Gravitational energy in spherical symmetry

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Abstract. Various properties of the Misner-Sharp spherically symmetric gravitational energy $E$ are derived, and known properties reviewed. In the Newtonian limit, $E$ yields the Newtonian mass to leading order and the Newtonian kinetic and potential energy to the next order. In vacuo, $E$ reduces to the Schwarzschild parameter. At null and spatial infinity, $E$ reduces to the Bondi-Sachs and Arnowitt-Deser-Misner energies respectively. In the small-sphere limit, the leading term in $E$ is the product of volume and the energy density of the matter. A sphere is trapped if $E > \frac{1}{2}r$, marginal if $E = \frac{1}{2}r$ and untrapped if $E < \frac{1}{2}r$, where $r$ is the areal radius. A central singularity is spatial and trapped if $E > 0$, and temporal and untrapped if $E < 0$. On an untrapped sphere, $E$ is non-decreasing in any outgoing spatial or null direction, assuming the dominant energy condition. It follows that $E \geq 0$ on an untrapped spatial hypersurface with regular centre, and $E \geq \frac{1}{2}r_0$ on an untrapped spatial hypersurface bounded at the inward end by a marginal sphere of radius $r_0$. All these inequalities extend to the asymptotic energies, recovering the Bondi-Sachs energy loss and the positivity of the asymptotic energies, as well as proving the conjectured Penrose inequality in spherical symmetry. Implications for general definitions of gravitational energy are discussed.

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I. Introduction

A massive source produces a gravitational field which has energy. In Relativity, the equivalence of mass and energy means that it is only the combined energy which may be measured at a distance. Moreover, the non-linearity of the gravitational field means that the material (or passive) mass and its gravitational and kinetic energy combine in a non-linear, non-local way to produce the effective (or active) mass-energy. In spherical symmetry, in vacuo, this effective mass-energy is just the Schwarzschild parameter. In general, there is no agreed definition of this energy, except at infinity in an asymptotically flat space-time, where one has the Arnowitt-Deser-Misner [1] mass-energy $E_{ADM}$ at spatial infinity and the Bondi-Sachs [2,3] mass-energy $E_{BS}$ at null infinity. One would therefore like a definition of energy which reduces to these asymptotic energies appropriately. Also, given the above physical motivation, one would like the energy to yield the Newtonian mass in the Newtonian limit, with the highest-order correction yielding the Newtonian energy. Remarkably, such an energy does exist in spherical symmetry: the Misner-Sharp [4] mass-energy $E$. Moreover, $E$ is intimately related to the characteristic strong-field gravitational phenomena, namely black and white holes and singularities.
This article lists various key properties of this energy. The main new results, apart from the Newtonian behaviour, are a monotonicity property of $E$ which leads to a positivity property and a lower bound for $E$ in terms of the area of a black or white hole, the so-called isoperimetric inequality. These properties extend to the asymptotic energies, in particular establishing the isoperimetric inequality for $E_{ADM}$, as conjectured by Penrose [5], and for $E_{BS}$. Known results are also reviewed, partly because some are prerequisites for the newer results, and partly because the existing literature on the subject is somewhat dispersed, so that it is not always appreciated that $E$ enjoys quite so many desirable properties.

In Section II, $E$ is defined geometrically and shown to have various purely geometrical properties related to trapped and marginal surfaces, central singularities and the asymptotic energies. In Section III, various dynamical properties are derived assuming the dominant energy condition, including the monotonicity, positivity and area-bound properties. In Section IV, the geometry is decomposed with respect to spatial hypersurfaces and the behaviour of $E$ in the Newtonian limit is found. The Conclusion discusses the implications for more general suggestions for gravitational energy. Since the whole article is concerned with spherical symmetry, this case will be assumed implicitly in the propositions, without repeated qualification. Similarly, all geometrical objects mentioned will be assumed to respect the spherical symmetry.

II. Geometrical properties

In spherical symmetry, the line-element may be written in double-null form as

$$ds^2 = r^2d\Omega^2 - 2e^{-f}d\xi_+ d\xi_-$$

where $d\Omega^2$ refers to the unit sphere and $r$ and $f$ are functions of the null coordinates $(\xi_+, \xi_-)$. This double-null form is natural in the sense that each symmetric sphere has two preferred normal directions, the null directions $\partial/\partial \xi_{\pm}$. One may also use one spatial and one temporal direction, as in Section IV, but there is no unique choice of such directions, which makes it more difficult to check coordinate invariance. The area of a symmetric sphere is $4\pi r^2$, so that $r$ is the areal radius, and will simply be called the radius. One may take $r \geq 0$, with $r > 0$ being a sphere and $r = 0$ being either a regular centre or a central singularity. Note that $r$ is invariant under diffeomorphisms

$$\xi_{\pm} \mapsto \hat{\xi}_{\pm}(\xi_{\pm})$$

but that $f$ is not. The geometry is given by the metric (1) modulo the null rescalings (2). It suffices to work in the two-dimensional space-time obtained by taking the quotient by the spheres of symmetry.

