QUANTUM STATISTICAL DISTRIBUTIONS FOR INTERACTING PARTICLES¹

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Abstract We consider quantum statistical distributions describing systems of particles obeying an exclusion or an inclusion principle together with Maxwell-Boltzmann distributions of classical particles. Particles belonging to any intermediate statistics can be described by a set of distributions, varying a parameter which is the degree of symmetrization or antisymmetrization. Components of astrophysical plasma with partial degeneracy can be usefully described by the proposed distributions.

In many physical fields of research we have often to deal with systems of particles which are with certainty bosons or fermions. We have to deal with classical particles when the thermal wave length is much smaller than the interparticle distance. For instance, in the case of a neutral plasma, the above condition is given by

$$\lambda_{ij} = \frac{\sqrt{2\pi}\hbar}{a_{ij}\sqrt{2\mu_{ij}kT}} \ll 1 \quad , \quad (1)$$

where the interparticle distance a_{ij} is

$$a_{ij} = \frac{a_i + a_j}{2} \quad , \quad a_i = \left(\frac{3Z_i}{4\pi n_e} \right)^{1/3} \quad , \quad (2)$$

Z_i is the charge of the i -th ion, n_e is the electron density and $\mu_{ij} = m_i m_j / (m_i + m_j)$ is the reduced mass. In high energy physics, high temperature superconductivity and in quantum Hall effect we have sometime to describe particles obeying statistics which are different from the pure Bose-Einstein (BE) or Fermi-Dirac (FD) statistics. This task can be accomplished or giving an ad hoc statistical distribution or by mixing with certain weights the two different BE and FD distributions. On the other hand, we have often particles behaving only partially as classical particles, because their distribution must have a quantal component (for instance a partially degenerate electron gas). Also in this case we can search for an ad hoc distribution function or we can mix the classical Maxwell-Boltzmann (MB) with the quantum statistical distributions. This kind of problems are very common in astrophysical dense plasma and particularly in the solar interior [1,2]. In the case of electron components

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in astrophysical plasma, there is a gradual transition from nondegeneracy toward complete degeneracy as the density rises. An approach to derive the distribution of partially degenerate electrons is well known [1].

In this paper we want to show how new distributions, recently proposed by us [3-5], can be used to completely describe the behavior of systems of particles in the particular situations mentioned above and with a particular interest in the plasma of the solar or stellar interiors.

Let us pose our attention to the function $n(t, \mathbf{v})$ which is the occupational number related to the velocity $\mathbf{v}$. We have recently shown [4] that $n(t, \mathbf{v})$ satisfies the following partial differential equation:

$$\frac{\partial n(t, \mathbf{v})}{\partial t} = \nabla \{ [\mathbf{J}(t, \mathbf{v}) + \nabla D(t, \mathbf{v})] n(t, \mathbf{v}) [1 + \kappa n(t, \mathbf{v})] + D(t, \mathbf{v}) \nabla n(t, \mathbf{v}) \} , \quad (3)$$

where $\mathbf{J}(t, \mathbf{v})$ is the drift coefficient and $D(t, \mathbf{v})$ is the diffusion coefficient in the velocity space. Eq.(3) is the continuity equation in the velocity space. We have used, to derive this equation, the following expression of the transition probability:

$$\pi(t, \mathbf{v} \rightarrow \mathbf{u}) = r(t, \mathbf{v}) n(t, \mathbf{v}) [1 + \kappa n(t, \mathbf{u})] \quad (4)$$

where $r(t, \mathbf{v})$ is the transition rate.

The quantity in the square brackets is an inhibition factor, when $\kappa = -1$: if the site $\mathbf{u}$ is fully occupied ($n(t, \mathbf{u}) = 1$), then the transition from $\mathbf{v}$ to $\mathbf{u}$ is forbidden. On the other hand, if the site is empty the transition probability is not affected by the occupation of the arrival site. In this case the stationary solution of Eq.(3) is the FD statistical distribution.

When $\kappa = 1$ the statistical distribution describes particles which are bosons. In this case the quantity in the square bracket in Eq.(4) represents an enhancement factor. The number of particles leaving the site $\mathbf{v}$ increases linearly with the population of the arrival site $\mathbf{u}$.

The two cases describe systems of particles submitted to an exclusion (fermions) or to an inclusion (bosons) principle. Both the principles induce a potential, repulsive or attractive respectively, acting on the particles that, consequently, must be considered as interacting particles.

Let us take κ as a continuous variable with values in the range $-1 \leq \kappa \leq +1$, and solve Eq.(3) in the case of Brownian particles; for $t \rightarrow \infty$ we find the following statistical distribution:

$$n(\mathbf{v}) = \frac{1}{\exp[\beta(E - \mu)] - \kappa} . \quad (5)$$

with $E = mv^2/2$ and $\beta = 1/kT$. The expression of $n(\mathbf{v})$ reduces to the standard FD ($\kappa = -1$) and MB ($\kappa = 0$), BE ($\kappa = 1$) distribution. We call κ the degree of symmetrization or antisymmetrization, μ is the chemical potential.

