \(|\vec{x}, j\rangle \in B_{q=1} \) be the vector with eigenvectors \( h_i = -j_i \), \((p \cdot p)_n = (p \cdot p)_{i+1} r_i = (p \cdot p)_{i} = M^2 r_i \). Define functions \( \pi_i(r_i, q) := [I_n q(i)] \) \( ([a] \) denotes the integral part of the number \( a \in \mathbb{R} \), and set \( \Pi_i := \sum_{[a]} \pi_i \). Note that \( \lim_{q \to 1} \pi_i = \infty \) whenever \( r_i < 1 \). If \( q < 1 \) (as we have assumed in all this chapter) and \( (1 - q) \) is sufficiently small then \(-j_i \leq \pi_i, \) and we can define

\[
|\psi_{\pi, j}\rangle := |\vec{x} = \vec{x}(r, q), -j < \infty A^\pi(p^0)^{n-1} \ldots (p^{n+1})^{n}.
\]

(170)

where \( J_i \) are related to \( j_i \) by formula (151). The weak sense which can be given to the limit \( \lim_{q \to 1} \Gamma_q = \Gamma_{q=1} \) is at least that for any \(|\vec{x}, j\rangle \in B_q \) and small \( \varepsilon > 0 \) we can find a \( q < 1 \) and a vector \(|\psi_{\pi, j}\rangle \in B_q \subseteq H_q \) such that the corresponding eigenvalues of the observables \((p \cdot p)_n\) differ by less than \( \varepsilon \): indeed, we only need to set \( q \equiv 1 - \varepsilon q^{n-1}, \), solve for \( q \) and define \(|\psi_{\pi, j}\rangle \) as in formula (170). This expresses in a precise form the fact that, roughly speaking, the tori \((p \cdot p)_n = c \), \( c \) \( \in \mathfrak{p} \), \( \times \), get “dense” in all the momentum space when \( q \to 1 \).

Note that the convergence of the \( q \)-deformed eigenvalues (selected in this way) to the classical ones is uniform only “on the states localized on the sphere \((p \cdot p)_n = c \)”, i.e. within each subspace characterized by a fixed value \((p \cdot p)_n = c \).

In the limit \( q \to 1 \) the “parity asymmetry” in the spectrum of the observables (115) noticed at the end of section 3.1.4 disappears, in the sense that the range of each \( j_i \) (as a function of the square momenta) becomes the whole set \( \mathfrak{s} \), whenever \( r_{i-1} < 1 \), i.e. \((p \cdot p)_{i-1} < (p \cdot p)_i \), i.e. “almost everywhere” in momentum space. (In fact, the condition \((p \cdot p)_{i-1} = (p \cdot p)_i \) fixes a cylinder in the classical momentum space \( \mathfrak{p} \), this is a subset of \( \mathfrak{p} \) of \( \infty \) of zero measure). The same is true also in the Irreps with highest weight \( \neq 0 \).

4 Appendix

4.1 Proof of Proposition 3

Since \( U_q^\mathfrak{p} \cong U_q^{-1}(so(N)) \) and \( Fun(SO_\mathfrak{p}^{-1}(N)) \) are known [7] to be Hopf dual, we only need to check the compatibility of the new commutation relations of either algebra with the coalgebra structure of its dual partner.

The pairing of the homogeneous Hopf sub-algebras on the generators \( L^\pm \) [7] takes the form

\[
< L^\pm_{\pi, j}, T^\pm_{\pi, k} > = R^\pm_{\pi, j} \delta_{\pi, k} \quad R^+_{\pi, j} = R_{\pi, j} P, \quad R^-_{\pi, j} = R^-_{\pi, j} P.
\]  