The space-time will be assumed time-orientable, and $\partial/\partial \xi_{\pm}$ will be assumed future-pointing. The *expansions* may be defined by

$$\theta_{\pm} = 2r^{-1} \partial_{\pm} r$$

where $\partial_{\pm}$ denotes the coordinate derivative along $\xi_{\pm}$. The expansions measure whether the light rays normal to a sphere are expanding or contracting, or equivalently, whether
the area of the spheres is increasing or decreasing in the null directions. Note that $\theta_\pm$ are not invariants of the geometry, though their signs are, but that the combination $e^f \theta_+ \theta_-$ is invariant. Indeed, the only invariants of the metric and its first derivatives are functions of $r$ and $e^f \theta_+ \theta_-$. The latter invariant has an important geometrical and physical meaning: a metric sphere is said to be (i) \textit{trapped} if $\theta_+ \theta_- > 0$, (ii) \textit{marginal} if $\theta_+ \theta_- = 0$, and (iii) \textit{untrapped} if $\theta_+ \theta_- < 0$. This terminology for surfaces will be extended to hypersurfaces and space-time regions. If $e^f \theta_+ \theta_-$ is a function with non-vanishing derivative, the space-time is divided into trapped and untrapped regions, separated by marginal hypersurfaces. The following subdivisions may be made [6].

(i) A trapped sphere is \textit{future} if $\theta_+ < 0$ and \textit{past} if $\theta_+ > 0$. Future and past trapped spheres occur in black and white holes respectively, and also in cosmological models.

(ii) A marginal sphere with $\theta_+ = 0$ is \textit{future} if $\theta_- < 0$, \textit{past} if $\theta_- > 0$, \textit{bifurcate} if $\theta_- = 0$, \textit{outer} if $\partial_- \theta_+ < 0$, \textit{inner} if $\partial_- \theta_+ > 0$ and \textit{degenerate} if $\partial_- \theta_+ = 0$. The closure of a hypersurface foliated by future or past, outer or inner marginal spheres is called a \textit{trapping horizon}. Future (respectively past) outer trapping horizons define black (respectively white) holes. Inner trapping horizons include cosmological horizons as well as the possible inner boundaries of black and white holes.

(iii) On an untrapped sphere, the orientation may be fixed by $\theta_+ > 0$ and $\theta_- < 0$. Then $\partial_+$ and $\partial_-$ may be described respectively as outgoing and ingoing null normal vectors. More generally, any spatial or null normal vector $\mathbf{z}$ is \textit{outgoing} if $g(z, \partial_+) > 0$ or $g(z, \partial_-) < 0$ and \textit{ingoing} if $g(z, \partial_+) < 0$ or $g(z, \partial_-) > 0$, where $g$ is the metric and the sign convention is that spatial metrics are positive definite. Then the area is increasing in any outgoing spatial or null direction, and decreasing in any ingoing spatial or null direction.

The Misner-Sharp spherically symmetric gravitational mass-energy, or simply the \textit{energy}, may be defined by

$$E = \frac{1}{2} r + e^f r \partial_+ r \partial_- r = \frac{1}{2} r + \frac{1}{4} e^f r^3 \theta_+ \theta_-.$$  \hspace{1cm} (4)

The form actually given by Misner & Sharp is derived in Section IV. Note that $E$ is an invariant. Indeed, the only invariants of the metric and its first derivatives are functions of $r$ and $E$, as explained above. This makes $r$ and $E$ natural variables to use, and $E$ has been rediscovered many times by different authors. It is remarkable that all of the key geometrical properties of spherically symmetric space-times are controlled by $r$ and $E$.

\textit{Proposition 1}: trapping. A metric sphere is trapped if $E > \frac{1}{2} r$, marginal if $E = \frac{1}{2} r$, and untrapped if $E < \frac{1}{2} r$.

\textit{Proof}. By definition.

This property is mathematically trivial given the definition in the above form, but is physically important because it shows that the ratio $E/r$ controls the formation of black and white holes, and trapped spheres generally. Note that the material (or passive) mass does not have this property; the sharpest relations between trapped spheres and the material mass [7] fall short of necessary and sufficient conditions. In other words, it is not the material mass which directly controls the formation of black and white holes, but the effective energy $E$. 

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The definitions of trapped and untrapped spheres, formulated for $r > 0$, can also be extended to $r = 0$. Suppose that $r = 0$ coincides with a smooth conformal boundary of the quotient space-time. A point $p$ in the boundary $r = 0$ will be called a regular centre if $e^f \partial_+ r \partial_- r = -\frac{1}{2}$ at $p$ and the components of the material mass tensor with respect to $\xi_\pm$ are bounded at $p$. A point in the boundary $r = 0$ which is not a regular centre will be called a central singularity. A point in the boundary $r = 0$ will be said to be trapped if it surrounded by a neighbourhood of trapped spheres, and untrapped if it surrounded by a neighbourhood of untrapped spheres. A regular centre is untrapped, but central singularities may be either trapped, as in the positive-mass Schwarzschild solution, or untrapped, as in the negative-mass Schwarzschild solution. Whether singularities are trapped or untrapped is relevant to the cosmic censorship hypothesis [8]. Also relevant is whether the singularity is causal or spatial, defined with respect to the conformal metric. It turns out that both features are controlled by $E$ [9,10]. One can evaluate $E$ at a singularity $p$ by taking the limit along a curve $\gamma$ approaching $p$. The limit may be direction-dependent or infinite, $\lim_{\gamma \to p} E = \pm \infty$.

