Non–Standard Fermion Propagators from Conformal Field Theory

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Abstract: It is shown that Weyl spinors in 4D Minkowski space are composed of primary fields of half-integer conformal weights. This yields representations of fermionic 2–point functions in terms of correlators of primary fields with a factorized transformation behavior under the Lorentz group. I employ this observation to determine the general structure of the corresponding Lorentz covariant correlators by methods similar to the methods employed in conformal field theory to determine 2– and 3–point functions of primary fields. In particular, the chiral symmetry breaking terms resemble fermionic 2–point functions of 2D CFT up to a function of the product of momenta.

The construction also permits for the formulation of covariant meromorphy constraints on spinors in 3+1 dimensions.

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1 Introduction

The quest for a four-dimensional notion of analyticity and the related problem to define four-dimensional analogues of two-dimensional conformal field theories is a subject of much interest and under intense study since many years. Already in 1956 Feza Gürsey employed the quaternionic formulation of the Dirac equation to derive a conformally invariant nonlinear spinor equation [1]. He later returned to the topic of quaternionic analyticity several times yielding important insights and results [2, 3]. Finally, his life-long fascination in this topic and the harmonic space approach developed by the Dubna group [4, 5] merged in a remarkable and beautiful recent paper with Mark Evans and Victor Ogievetsky [6], see also [7]. In this paper the relevance and applicability of quaternionic analyticity in the framework of self-dual theories is worked out very clearly.

Notions of quaternionic analyticity arise in a similar manner also in the twistor approach to space–time [8].

Other Ansätze, which were developed recently, employ direct transfers of methods and notions of 2D conformal field theory to 4D conformal field theories. This led e.g. to the discovery of structures reminiscent of Zamolodchikov’s c–theorem [9, 10], and to new results on correlation functions in 4D conformal field theories, including in particular an extension of the central charge of 2D conformal field theory to a triple of central charges in 4 dimensions [11, 12]. Of related interest is the impressive list of recent results on quasi–primary fields in the $O(N)$ $\sigma$–model for $2 < d < 4$ [13].

Here I would like to introduce still another approach to analyticity in 3+1 dimensions: I would like to point out that left or right handed massless spinors in 3+1 dimensions can be interpreted as half–differentials on spheres in momentum space. This implies the possibility to formulate covariant phase space constraints on spinors of definite helicity in terms of (anti–)meromorphy constraints. More specifically, the entries of a spinor of negative helicity are shown to yield local representations $\psi(z, \bar{z}, E)$ of a primary field of weight $(\frac{1}{2}, 0)$, where $z(p)$ denotes stereographic coordinates in momentum space:

$$z = \frac{p_+}{|p| - p_3}$$

The Weyl equation then appears as a particular consequence of the transformation behavior under holomorphic reparametrizations:

$$\psi'(z', \bar{z}', E') = \psi(z, \bar{z}, E) \left(\frac{\partial z'}{\partial z}\right)^{-\frac{1}{2}}$$

(1)

Covariance of the construction follows because Lorentz transformations induce via $SL(2, \mathbb{C})$ holomorphic transformations of spheres in momentum space, and the resulting transformation behavior of left handed spinors complies with the corresponding transformations of half-differentials. The construction implies in particular, that left
handed spinors can be subjected to covariant constraints

\[ \frac{\partial \psi}{\partial \bar{z}} = 0 \]  

stating that a left handed spinor which does not depend on \( \bar{z}(p) \) in one inertial frame will also remain independent of this particular combination of momenta in any other inertial frame.

Another source of motivation for the present work besides the formulation of covariant analyticity constraints is due to applications to low energy QCD:

The relevance of methods of 2D field theory in certain kinematical regimes or large \( N \) expansions of QCD has been noticed in many places, and applications of notions or techniques of 2D field theory have proven fruitful recently. In particular, constructions of effective 2D field theories to describe high energy scattering in QCD were given in [14, 15].