(171)
here one uses the basis \( \{ L^k \} \) of the subalgebra \( \text{Fun}(SO_{q^{-1}}(N))_{\text{reg}} \subset U_{q^{-1}} so(N) \), in the notation of Ref. [7]. Using the transformation [7] from these generators to the Drinfel'd-Jimbo ones, and the transformation [11] from the latter to our generators \( \{ L, k \} \), one can easily arrive at some useful pairing relation between the \( T^s \)s and the \( L, k \):

\[
< k^i, T^b_j > = \delta^b_j q^{2\delta^i_n - \delta^i_k} \tag{172}
\]

\[
< L^{m,m-1}, T^j_i > = \frac{q^{\delta^m_n - 1}}{1 - q^{\delta^m_n}} R_{q^{-1} \delta^m_n}^{-1} L^{1-m, j} q^{\delta^m_n - 1} - q^b
\]

\[
< L^{1-m, m}, T^b_j > = \frac{q^{\delta^m_n - 1}}{1 - q^{\delta^m_n}} R_{q^{-1} \delta^m_n}^{-1} L^{1-m, b} q^{\delta^m_n - 1} - q^b
\]

Note that the matching of conventions of Ref. [7] with ours requires: 1) shifting the \( i \) indices in [7], all indices by \( -n - 1 \) in the \( N = 2n + 1 \) case, the first \( n \) indices by \( -n - 1 \) and the remaining by \( -n \) in the \( N = 2n \) case; 2) the inverse ordering of the spots of the \( so(N) \) Dynkin diagrams.

Let us consider the unbarred Hopf algebras. As a first step we consider the compatibility between the algebraic relations within \( U_q(e^N) \) and the coalgebra structure of \( \text{Fun}(E_{q^{-1}}^N) \), starting from relation (69):

\[
< \Lambda \partial_n - q^{-1} \partial_n \Lambda, \left\{ w \right\} T^j_i >= < \Lambda \otimes \partial_n - q^{-1} \partial_n \otimes \Lambda, \left\{ w \otimes \right\} T^b_j >= 0 \tag{174}
\]

trivially, whereas

\[
< \Lambda \partial_n - q^{-1} \partial_n \Lambda, y^i >= < \Lambda \otimes \partial_n - q^{-1} \partial_n \otimes \Lambda, y^j \otimes 1 + w^{-1} T^j_i \otimes y^b >= < \Lambda, w^{-1} > - q^{-1} \delta^i_n \delta^i_j \tag{175}
\]

hence one easily realizes that the check with higher powers in \( w, T, y \) in the RHS is trivial, in other words \( < \Lambda \partial_n - q^{-1} \partial_n \Lambda, \text{Fun}(E_{q^{-1}}) >= 0 \). Now we go on by writing down explicitly only the nontrivial checks. The check of relations (69), (69)_b is trivial, the check of relations (69)_d is trivial when these relations are paired with \( w^{\pm 1}, T^j_i \), whereas:

\[
< k^i, \partial_n - q^{2\delta^i_n} \partial_n k^i, y^j >= < k^i, T^j_i > < \partial_n, y^j > - q^{2\delta^i_n} < \partial_n, y^j > < k^i, 1 > \tag{176}
\]

\[
< [L^{-i,i-1}, \partial_n]_{i \in \mathbb{N}}, y^j >= < L^{-i,i-1}, T^j_i > < \partial_n, y^j > \tag{177}
\]

\[
< [L^{-m,m}, \partial_n], y^j >= < L^{-m,m}, T^j_i > < \partial_n, y^j > \tag{178}
\]

\[
< [L^{-n,n}, \partial_n], y^j >= < (L^{-n,n})^2, T^j_i > < \partial_n, y^j > \tag{179}
\]