**Proposition 2:** central singularities. A central singularity $p$ is spatial (respectively temporal) and trapped (respectively untrapped) if $\lim_{\gamma \to p} E > 0$ (respectively $\lim_{\gamma \to p} E < 0$) along all curves $\gamma$ approaching $p$. This includes the case $\lim_{\gamma \to p} E = \infty$ (respectively $\lim_{\gamma \to p} E = -\infty$).

**Proof.** (i) For a central singularity $p$, $\lim_{\gamma \to p} E = \lim_{\gamma \to p} \frac{e^f}{4} r^3 \theta_+ \theta_-$. If $\lim_{\gamma \to p} E > 0$ along a curve $\gamma$, then $\lim_{\gamma \to p} r^3 \theta_+ \theta_- > 0$, so $\theta_+ \theta_- > 0$ in a neighbourhood of $p$ in $\gamma$. So if $\lim_{\gamma \to p} E > 0$ along all curves $\gamma$, then $\theta_+ \theta_- > 0$ in a neighbourhood of $p$, so that all spheres in the neighbourhood are trapped. (ii) The tangent vector $z = \partial / \partial \zeta$ to the singularity is a linear combination $z = \beta \partial_+ - \alpha \partial_-$ of the null normals $\partial_\pm$, so that $0 = \partial r / \partial \zeta = \beta \partial_+ r - \alpha \partial_- r$. If $\lim_{\gamma \to p} E > 0$ then $\lim_{\gamma \to p} r \partial_+ r \partial_- r > 0$, so $\alpha \beta > 0$, which means that $z$ is spatial. Similarly for the case $\lim_{\gamma \to p} E < 0$.

Combined with the property $E \geq 0$, which will be derived later under certain assumptions, the above result is almost a proof of cosmic censorship. The missing link is the case where $\lim_{\gamma \to p} E = 0$, in which case $r = 0$ could be a spatial, null or temporal singularity, or a regular centre. Specifically, if $\lim_{\gamma \to p} 2E/r > 1$ the singularity is spatial and trapped, and if $\lim_{\gamma \to p} 2E/r < 1$ the singularity is temporal and untrapped. If $\lim_{\gamma \to p} 2E/r = 1$, one must look at higher orders, $\lim_{\gamma \to p} (2E/r - 1)/r$, and so on. Exactly which possibility occurs seems to depend on the matter field. According to Christodoulou [11], for a massless scalar field it is possible to obtain causal central singularities, but such configurations are non-generic with respect to initial data. Conversely, for pure radiation (or null dust), a sufficiently weak wave travelling into an initially flat space-time necessarily creates a null singularity which is at least locally visible [12,13]. Such visible null singularities are also possible for dust [14-16]. Despite such material-dependent differences, one useful fact remains: if $E \geq 0$, a central singularity which is either causal or untrapped must be massless, $\lim_{\gamma \to p} E = 0$. (For instance, the analysis of Joshi & Dwivedi [17] is restricted to massless singularities). This at least constrains counter-examples to cosmic censorship. Indeed, it has been suggested [18] that massless singularities are non-gravitational and do not conflict with the spirit of the cosmic censorship hypothesis. Beyond spherical
symmetry, there is some evidence for a weakened form of cosmic censorship in which massive singularities are censored [19].

*Proposition 3: asymptotics.* In an asymptotically flat space-time, $E$ coincides with the Arnowitt-Deser-Misner mass-energy $E_{ADM}$ at spatial infinity, and with the Bondi-Sachs mass-energy $E_{BS}$ at null infinity.

*Proof.* By definition [2,3,9,20]:

$$E_{BS} = \lim_{A \to \infty} \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int \mu(\mathcal{R} + e^f \theta_+ \theta_-)$$

(5)

where $\mathcal{R}$ is the Ricci scalar, $\mu$ the area form and $A = \int \mu$ the area of a family of affinely parametrised surfaces lying in a null hypersurface approaching $I^\pm$. In spherical symmetry, $\int \mu = 4\pi r^2$ and $\mathcal{R} = 2/r^2$, so that

$$E_{BS} = \lim_{\gamma \to p} E$$

(6)

for a null curve $\gamma$ approaching $p \in I^\pm$. Similarly, by definition [1,9,21]:

$$E_{ADM} = \lim_{A \to \infty} \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int \mu(\mathcal{R} + e^f \theta_+ \theta_- - \frac{1}{2} e^f (\sigma_+^-, \sigma_-^-))$$

(7)

for a family of surfaces parametrised by area radius $r = \sqrt{A/4\pi}$ lying in a spatial hypersurface approaching $i^0$, where $\sigma_{\pm}$ are the shears corresponding to the expansions $\theta_{\pm}$. In spherical symmetry, $\sigma_{\pm} = 0$, so that

$$E_{ADM} = \lim_{\gamma \to p} E$$

(8)

for a spatial curve $\gamma$ approaching $p = i^0$.