These two quantities can be determined (as explained below) through normalization conditions, using the known density and pressure of the particle system under consideration.

To make an example let us fix our attention to a one or two component dense plasma with an electron density n_e satisfying the neutrality condition

$$(6) \quad \sum_i Z_i n_i = n_e .$$

Degrees of Fermi degeneracy for electrons at temperature T are measured by the ratio

$$(7) \quad \Theta = \frac{E_F}{kT} ,$$

where

$$(8) \quad E_F = m_e c^2 \left(\sqrt{1 + x_F^2} - 1 \right) ,$$

$$(9) \quad x_F = \frac{m_e c}{\hbar k_F} = \frac{m_e c}{\hbar (3\pi^2 n_e)^{1/3}} ,$$

When $\Theta > 0.1$ we have complete degeneracy, when $0.1 \leq \Theta \leq 10$ we have intermediate degeneracy (quantum and classical effects coexist), when $\Theta > 10$ we have not degeneracy and the particles can be considered classical. For instance we have in solar interior (SI) $\Theta = 2.23$, in brown dwarf (BD) $\Theta = 0.01$, in giant planet (GP) $\Theta = 0.023$, in inertial confinement fusion (ICF) $31.8 \leq \Theta \leq 387$ [2]. To be more explicit, let us consider the solar interior. The correct normalization conditions are (non relativistic case):

$$(10) \quad 8\pi \int_0^\infty \left(\frac{m_e}{\hbar} \right)^3 \frac{v^2 dv}{\exp[\beta(E - \mu)] - \kappa} = n_e ,$$

$$(11) \quad 8\pi m_e \left(\frac{m_e}{\hbar} \right)^3 \int_0^\infty \frac{v^4 dv}{\exp[\beta(E - \mu)] - \kappa} = P_e .$$

We know temperature, density and pressure in SI, then we may derive from Eqs (10) and (11) the value of the chemical potential μ and the degree of antisymmetrization $-1 \leq \kappa \leq 0$ [5]. Let us define now the fuction $F_l(x)$

$$(12) \quad F_l(x) = \int_0^\infty \frac{t^l dt}{\exp(t+x) + 1} ,$$

from Eqs (10) and (11) we obtain the following expressions of the density and pressure:

$$(13) \quad n_e = \frac{4\pi}{h^3} (2mkT)^{3/2} \frac{1}{\hbar} F_{1/2} \left(\ln \frac{|\kappa|}{1} - \frac{\mu}{kT} \right) ,$$

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References

$$n(\mathbf{v}) = \left\{ \exp \left[\sum_{i=1}^3 \int \frac{D(v_i)}{D(v_i)} J_i(v_i) + \frac{\partial D(v_i)}{\partial v_i} \right] - \beta \mu \right\}^{-1} . \quad (19)$$

distribution function in three dimensions is given by:
 If, in addition to the repulsive and attractive potentials due to the exclusion and inclusion principles, we suppose that other internal and external potentials are active among the particles, then the diffusion and the drift coefficients are different from the Brownian case; the expression of the stationary

$0 \leq \kappa \leq 1$ can be derived.
 If, in addition to the repulsive and attractive potentials due to the exclusion and inclusion principles, we suppose that other internal and external potentials are active among the particles, then the diffusion and the drift coefficients are different from the Brownian case; the expression of the stationary distribution function in three dimensions is given by:

$$\Lambda_{-1} \left(\frac{kT}{\mu} \right) = \frac{2}{3} F_{3/2} \left(\frac{kT}{\mu} \right) / F_{1/2} \left(\frac{kT}{\mu} \right) . \quad (18)$$

while in the case of a fermionic system ($\kappa = -1$) from Eq.(15) we get

$$\Lambda_0 \left(\frac{kT}{\mu} \right) = 1 , \quad (17)$$

In the case of completely classical particles ($\kappa = 0$) we have:

$$P_e = n_e kT \Lambda_\kappa \left(\frac{kT}{\mu} \right) . \quad (16)$$

particles:
 we obtain the following state equation for the partially quantal system of

$$\Lambda_\kappa \left(\frac{kT}{\mu} \right) = \frac{2}{3} F_{3/2} \left(\ln \frac{|\kappa|}{1} - \frac{kT}{\mu} \right) / F_{1/2} \left(\ln \frac{|\kappa|}{1} - \frac{kT}{\mu} \right) , \quad (15)$$

we define the quantity
 can derive the chemical potential μ and the degree of symmetrization κ . If
 By using Eqs (13) and (14) if we know the values of density and pressure we

$$P_e = \frac{8\pi kT}{3h^3} (2mkT)^{3/2} \frac{1}{1} F_{3/2} \left(\ln \frac{|\kappa|}{1} - \frac{kT}{\mu} \right) . \quad (14)$$