Here we have used the explicit form of \( R_{q^{-1}}[7, 2] \). The second step is to consider the compatibility between the algebraic relations in \( \text{Fun}(E_{q^{-1}}^N) \) and the coalgebra structure of \( U_q(e^N) \). The nontrivial checks follow:

\[
< \partial_n, w y^j > = < \Lambda, w > - q \delta^j_n \tag{180}
\]

\[
< \partial_n, y^j T^j_i - R_{q^{-1}}^{-1} ij L^i \otimes y^j > = < \partial_n, y^j > - (P^R)^j_i T^j_i > R_{q^{-1}}^{-1} ij \tag{181}
\]

\[
< \partial_n \partial_n, \mathcal{P} A_i \delta^j_k y^k > xl \mathcal{P} A_i \delta^j_k y^k = 0 \tag{182}
\]

The proof for the barred Hopf algebras is analogous.
4.2 Proof of Theorem 5

We will denote by \( \mathcal{H}_{\alpha, \beta} \subseteq \mathcal{H}_\alpha \) the eigenspace of all \( k' \)'s with eigenvalues \( \mu_i \). We can easily show that \( \mathcal{H}_{\alpha, \beta} \) is finite dimensional \( \forall \mu \in \mathbb{R}^n \). In fact, if \( \phi > \mathcal{H}_{\alpha, \beta}, |\phi| > 0 \), the whole \( \mathcal{H}_{\alpha, \beta} \) has to be obtained by applying zero-graded (w.r.t. the grading of \( k' \)) operators obtained as polynomials of elements \( L_{+a}, L_{-a} \) (\( a \) are the roots of \( U^N_a \)) of the Poincaré-Birkhoff-Witt bases of the subalgebras \( \mathfrak{u}_{+}^{N}, \mathfrak{u}_{-}^{-N} \) introduced in section 2.3; But these operators make up a finite-dimensional vector space \( \mathcal{L}_a \) within each irrep. Indeed, note that the introduced \( a \) belong to \( \mathcal{L}_a \); they are proportional to the identity operator within each irrep. Then, one can easily realize that within each irrep there is only a finite number of linearly independent polynomial operators of the kind mentioned, since one can iteratively mod out casimirs from polynomials of higher degrees in the roots, so as to reduce them to polynomials of lower degree, until one reduces them to combinations of the elements of a finite "basis" of low degree polynomials.

Assume for brevity that all eigenvalues \( \mu_i \) are positive. Let \( L \) be one of the positive roots, \( \mathcal{L}^+ \) its Cartan-Weyl partner; then their commutation relation is of the form

\[
[L, \mathcal{L}^+] = \frac{(1 + d) - k}{1 - a^{-1}}, \quad 0 < a < 1, c > 0
\]

where \( k \) is an element of the Cartan subalgebra, and in the sequel

\[
a = \begin{cases} 
q & \text{if } N = 2n + 1 \text{ and } L = L^{L_1} \\
q^2 & \text{otherwise}
\end{cases}
\]

Lemma 2 Let \( L, \mathcal{L}^+ \), \( k \) satisfy relations (183) and let \( \mathcal{L}^+ L \) be hermitean positive definite (both conditions are satisfied if \( L \) is a simple roots, for instance). For any \( \mu \) there \( \exists m \in \mathbb{N} \) such that \( (L)^m \mathcal{H}_{\alpha, \beta} = 0 \); if \( 1 + d < 0 \), then \( \exists m' \in \mathbb{N} \) such that \( (L)^{m'} \mathcal{H}_{\alpha, \beta} = 0 \)

Proof. Let \( \{l_i \}_{i \in I} \) be a basis of \( \mathcal{H}_{\alpha, \beta} \) consisting of eigenvectors of \( \mathcal{L}^+ L \) and let \( l_i \) be the corresponding eigenvalues. By recursively applying commutation relation (7.62) we find that \( (L^m)|l_i >, m \in \mathbb{N} \), is an eigenvector of the positive definite operator \( \mathcal{L}^+ L \):

\[
(L^L)(L^m)|l_i > = l(l_i, m)(L^m)|l_i >
\]

\[
l(l_i, m):= \left[ i(a^{-m-1} - (m + 1)a^{-m} + (1 + d) - \lambda a^{-m}) \right].
\]

where \( \lambda \) denotes the eigenvalue of \( k \) in \( \mathcal{H}_{\alpha, \beta} \). We see that \( l(l_i, m) \leq l(b, m) \) (\( b \geq l_i \), \( \forall i \)), and that \( l(b, m) \) would get negative for large \( m \), unless there exists a \( m \in \mathbb{N} \) such that \( L^m|l_i > = 0 \forall i \in I \). Similarly one proves the second part of the Lemma.