The result shows that the asymptotic energies are just special cases of $E$, defined at infinity in an asymptotically flat space-time. It is usual to interpret the asymptotic energies as measuring the total mass-energy of an asymptotically flat space-time, whereas $E$ provides a more general definition of energy which applies locally as well as asymptotically.

**III. Dynamical properties**

Having derived various purely geometrical properties of $E$, consider now applying the Einstein equations. The most general form of the Einstein tensor in spherical symmetry determines the most general material mass tensor $T$, given by the line-element

$$pd\Omega^2 + \phi_+ d\xi_+^2 + \phi_- d\xi_-^2 + 2\rho d\xi_+ d\xi_-.$$  

(9)

The Einstein equations are

$$\partial_+ \partial_+ r + \partial_+ f \partial_+ r = -4\pi \rho \phi_+$$  

(10a)

$$r \partial_+ \partial_- r + \partial_+ r \partial_- r + \frac{1}{2} e^{-f} = 4\pi r^2 \rho$$  

(10b)

$$r^2 \partial_+ \partial_- f + 2\partial_+ r \partial_- r + e^{-f} = 4\pi r^2 \rho$$  

(10c)
with units $G = 1$. The variation of $E$ is determined by these equations as

$$
\partial_\pm E = 4\pi e^f r^2 (\rho \partial_\pm r - \phi_\pm \partial_\mp r) = 2\pi e^f r^3 (\rho \theta_\pm - \phi_\pm \theta_\mp).
$$

This can also be written in a manifestly covariant form [22].

**Proposition 4:** vacuum. In vacuo, $E$ is constant and the only solution is the Schwarzschild solution with parameter $E$.

**Proof.** In vacuo, $\partial_\pm E = 0$, so $E$ is constant. The vacuum Einstein equations are

$$
\begin{align}
\partial_\pm (e^f \partial_\pm r) &= 0 \quad (12a) \\
\partial_+ \partial_- r &= -E e^{-f} r^{-2} \quad (12b) \\
\partial_+ \partial_- f &= -2 E e^{-f} r^{-3}. \quad (12c)
\end{align}
$$

Noting that $\partial_+ \partial_- (f - \log r - r/2E) = 0$, and that the rescaling freedom (2) consists of $f \mapsto f + f_+ + f_-$ for functions $f_\pm(\xi_\pm)$, one may fix

$$
e^f = re^{r/2E}/16E^3 \quad (13)$$

if $E \neq 0$. ($E = 0$ gives flat space-time). The equations (12a) integrate to $e^f \partial_{\pm} r = \eta_\mp$ for some functions $\eta_\pm(\xi_\pm)$ of integration. Differentiating using (12b), $\partial_\pm \eta_\pm = -1/4E$, which integrates to $\eta_\pm = -\xi_\pm/4E$, fixing the zero. Thus

$$
\xi_+ \xi_- = (1 - r/2E)e^{r/2E} \quad (14)
$$

which implicitly determines $r$ (and hence $f$) as a function of $\xi_+ \xi_-$. This is the Kruskal form [23] of the Schwarzschild solution with parameter $E$, which can be put in static form in terms of $t_\pm = 2E \log(-\xi_\pm/\xi_\mp)$.

This is a proof of Birkhoff’s theorem: a vacuum, spherically symmetric space-time must be the Schwarzschild solution. The proof improves on the usual one [24] in that a global coordinate patch is obtained automatically, so that one does not have to subsequently join the $r > 2E$ and $r < 2E$ patches.

The Schwarzschild solution provides an example of Propositions 1–3. For the case $E > 0$ there are trapped spatial singularities, while for the case $E < 0$ there is an untrapped temporal singularity. In the former case, there are trapped spheres in the black-hole and white-hole regions $E > 1/2r$, and the event horizons coincide with the trapping horizons $E = 1/2r$.

Consider now the non-vacuum cases. In order to obtain results which are as general as possible, the type of matter will not be fixed but energy conditions will be imposed instead. Three useful energy conditions are as follows [24,25]. The null energy (or convergence) condition states that a ‘null observer’ measures non-negative energy:

$$
\text{NEC: } g(u, u) = 0 \Rightarrow T(u, u) \geq 0. \quad (15)
$$
The weak energy condition states that a causal observer measures non-negative energy:

\[ \text{WEC: } g(u, u) \leq 0 \Rightarrow T(u, u) \geq 0. \]  \hspace{1cm} (16)

The dominant energy condition states that a future-causal observer measures future-causal momentum:

\[ \text{DEC: } g(u, u) \leq 0, g(v, v) \leq 0, g(u, v) \leq 0 \Rightarrow T(u, v) \geq 0. \]  \hspace{1cm} (17)

Clearly

\[ \text{DEC } \Rightarrow \text{ WEC } \Rightarrow \text{ NEC.} \]  \hspace{1cm} (18)

All this applies to general space-times. In the spherically symmetric case,

\[ \text{NEC } \Rightarrow \phi_\pm \geq 0 \]  \hspace{1cm} (19)

and

\[ \text{DEC } \Rightarrow \phi_\pm \geq 0, \rho \geq 0. \]  \hspace{1cm} (20)

*Proposition 5: monotonicity.* If the dominant energy condition holds on an untrapped sphere, \( E \) is non-decreasing (respectively non-increasing) in any outgoing (respectively ingoing) spatial or null direction.