In the present context, the expansion of massless spinors in terms of half-differentials may provide new insights into the issue of chiral symmetry breaking in low energy QCD. For an explanation of this note that the isomorphy between chiral Weyl spinors on the one hand and half-differentials on the other hand offers the possibility to write correlators of massless fermions as a sum of correlators of primary fields with a factorized transformation behavior under Lorentz transformations. If one employs the hypothesis, that any massless fermion propagator has a representation in terms of correlators of Weyl spinors, as specified in Eq. (40) below, then this offers a possibility to apply techniques of 2D conformal field theory to determine the structure of the propagators from Lorentz covariance. It turns out that the general Lorentz covariant propagator in the massless limit is determined up to 2 functions \( f_1 \) and \( f_2 \) which depend on single, but different arguments [16]:

\[ \langle \Psi(\bar{p}) \overline{\Psi}(\bar{p}') \rangle = \]  

\[ + \left( \begin{array}{ll} 0 & 1 \\ 0 & 0 \end{array} \right) \otimes \left( \begin{array}{l} z \bar{z}' \\ \bar{z}' \end{array} \right) \langle \phi(\bar{p}) \phi^+(\bar{p}') \rangle + \left( \begin{array}{ll} 1 & 0 \\ 0 & 0 \end{array} \right) \otimes \left( \begin{array}{l} 1 - z \bar{z}' \\ -z \bar{z}' \end{array} \right) \langle \psi(\bar{p}) \psi^+(\bar{p}') \rangle \]

\[ + \left( \begin{array}{ll} 1 & 0 \\ 0 & 0 \end{array} \right) \otimes \left( \begin{array}{l} \bar{z} - z \bar{z}' \\ 1 - \bar{z}' \end{array} \right) \langle \phi(\bar{p}) \psi^+(\bar{p}') \rangle + \left( \begin{array}{ll} 0 & 0 \\ 0 & 1 \end{array} \right) \otimes \left( \begin{array}{l} z' - 1 \\ -z' \end{array} \right) \langle \psi(\bar{p}) \phi^+(\bar{p}') \rangle \]

\[ \langle \psi(\bar{p}_1) \psi^+(\bar{p}_2) \rangle = \langle \phi(\bar{p}_1) \phi^+(\bar{p}_2) \rangle = f_1 \left( \frac{|\bar{p}_1|}{|\bar{p}_2|} \right) \frac{1 + z_1 z_2}{\sqrt{|\bar{p}_1||\bar{p}_2|}} \delta_{z_1 z_2} \]

\[ \langle \psi(\bar{p}_1) \phi^+(\bar{p}_2) \rangle = \langle \phi(\bar{p}_2) \psi^+(\bar{p}_1) \rangle = \frac{1}{z_1 - z_2} f_2 \left( \frac{|\bar{p}_1||\bar{p}_2|}{\sqrt{|\bar{p}_1||\bar{p}_2|}} \right) \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + z_1 \bar{z}_1)(1 + z_2 \bar{z}_2)} \]

Applications of this result to low energy QCD arise from expectations that chiral symmetry is broken even in the massless limit of QCD, since otherwise it would be hard to understand that the chiral condensates of all the light flavours seem to have
the same order of magnitude [17]. Now the derivation of Eq. (3) given in Sec. 4 implies that the massless limit of the light quark propagators must be of this form with non-vanishing $f_2$-terms, since the terms containing $f_2$ are the only terms which comply both with Lorentz covariance and chiral symmetry breaking. Note the consistency of this result: Since (3) provides the general form of a Lorentz covariant massless propagator, those parts of it which break chiral symmetry necessarily must also break translational invariance. This is in agreement with Eq. (5), since the right hand side of this equation cannot accommodate for a $\delta$-function in external momenta. The $f_1$-terms in turn preserve chiral symmetry: They do not contribute to a chiral condensate and anticommute with $\gamma_5$. Consistency of the result in this sector is expressed by the fact that these terms contain a $\delta$-function which restricts the correlator to parallel momenta.