Applying repeatedly Serre relations (50)-(52) one can prove the following

Lemma 3 Let \( |j| < k, k \geq i \geq h + 1 \).

\[
[L^{j-i}, L^{i-j}] = 0 \quad a = \begin{cases} 
q & \text{if } i = k, \ j \neq 1 - i, \ i - 1 \\
q^{-1} & \text{if } i < k, \ j = i, 1 - i \\
1 & \text{if } i < k, \ j \neq \pm i, \pm (i - 1)
\end{cases}
\]

as a consequence, if \( k > j > i \geq h \)

\[
[L^{j-i}, L^{i-j}] = 0 \quad [L^{j-i}, L^{i-j}] = 0.
\]

The same holds if we replace the \( L \) by the \( L \) roots.
Proof of theorem 5. Because of proposition 8, it is sufficient to prove the theorem within $\mathcal{H}_\delta$. We prove only the first part of the thesis, stating the existence of a highest weight vector and its uniqueness. The rest of the proof will be given elsewhere [13], where we will use some explicit knowledge about the casimirs of $\hat{u}_q(e^N)$.

Given any $|\psi_0\rangle \in \mathcal{H}_\delta$, let $\widetilde{k}|\psi_0\rangle = \lambda |\psi_0\rangle$ and apply lemma 9 to $\mathcal{H}_{\delta,\lambda}$ by setting $L \equiv L^{01}$ if $N = 2n + 1$ and $L \equiv L^{k,1,2}$ if $N = 2n$; we will respectively determine integers $p, p_k \in \mathbb{Z}$ such that

$$\langle L^{01} \rangle^{p+1} |\psi_0\rangle = 0 \quad \langle L^{01} \rangle^p |\psi_0\rangle \neq 0 \quad \text{if } N = 2n + 1$$

$$\left\{ \begin{array}{ll}
\langle L^{01} \rangle^{p+1}(L^{01})^p |\psi_0\rangle = 0 = \langle L^{1,2} \rangle^p(L^{01})^{p+1} |\psi_0\rangle, \\
\langle L^{1,2} \rangle^p(L^{01})^{p+1} |\psi_0\rangle \neq 0 & \text{if } N = 2n.
\end{array} \right. \quad (187)$$

If we define

$$|\psi_1\rangle := \left\{ \begin{array}{ll}
\langle L^{01} \rangle^p |\psi_0\rangle \\
\langle L^{1,2} \rangle^p(L^{01})^p |\psi_0\rangle
\end{array} \right. \quad (188)$$

this means that $|\psi_1\rangle$ is annihilated by $U_q^{+,\delta}$ (resp. $U_q^{+,\delta}$).

Now the proof goes on by induction. Assume that we have determined a nontrivial vector $|\psi_{j-1}\rangle \in \mathcal{H}_\delta$ such that $U_q^{+,2j-1+k}|\psi_{j-1}\rangle = \{0\}$, $j = 2, \ldots, n$. Moreover, assume that we have determined integers $p_i \in \mathbb{Z}$, $i = j - 1, j - 2, \ldots, i, j - 1 \geq i \geq 2 - j$ such that

$$\left\{ \begin{array}{ll}
\langle L^{i,2} \rangle^p(L^{j,2})^p |\psi_{j-1}\rangle \neq 0, \\
\langle L^{i,2} \rangle^p(L^{j,2})^p |\psi_{j-1}\rangle = 0 & \text{if } N = 2n, \ (189)
\end{array} \right.$$}