*Proof.* Fix the orientation of the untrapped sphere by \( \theta_+ > 0 \) and \( \theta_- < 0 \). The the variation formula (11) and dominant energy condition (20) yield \( \partial_+ E \geq 0 \) and \( \partial_- E \leq 0 \), i.e. \( E \) is non-decreasing (respectively non-increasing) in the outgoing (respectively ingoing) null direction. If \( z = \partial/\partial \zeta \) is an outgoing spatial vector, then \( z = \beta \partial_+ - \alpha \partial_- \) with \( \alpha > 0 \) and \( \beta > 0 \), which yields \( \partial E/\partial \zeta = \beta \partial_+ E - \alpha \partial_- E \geq 0 \). Similarly for an ingoing spatial direction.

The proof illustrates the economy of the double-null approach: monotonicity in any spatial direction follows immediately from monotonicity in the null directions. This monotonicity property has the physical interpretation that the energy contained in a sphere is non-decreasing as the sphere is perturbed outwards. Note that the result is for untrapped spheres only, though a similar result for marginal spheres will be given later. There is no possibility of a similarly general monotonicity result for trapped spheres, since they do not have a preferred orientation.

*Proposition 5A: asymptotic monotonicity.* If the dominant energy condition holds at \( \mathcal{I}^{\pm} \) (respectively \( \mathcal{I}^{-} \)), then \( E_{BS} \) is non-increasing (respectively non-decreasing).

*Proof.* By Propositions 3 and 5.

At \( \mathcal{I}^{+} \), this is the Bondi-Sachs energy-loss property, which is usually interpreted as describing a loss of energy due to outgoing radiation. Similarly, the more general monotonicity property of \( E \) may be interpreted as being due to ingoing and outgoing radiation. Monotonicity also leads to positivity, as follows.
Proposition 6: positivity. If the dominant energy condition holds on an untrapped spatial hypersurface with regular centre, then \( E \geq 0 \) on the hypersurface.

Proof. By Proposition 5, since \( E = 0 \) at a regular centre.

Moreover, denoting the tangent vector to the hypersurface as before by \( \partial / \partial \zeta = \beta_+ - \alpha_\zeta \), with \( \zeta = 0 \) being the centre, one can write explicitly

\[
E(\zeta) = 4\pi \int_0^\zeta e^f r^2 \{(\beta_\rho + \alpha_\phi_\zeta) \partial_+ r - (\alpha_\rho + \beta_\phi_\zeta) \partial_\zeta r \} d\zeta.
\]

(21)

The positivity property has the physical interpretation that under the stated circumstances, total energy cannot be negative. This is not immediately obvious even given an energy condition on the matter, since gravitational potential energy tends to be negative. The result shows that the total energy \( E \), including potential energy, cannot be negative. Note that the result concerns untrapped hypersurfaces, but that \( E \) is automatically positive for trapped and marginal spheres, by Proposition 1.

Proposition 6A: asymptotic positivity. If the dominant energy condition holds on an untrapped spatial hypersurface which has a regular centre and extends to \( I^\pm \) (respectively \( i^0 \)), then \( E_{BS} \geq 0 \) (respectively \( E_{ADM} \geq 0 \)) there.

Proof. By Propositions 3 and 5.

This is the famous positive energy theorem for the spherically symmetric case. Note that the result would not be true without the assumption of a complete spatial hypersurface on which the dominant energy condition holds. For instance, \( E_{BS} < 0 \) and \( E_{ADM} < 0 \) for the negative-mass Schwarzschild solution.

Proposition 7: area inequality. If the dominant energy condition holds on an untrapped spatial hypersurface bounded at the inward end by a marginal sphere of radius \( r_0 \), then \( E \geq \frac{1}{2} r_0^2 \) on the hypersurface.

Proof. By Propositions 1 and 5.

Since \( r_0 > 0 \), this is a stronger result than mere positivity of \( E \): there is a positive lower bound on \( E \). The physical interpretation is that if there is a black or white hole of area \( 4\pi r_0^2 \), then the energy measured outside the hole is at least \( \frac{1}{2} r_0 \). As for the positivity result, one can write an explicit formula

\[
E(\zeta) = \frac{1}{2} r_0^2 + 4\pi \int_0^\zeta e^f r^2 \{(\beta_\rho + \alpha_\phi_\zeta) \partial_+ r - (\alpha_\rho + \beta_\phi_\zeta) \partial_\zeta r \} d\zeta
\]

(22)

where \( \zeta = \zeta_0 \) is the marginal surface.

Proposition 7A: asymptotic area inequality. If the dominant energy condition holds on an untrapped spatial hypersurface which is bounded at the inward end by a marginal sphere of radius \( r_0 \), and which extends to \( I^\pm \) (respectively \( i^0 \)), then \( E_{BS} \geq \frac{1}{2} r_0 \) (respectively \( E_{ADM} \geq \frac{1}{2} r_0 \)) there.
Proof. By Propositions 3 and 7.