Thus our result lends some support to conjectures that chiral symmetry breaking may appear as a consequence of confinement: Accompanying valence quarks yield background gauge fields from the point of view of the quark described by the propagator (3), and terms which result from breaking of translational invariance also break chiral symmetry.

Remarkable progress in the study of confinement has been achieved recently due to the work of Seiberg and Witten, who provide strong evidence for monopole condensation if $N = 2$ supersymmetric Yang–Mills theory is broken down to $N = 1$ supersymmetry [18].

The notion of primary fields will be introduced in a fully covariant setting in Sec. 2, while the isomorphy between Weyl spinors of definite helicity and half–differentials will be the topic of Sec 3. Sec. 4 essentially comprises a group theoretical investigation of propagators of massless fermions by means of primary field techniques to yield the result (3). Sections 3 and 4 can be read independently of Sec. 2, if the reader is familiar with the notion of half–differentials and not interested in a covariant definition of primary fields. Other readers with primary interest in the derivation of the massless propagator may be willing to take Eq. (1) as a definition and refer to Sec. 2 only if necessary.

## 2 Covariant Primary Fields

In two–dimensional field theories two apparently different formulations of covariance existed in parallel for several years. On the one hand two–dimensional field theories can be formulated covariantly in the usual way employing tensor and spinor fields, while on the other hand it is known that in a conformal gauge primary fields can be employed to ensure covariance with respect to the conformal remnant of the diffeomorphism group [19]. This was puzzling, because there exist primary fields of half–integral order on two–manifolds, and it was not clear in what sense these could
be considered as remnants of tensor or spinor fields in a conformal gauge\textsuperscript{2}. Furthermore, it was unclear how half-differentials should transform under non-conformal transformations, or how they could be defined outside the realm of conformal gauge fixing. The puzzle was partially solved by the introduction of a covariant definition of primary fields \cite{20}, thus demonstrating that primary fields yield factorized representations of the full two-dimensional diffeomorphism group. This work also included a demonstration of isomorphy between tensor fields and covariant primary fields of integer weight. However, the exact relation between spinors in two dimensions and the covariant half-differentials of \cite{20} was given only recently in \cite{21}, where the formalism was further developed and applied to two-dimensional supergravity.

Initially primary fields $\Phi$ of conformal weight $(\lambda, \bar{\lambda})$ on a two-manifold $\mathcal{M}$ are defined by their transformation behavior under a holomorphic change of charts $z \to u(z)$ \cite{19}:

$$
\Phi(u, \bar{u}) = \Phi(z, \bar{z}) \cdot \left( \frac{\partial u}{\partial z} \right)^{-\lambda} \cdot \left( \frac{\partial \bar{u}}{\partial \bar{z}} \right)^{-\bar{\lambda}}
$$

(6)

where I employed the usual convention to denote the weight for the complex conjugate sector of coordinates by $\bar{\lambda}$.

The scaling dimension of the field $\Phi$ is $\Delta = \lambda + \bar{\lambda}$ and the spin\textsuperscript{3} is $\sigma = \lambda - \bar{\lambda}$. A cohomological investigation reveals that $\sigma$ is restricted to integer or half-integer values, while no similar restriction is imposed on the scaling dimension. We will demonstrate this in the more general setting of covariant primary fields below.

The factorized transformation behavior makes primary fields particular convenient for the formulation of two-dimensional field theories and the investigation of short distance expansions. However, this definition of primary fields works only in a conformal gauge, i.e. in an atlas with holomorphic transition functions. This causes no problem for integer values of $\lambda$ and $\bar{\lambda}$, because the corresponding primary fields might be considered as remnants of tensor fields in the conformal gauge. However, such an interpretation is not possible for fractional conformal weights. Furthermore, if the metric of the two-manifold $\mathcal{M}$ is considered as a dynamical degree of freedom it is very inconvenient to switch to a conformal gauge, because this implies that two degrees of freedom of the metric corresponding to the Beltrami-parameters (see below) are hidden in the holomorphic transition functions. Therefore, in a conformal gauge it is impossible to formulate the dynamics of the metric in terms of local fields.