Then we can determine an integer $p_{j-1} \in \mathbb{Z}$ such that

$$\left\{ \begin{array}{ll}
\langle L^{i,2} \rangle^p(L^{j,2})^{p+1} |\psi_{j-1}\rangle \neq 0, \\
\langle L^{i,2} \rangle^p(L^{j,2})^{p+1} |\psi_{j-1}\rangle = 0 & \text{if } N = 2n, \ (190)
\end{array} \right.$$}

In fact, on one hand we set $L \equiv L^{-1,2}$ and try to apply lemma 3. It is well known that when $L$ is not simple the Cartan-Weyl partner of $L$, $L^-$, differs from $(L^{01})^\dagger \propto L^{-1,1}$; its polynomial expression in terms of simple negative roots can be obtained from the one of $L^{-1,1}$ by the replacement $q \rightarrow q^{-1}$. Vice versa, the polynomial expression in terms of simple positive roots of $L^\dagger := (L^-)^\dagger$ can be obtained from the one of $L^{-1,2}$ by the same replacement. For instance,

$$L^\dagger := L^{-1,3} \propto \left[L^{-1,2}, L^{-3,2}\right]_q, \quad (L^\dagger)^\dagger = L^{-5,3} \propto \left[L^{-3,2}, L^{-2,3}\right]_q; \quad (191)$$

$$L^- \propto \left[L^{-2,3}, L^{-3,2}\right]_{q^{-1}}, \quad L^\dagger \propto \left[L^{-1,2}, L^{-3,3}\right]_{q^{-1}}. \quad (192)$$

One can easily verify that in any case we can find $u' \in U_q^{+,N-2}$, $u \in u_q^{+,N}$ $a > 0$ such that

$$L' = aL + uu' \quad \Rightarrow \quad L^-L' = aL^-L + L^-uu'. \quad (193)$$

Now the second term in the RHS can be neglected because it always gives zero when applied to $(L^{-1,2})^p |\psi_{i,j}\rangle$, because of the induction hypothesis (189). Therefore we are in the conditions to apply lemma 3 to the subspace $\mathcal{H}_{\delta,\lambda}$ containing $|\psi_{i,j}\rangle$, since $L^-L'$ is positive definite and $L, L^-$ belong to a Cartan-Weyl triple.

On the other hand, one can easily show that the generators $L^{k,1,2}$ $(k = h+1, h+2, \ldots, j-1)$ of $u_q^{+,2j-1+k}$ annihilate $|\psi_{i-1,j}\rangle$ as well, by application of the commutation relations (185),(186), their consequences

$$L^{k,1,2}(L^{i,j})^p = q^p(L^{i,j})^p L^{k,1,2} + aq^p(L^{i,j})^{p-1}L^{1-k,j} \quad a > 0. \quad (194)$$
\[ L^{1-k,i}(L^{j-1,k})^p = q^i(L^{k,i})^p L^{1-k,i} + d' q^i(L^{k,i})^p-1 L^{k,i} \quad \text{for} \quad d' > 0 \]  \hspace{1cm} (195)

and the induction hypothesis. In the least simple case, \( i - 1 \leq -k \), for instance,

\[ L^{1-k,i}|\psi_{j-1,i} > \approx \sum_{a=0}^{(185)} \ldots L^{1-k,i} (L^{j-1,k})^p \ldots (L^{1-k,i})^p-1 |\psi_{j-1} > \]

\[ \approx \ldots [(L^{k,i})^p-1 L^{j-1,i} + a(L^{k,i})^p-1 L^{j-1,i}] (L^{k,i})^p \ldots (L^{j-1,i})^p-1 |\psi_{j-1} > \]

\[ \approx \ldots [(L^{j-1,i})^p-1 L^{k,i} + d'(L^{j-1,i})^p \ldots (L^{j-1,i})^p-1] (L^{j-1,i})^p \ldots (L^{j-1,i})^p-1 |\psi_{j-1} > \]

\[ \approx \ldots \ldots (L^{j-1,i})^p \ldots (L^{j-1,i})^p-1 |\psi_{j-1} > = \approx 0. \]  \hspace{1cm} (196)

The dots stand for the same powers of the roots \( L \) which appear in the definition (189), of \( |\psi_{j-1,i} > \).