The result is the spherically symmetric case of the isoperimetric inequality conjectured by Penrose [5]. Establishing this result even in spherical symmetry appears to be new. Malec & Ó Murchadha [26] recently showed this for maximal hypersurfaces in spherical symmetry.

The properties 5A, 6A and 7A of the asymptotic energies are of interest in their own right. Nevertheless, they are just special cases of properties of $E$. If these properties of the asymptotic energies are accorded their usual conceptual and physical importance, then the more general properties of $E$ are of even greater importance. The idealisation of asymptotic flatness is no longer necessary for the formulation of such ideas about energy.

Proposition 8: second law. If the null energy condition holds on a future (respectively past) outer trapping horizon, or on a past (respectively future) inner trapping horizon, then $E = \frac{1}{2} r$ is non-decreasing (respectively non-increasing) along the horizon [6].

Proof. Denote the tangent to the horizon by $\partial / \partial \zeta = \beta \partial_\theta - \alpha \partial_\phi$ and fix the orientations by $\theta_+ = 0$ and $\beta > 0$ on the horizon. Then $0 = \partial \theta_+ / \partial \zeta = \beta \partial_\theta \theta_+ - \alpha \partial_\phi \theta_+$ yields $\partial r / \partial \zeta = -\alpha \partial_\phi r = -\beta r \theta_- \theta_+ / 2 \partial_\theta \theta_+$. The focussing equation (10a) and null energy condition (19) yield $\partial_+ \theta_+ \leq 0$, and the signs of $\theta_-$ and $\partial_\phi \theta_+$ are determined by the definition of future or past, outer or inner trapping horizons. Thus $\partial r / \partial \zeta \geq 0$ for future outer or past inner trapping horizons, and $\partial r / \partial \zeta \leq 0$ for past outer or future inner trapping horizons.

Propositions 1–8 alone give a quite coherent picture of gravitational collapse, which may be further refined using related results [6,27]. Suppose there exists an untrapped spatial hypersurface with regular centre, to the future of which a future outer trapping horizon forms. The horizon starts at the centre and develops outwards with non-decreasing $E = \frac{1}{2} r$, and is also spatial or null [6]. Outside the horizon is an untrapped region in which $0 \leq E < \frac{1}{2} r$. Inside the horizon is a trapped region in which $E > \frac{1}{2} r$. If $r$ becomes zero inside the trapped region, and $E$ is non-zero there, then this will be a singularity which is spatial and trapped. This tends to confirm the cosmic censorship hypothesis [8]. The only ways in which this scenario could change are (i) if the central singularity is massless [11–19], (ii) if there is a non-central singularity [28], or (iii) if an inner trapping horizon forms inside the trapped region. The last possibility occurs for the Reissner-Nordström solution, but such a horizon appears to be unstable [29].

IV. Spatial hypersurfaces and the Newtonian limit

Consider any spatial hypersurface $\Sigma$. Set up coordinates $(\tau, \zeta)$ such that $\Sigma$ is given by $\tau = 0$ and $\partial / \partial \zeta$ is tangent to $\Sigma$. Choosing $\tau$ to be an affine parameter,

$$\ g(\partial / \partial \tau, \partial / \partial \tau) = -1. \tag{23}$$

Define a function $\lambda$ by

$$\ e^\lambda = g(\partial / \partial \zeta, \partial / \partial \zeta). \tag{24}$$
For any such coordinates \((\tau, \zeta)\), the rescaling freedom (2) can be used to fix \(\xi_\pm\) such that
\[
\sqrt{2} e^{-f/2} \, d\xi_\pm = d\tau \pm e^{\lambda/2} d\zeta.
\]  
(25)
The line-element (1) transforms to
\[
ds^2 = r^2 d\Omega^2 + e^{\lambda} d\zeta^2 - d\tau^2
\]  
(26)
where \(r\) and \(\lambda\) are functions of \((\tau, \zeta)\). Denote \(\dot{\varphi} = \partial\varphi/\partial\tau\) and \(\varphi' = \partial\varphi/\partial\zeta\). Then the definition (4) of \(E\) can be rewritten as
\[
1 - \frac{2E}{r} = e^{-\lambda(r')}^2 - \dot{r}^2
\]  
(27)
which is the form actually given by Misner & Sharp [4], with different notation.
The Einstein equations may be transformed to these coordinates, but for the following results it suffices to find the corresponding variation formulas for \(E\), which are
\[
E' = 4\pi r^2 (T_{00} r' - T_{01} \dot{r})
\]  
\[
\dot{E} = 4\pi r^2 e^{-\lambda}(T_{01} r' - T_{11} \dot{r})
\]  
(28a)  
(28b)
where
\[
T_{00} = T(\partial/\partial\tau, \partial/\partial\tau)
\]  
(29a)
\[
T_{01} = T(\partial/\partial\tau, \partial/\partial\zeta)
\]  
(29b)
\[
T_{11} = T(\partial/\partial\zeta, \partial/\partial\zeta).
\]  
(29c)

**Proposition 9: small spheres.** Near a regular centre with tangent \(\partial/\partial\tau\),
\[
E = \frac{4}{3} \pi r^3 T_{00} + O(r^4).
\]  
(30)

**Proof.** Near a regular centre \(r = 0\), \(T_{00} = O(1)\), \(T_{01} = O(1)\) and \(r' = O(1)\). If \(\partial/\partial\tau\) is tangent to the centre, then \(\dot{r} = O(r)\). Thus \(E' = 4\pi r^2 T_{00} r'\), which integrates along the hypersurface \(\Sigma\) to the above result.

In other words, the leading term in \(E\) is the product of volume \(\frac{4}{3} \pi r^3\) and density \(T_{00}\), as would be expected physically.

Misner & Sharp [4] derived a useful formula for \(E\) which will be derived below and used to find the Newtonian limit. Misner & Sharp considered a perfect fluid with energy density \(\varepsilon\), pressure \(p\) and velocity \(\partial/\partial\hat{r}\):
\[
T = (\varepsilon + p) d\hat{r} \otimes d\hat{r} + pg
\]  
(31a)
\[
g(\partial/\partial\hat{r}, \partial/\partial\hat{r}) = -1.
\]  
(31b)
For any chosen spatial hypersurface $\Sigma$, adapt the $(\tau, \zeta)$ coordinates to the fluid by taking $\tau = \hat{\tau}$. Then the variation formulas (28) for $E$ reduce to

\begin{align}
E' &= 4\pi r^2 \varepsilon r' \\
\dot{E} &= -4\pi r^2 p r'.
\end{align}

(32a) (32b)

The second equation (32b) expresses the rate $\dot{E}$ of work done by the force $4\pi r^2 p$ due to pressure. The first equation (32a) allows $E$ to be expressed as an integral of $\varepsilon$ over $\Sigma$. Specifically, if $\Sigma$ has a regular centre $\zeta = 0$, (32a) integrates to

$$E(\zeta) = \int_0^\zeta 4\pi r^2 \varepsilon r' d\zeta.$$ 

(33)

Since the volume form $*1$ of $\Sigma$ is given by (26) as $\int_{\Sigma} *\varphi = 4\pi \int_0^\zeta \varphi r^2 e^{\lambda/2} d\zeta$, this may be rewritten as

$$E = \int_{\Sigma} *\varepsilon e^{-\lambda/2} r' = \int_{\Sigma} *\varepsilon \left(1 + r'^2 - \frac{2E}{r}\right)^{1/2}$$

(34)

where the second expression follows from (27). Comparing with the material (or passive) mass

$$M = \int_{\Sigma} *\varepsilon$$

(35)

it can be seen that the integrand for $E$ differs from that of $M$ by a factor which Misner & Sharp interpreted as being due to kinetic and potential energy. This can be made precise in the Newtonian limit in terms of the Newtonian kinetic energy $K$ and gravitational potential energy $V$, defined by

\begin{align}
K &= \int_{\Sigma} *\frac{1}{2} \varepsilon r^2 \\
V &= -\int_{\Sigma} *\frac{M \varepsilon}{r}.
\end{align}

(36a) (36b)

Factors of the speed of light $c$ may be introduced on dimensional grounds by the formal replacements $\tau \mapsto c\tau, (r, *1) \mapsto (r, *1), (\varepsilon, M) \mapsto c^{-2}(\varepsilon, M), (p, K, V, E) \mapsto c^{-4}(p, K, V, E)$, assumed henceforth. These factors are determined simply by the desired interpretation of the various quantities as time, length, mass etc.

**Proposition 10: Newtonian limit.** For a perfect fluid on a spatial hypersurface $\Sigma$ with regular centre: if $(*1, r, \hat{r}, \varepsilon) = O(1)$ as $c \to \infty$, then $(M, K, V) = O(1)$ and

$$E = Mc^2 + K + V + O(c^{-2}).$$

(37)

Additionally, if $p = O(1)$ then

$$\dot{M} = O(c^{-2})$$

(38)

so that Newtonian conservation of mass is recovered.
Proof. Inserting the factors of $c$, (35) and (36) take the same form, so that $(M, K, V) = O(1)$, while (34) takes the form

$$E = c^2 \int_{\Sigma} * \varepsilon \left( 1 + \frac{r^2}{c^2} - \frac{2E}{c^4r} \right)^{1/2}.$$  \hspace{1cm} (39)$$

Thus $E = c^2 \int_{\Sigma} * \varepsilon + O(1) = Mc^2 + O(1)$ to leading order. Expanding the square root, $E = Mc^2 + \int_{\Sigma} * \varepsilon(\frac{1}{2}r^2 - E/r^2) + O(\varepsilon^{-2}) = Mc^2 + K + V + O(\varepsilon^{-2})$. Similarly, (32b) expands to $-4\pi r^2 \rho = \dot{M}c^2 + O(1)$.