Finally, let us define

\[ |\psi_i > := (L^{1-j,i})^p \ldots (L^{j-1,i})^p \ldots (L^{j-1,i})^p-1 |\psi_{j-1} > ; \]

it is easy to show the relation \( U_q^{j+1+\delta} |\psi_i > \seteq \{ 0 \} \), i.e. the induction hypothesis for the subsequent step.

In fact, on one hand \( L^{j-1,i}|\psi_i > = 0 \) trivially, because of equation (190) and the definition (197); on the other, \( U_q^{j+1+\delta} |\psi_i > \seteq \{ 0 \} \) because \( U_q^{j+1+\delta} \) annihilates by construction each of the vectors (189).a

The first part of the thesis is proved if we set \( |\phi > \seteq |\psi_{1,n,n} > \).

To prove the uniqueness of the highest weight vector in the case \( N = 2n + 1 \) (resp. the structure of \( \mathcal{H}_G \) in the case \( N = 2n \)) we can proceed as follows. Let \( |\phi >, |\phi' > \) the highest weight vectors constructed by the above procedure starting from two vectors \( |\psi_0 >, |\psi_0' > \in \mathcal{H}_G \). The \( n - h \) casimirs of \( \psi_0 (e^N) \) different from \( (p \cdot p_j)_j \) are independent functions of the generators and when acting on highest weight vectors as \( |\phi >, |\phi' > \) will reduce to \( n - h \) independent functions of \( k \) only. In the casimirs are sum of zero-graded (w.r.t. \( k \)) monomials in \( L, k \) which can be written in a form where \( L \in \mathfrak{a}^+ \mathfrak{n} \) stand at the right of \( L \in \mathfrak{a}^+ \mathfrak{n} \), and the former annihilate \( |\phi >, |\phi' > \). But since these casimirs take the same values on \( \mathcal{H}_G \), this implies that \( |\phi >, |\phi' > \) have the same \( k \) eigenvalues \( \forall i \) if \( N = 2n + 1 \), \( \forall i > 1 \) if \( N = 2n \) (in fact, \( (p^{k+1})^p |\phi > \) are highest weight eigenvectors \( \forall \psi \in \psi \)). This implies the claim on the dimensionality of \( \mathcal{H}_G \), since as a consequence \( \mathcal{C} |\phi > = |\phi' > \) implies that \( \mathcal{O} \) is a zero-graded (w.r.t. to \( k^i \), \( i = h + 1, h + 2, \ldots, n \)) operator, and up to a constant such an operator acts on a highest weight vector as the identity, in the case \( N = 2n + 1 \), and as some powers of \( p^\pm \) in the case \( N = 2n \). \( \diamond \)

### 4.3 Proof of theorem 6

It is immediate to check that commutation relations (147), (148) hold when applied to \( |\bar{0}, \bar{0} > \).

Let \( H_{\bar{0}, i,j} = \text{Span} \{ |\bar{0}, \bar{l}, \ldots > \text{ } | \bar{l} > \bar{j} \} \). The proof of these equations is by induction in the weights \( j \), i.e. assuming that they hold for \( H_{\bar{0}, i,j} \) we show that they hold when applied on \( H_{\bar{0}, i,j+1+\epsilon} \).

As a consequence of formulae (147), (148) one can easily show that within \( H_{\bar{0}, i,j} \) the

\[ L^{1-i-j}(L^{j-1,i})^p = \begin{cases} q^{-i} \frac{[i+j]_{q^2-1}}{[i+j]_{q^2-1}} & \text{if } N = 2n + 1 \text{ and } i = 1 \\ q^{-i} \frac{[i+j]_{q^2-1}}{[i+j]_{q^2-1}} & \text{if } N = 2n \text{ and } i = 2 \\ q^{-i} \frac{[i+j]_{q^2-1}}{[i+j]_{q^2-1}} & \text{otherwise.} \end{cases} \]  \hspace{1cm} (198)
First part of the thesis: if $|\chi\rangle$ is singular, then for any $u \in \tilde{u}_{q}^{-N}$ $u|\chi\rangle$ is. It is sufficient to consider only $u = p^{k}$ or $u = L^{-i-1}$, and vectors $|\chi\rangle$ of the form $|\chi\rangle = v_{1}v_{2}|0\rangle$, where $v_{1}, v_{2}$ are two ordered polynomials in the generators of $\tilde{u}_{q}^{-N}$; in fact, by the induction hypothesis the latter are singular. Any other singular vector can be expressed as a combination of singular vectors of this form.

We consider first relations (147) in the case $N = 2n$. Since $[g_{1}, u_{q}^{-N}] = 0$, $|\chi\rangle$ is of the form $|\chi\rangle = v_{1}v_{2}|0\rangle$. The vector $|\chi\rangle := L^{-1,1}|\chi\rangle$ is trivially orthogonal to any vector corresponding to a different weight. In addition it is orthogonal to the only two independent vectors $|1\rangle, |2\rangle$ with the same weight, which are (according to formula (149) of the induction hypothesis)

$$
|1\rangle := L^{-2,1}|1\rangle := L^{-2,1}vL^{-2,1}|0,0\rangle, \quad |2\rangle := L^{-2,1}|2\rangle := \frac{L^{-2,1}vL^{-2,1}q^{-2}p^{-1}p^{-1}}{(1 - q^{2})(p, p)}|0,0\rangle.
$$

In fact, one can easily check that for any $v$ as considered above,

$$(v^{*}L^{-1,2})L^{-2,1} = v L^{-2,1}(v^{*}L^{-1,2}) + kv^{*}$$

$k$ being an element of the Cartan subalgebra), $[v^{*}, L^{-1}] = 0$, implying

$$<1|\chi\rangle > \propto \langle 0,0|L^{1,2}(v^{*}L^{-1,2})L^{-2,1}|\chi\rangle > \propto \langle 0,0|L^{1,2}L^{-2,1}(v^{*}L^{-1,2})|\chi\rangle > + b < 0,0|L^{1,2}v|\chi\rangle >$$

$$= < 0, 0 |L^{-2,1}L^{1,2}|\chi\rangle > + c < 2|\chi\rangle > + d' < 1|\chi\rangle > = 0;$$

in the last identity, we have used the singularity of $|\chi\rangle$ and the fact that $< 0,0|L^{-2,1} = 0$. Similarly one shows that $< 2|\chi\rangle = 0$. Summing up, $|\chi\rangle >$ is singular. The corresponding result for $g_{1}$ is a direct consequence of the above.

Now we consider the remaining commutation relations (148) (for even and odd $N$ at the same time). One can easily check that they imply the relations

$$L^{-m,m-1}g_{i} = g_{i}f_{i}L^{-m,m-1}, \quad f_{m} := \begin{cases} 
q^{1 - \frac{m-2}{2}} & \text{if } m = i, i + 1 \\
q^{1 - \frac{m-2}{2} + 1} & \text{if } m = i + 2 \\
q^{1 - \frac{m-2}{2} - 1} & \text{if } m = i - 1 \\
1 & \text{otherwise};
\end{cases}$$

since they hold in $H_{\tilde{g}_{i}(k),<}$, we can reduce any singular vector to one of the form $|\chi\rangle = g_{i}u|0\rangle$. Using commutation relations (148) and relations (149) only in $H_{\tilde{g}_{i}(k),<}$ one can easily show that

$$L^{-i-1}L^{1-i-1}g_{i} = s_{i}(k)g_{i}; \quad \text{or equivalently:} \quad L^{1-i-1}L^{-i-1}g_{i} = s_{i}(k)g_{i}. \quad (202)$$