In words, $E$ yields the Newtonian mass $M$ to leading order and the Newtonian kinetic energy $K$ and gravitational potential energy $V$ to next highest order. This illustrates how $E$ measures the total energy including contributions from mass, kinetic energy and potential energy. Note also that the quantities $M$, $K$ and $V$ are all defined in the full theory rather than just the Newtonian limit. Of these, $M$ may always be interpreted as the material (or passive) mass, whereas the interpretation of $K$ and $V$ as kinetic and potential energy makes sense in the Newtonian limit only. In general, $E$ cannot be expressed as a sum of individually meaningful energies, as the form (34) indicates. It is only the total energy which is meaningful.

Incidentally, the above considerations provide the reason for referring to $E$ as an energy rather than a mass, though the latter is more common. Although mass and energy are formally equivalent in Relativity, the two words carry connotations inherited from their status as distinct concepts in Newtonian theory. Specifically, mass is a measure of matter whereas energy exists in many different forms. So it is reasonable to describe $M$ as mass, since it is simply the integral (35) of the material density, and to describe $E$ as energy, since it contains contributions from kinetic and potential energy.

Under the assumptions of Proposition 10, the inverse metric is Euclidean to leading order, $g^{-1} = \delta^{-1} + O(\varepsilon^{-2})$, where the flat metric $\delta$ is given by $r^2d\Omega^2 + dr^2$, so that $r$ is a standard radial coordinate. Thus the flat space of Newtonian theory is recovered.

Note that it has not been necessary to introduce the usual coordinate conditions [30] required to obtain an inertial frame in which Newton’s laws hold. Such conditions are necessary if one wishes to obtain Newtonian equations from the Einstein equations, or to obtain Newtonian solutions from solutions to the Einstein equations [31]. In contrast, the Newtonian behaviour of $E$ may be obtained simply by expanding a formula for $E$ in powers of $c$, with no special coordinate conditions. In this sense, the recovery of Newtonian mass and energy from $E$ is more robust than the complete recovery of Newton’s gravitational theory from Einstein’s theory. This could be interpreted as meaning that energy is a fundamental concept which connects Newtonian theory with Relativity. Certainly this is consistent with the key role that the equivalence of mass and energy played in the historical development of Relativity.

V. Remarks: gravitational energy in general

The Misner-Sharp energy $E$ has an impressive variety of useful properties, ranging from the Newtonian limit to the black holes and singularities characteristic of strong gravitational
fields. These properties have an exact geometrical character and are simultaneously of direct physical relevance. In particular, $E$ has quite general monotonicity and positivity properties, determines the causal nature of central singularities, and determines when trapping occurs. This makes $E$ useful in many different applications, particularly regarding black holes and singularities, and indeed $E$ has been rediscovered by many authors and often plays a key role in their analyses [11–16,22,29]. One aspect which has not been explored here is quantisation of the gravitational field, where it may also be expected that the radius $r$ and energy $E$ play a key role [32–33].

The conceptual and physical importance of $E$ in spherical symmetry encourages the search for a more general definition of gravitational energy. It is widely accepted that such a general definition should reduce to $E$ in spherical symmetry, as occurs for the definition of Penrose and variations thereof [34–38], and for the definition of Hawking and variations thereof [9,39,40]. Exceptions include the Brown-York energy [41], which gives a value different from $E$ for the Schwarzschild solution, and the Bartnik energy [42], which is undefined for trapped surfaces. Whether such definitions have some other meaning is unclear, but they do not represent gravitational energy in the sense of Propositions 1–10.

Familiarity with the spherically symmetric case also yields other guidelines to more general definitions. Firstly, it is noteworthy that $E$ is defined on spheres rather than hypersurfaces. One can write an implicit expression for $E$ involving an integral (34) over a hypersurface with regular centre, if such a hypersurface exists, but the general definition (4) or (27) is a function of spheres only. This is quite different to the definition of mass $M$ as an integral (35) over a hypersurface. So rather than looking for a definition of gravitational energy as a hypersurface integral, as would be natural in Newtonian theory, one should look for a surface integral. Specifically, one wants an invariant of the intrinsic and extrinsic curvature of an embedded surface, for which Penrose introduced the name quasi-local mass or energy [34].

Another guideline to more general definitions is that one would like analogous positivity and monotonicity properties, and relations to black holes and conformal singularities. The Hawking energy [39] takes the value $\sqrt{A/16\pi}$ on a spherical marginal surface of area $A$, and so automatically generalises Propositions 1 and 8. The Hawking energy also has the same small-sphere behaviour as in Proposition 9 [43]. Generalisations of the positivity and monotonicity Propositions 5–7 for the Hawking energy have also been found [44]. A warning should be sounded that such positivity and monotonicity theorems involve certain assumptions without which the conclusions are invalid. Even the Schwarzschild parameter may be negative. Thus there is little point in searching for a definition of gravitational energy which is always non-negative by definition, as is sometimes suggested [42].

The final lesson of the spherically symmetric case is that a general definition of gravitational energy should give sensible results in the Newtonian limit. In particular, one would like to obtain the Newtonian mass to first order in $c$, with corrections interpretable as Newtonian energies. This is possible for the Penrose energy in certain cases [45]. This brings the discussion back to the physical meaning of gravitational energy as expressed in the Introduction: an effective energy which is measurable on a surface and which is produced by the non-local, non-linear interaction of the mass of sources with the energy of the consequent gravitational field.
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