The vector $|\chi\rangle := L^{-1,1}|\chi\rangle >$ is trivially orthogonal to any vector corresponding to a different weight; in addition it has zero norm

$$<\chi|\chi\rangle > \propto <\chi|s_{i}^{*}|\chi\rangle > = 0$$

because of the preceding relation and the fact that $|\chi\rangle >$ is singular; therefore $|\chi\rangle >$ is singular itself.

Second part of the thesis: if $|\chi\rangle \in H_{\tilde{g}_{i}(k),<}$ is nonsingular, then $|\chi\rangle > := g_{i}|\chi\rangle >$ is singular.

Again, we consider first relations (147) in the case $N = 2n$. As already noticed, it is easy to write this vector in the form $|\chi\rangle > = v_{1}v_{2}|0,0\rangle$; then the proof that $|\chi\rangle >$ is singular goes as for $|\chi\rangle >$.
As for the remaining relations, using formulae (132), (130) we find, when \( i > h + 1 \)

\[
[L^{-i,i+1}L^{1-i,i}, g_i]_q = q^{2^{p+1}k^i} \frac{1 - q^2(k^{i+1})^{-1}}{1 - q^{-2}k^i} L^{-i,i+1}L^{1-i,i} + q^{2^{p+1}k^i} \frac{1 - q^2(k^{i+1})^{-1}}{1 - q^{-2}k^i} ;
\]

\[
q^{2^{p+1}k^i} \frac{1 - q^{-2}(k^{i+1})^{-1}}{1 - q^{-2}k^i} \left( q^2 L^{-i,i+1}L^{1-i,i} + q^{2^{p+1}k^i} \frac{1 - q^2(k^{i+1})^{-1}}{1 - q^{-2}k^i} \right) ;
\]

(204)

within \( \mathcal{H}_{\tilde{q},(i+1)} \), we can replace the operators \( L^{-i,i+1}L^{1-i,i}, L^{-i,i+1}L^{1-i,i} \) in the RHS by their expressions (202), and we find that the RHS vanishes. A similar argument can be used when \( i = h + 1 \), and also when the order of the two positive roots appearing in the LHS is reversed. We find that on \( \mathcal{H}_{\tilde{q},(i)} \),

\[
[L^{-i,i+1}L^{1-i,i}, g_i]_a = 0 \quad [L^{1-i,i}L^{-i,i+1}, g_i]_a = 0 \quad a = \begin{cases} q & \text{if } N = 2n + 1, \ i = 1 \\ q^2 & \text{otherwise.} \end{cases}
\]

(205)

Now consider the vector \( g_i |\overline{0}, -\overline{j} > \). On one hand, it is trivially orthogonal to any other vector of \( \mathcal{H}_0 \) of different weight; on the other, it is orthogonal both to \( |1| := L^{-i,i+1}L^{1-i,i} |\overline{0}, -\overline{j} > \) and \( |2| := L^{-i,i+1}L^{1-i,i} |\overline{0}, -\overline{j} > \)

\[
< 2 |g_i |\overline{0}, -\overline{j} > = 0 \quad < 1 |g_i |\overline{0}, -\overline{j} > = 0,
\]

(206)

where we have used equations (204), the fact that the vectors \( g_i L^{-i,i+1}L^{1-i,i} |\overline{0}, -\overline{j} >, g_i L^{1-i,i+1}L^{-i,i} |\overline{0}, -\overline{j} > \) belong to \( \mathcal{H}_{\tilde{q},i} \), and the induction hypothesis. Therefore the vector \( g_i |\overline{0}, -\overline{j} > \) is singular in \( \mathcal{H}_0 \) and must be set equal to zero. \( \diamond \)

References