To avoid the restriction to conformal atlases requires a generalization of equation (6) to diffeomorphisms $z \to u(z, \bar{z})$, i.e. we will define primary fields for arbitrary atlases on smooth two-manifolds, thereby introducing a covariant definition of half-differentials. Hence, in the sequel $z, w$ and $u$ will denote complex local coordinates, but no holomorphy conditions on transformations will be assumed any more. To define covariant primary fields it is convenient to switch to a Beltrami-parametrization

\textsuperscript{2}In this section, spinor refers to 2D spinors

\textsuperscript{3}We distinguish between the spin $\sigma$ referring to rotations induced by diffeomorphisms of $\mathcal{M}$ and the spin $s$ referring to rotations of tangent frames.
of the metric:

\[(ds)^2 = \frac{2g_{\bar{z}z}}{1 + \mu_{\bar{z}}^z \cdot \mu_{z}^\bar{z}} \cdot |dz + \mu_{\bar{z}}^z \cdot d\bar{z}|^2\]  

(7)

i.e. the Beltrami-parameters \(\{\mu_{\bar{z}}^z, \mu_{z}^\bar{z}\}\) specify the metric modulo scaling transformations:

\[
\mu_{\bar{z}}^z = \frac{g_{\bar{z}z} - \sqrt{g_{\bar{z}z}^2 - g_{z\bar{z}}g_{\bar{z}\bar{z}}}}{g_{z\bar{z}}} = \mu_{\bar{z}}^z \sqrt{g_{\bar{z}z} + \sqrt{g_{\bar{z}z}^2 - g_{z\bar{z}}g_{\bar{z}\bar{z}}}}
\]

(8)

\[
\mu_{z}^\bar{z} = \frac{2\mu_{\bar{z}}^z}{g_{z\bar{z}}} \left(1 + \mu_{\bar{z}}^z \cdot \mu_{z}^\bar{z}\right)
\]

(9)

The Beltrami-parameters satisfy \(\mu_{\bar{z}}^z \cdot \mu_{z}^\bar{z} < 1\) and have a subtle transformation behavior under reparametrizations \(z \rightarrow u(z, \bar{z})\) with \(|\partial_z u| > |\partial_{\bar{z}} u|\):

\[
\mu_{u}^v = \frac{\mu_{\bar{z}}^z \cdot \partial_z u - \partial_{\bar{z}} u}{\partial_{\bar{z}} u - \mu_{\bar{z}}^z \cdot \partial_z \bar{u}} = \frac{\partial_{\bar{z}} u + \mu_{\bar{z}}^z \cdot \partial_z \bar{u}}{\partial_z u + \mu_{\bar{z}}^z \cdot \partial_{\bar{z}} u}
\]

(10)

This transformation law implies in particular

\[
\partial_z - \mu_{\bar{z}}^z \partial_{\bar{z}} = (\partial_{\bar{z}} \bar{u} - \mu_{\bar{z}}^z \partial_z \bar{u})(\partial_z - \mu_{\bar{z}}^z \partial_{\bar{z}} u) = \frac{1}{\partial_{\bar{z}} u - \mu_{\bar{z}}^z \partial_z \bar{u}}(\partial_z - \mu_{\bar{z}}^z \partial_{\bar{z}} u)
\]

This observation motivates the introduction of particular non-holonomic bases of vector fields and differentials on two-manifolds \(\mathcal{M}\):

\[
\mathcal{D}_z = \partial_z - \mu_{\bar{z}}^z \cdot \partial_{\bar{z}}
\]

(11)

\[
\mathcal{D}_{\bar{z}} = \frac{1}{1 - \mu_{\bar{z}}^z \cdot \mu_{z}^\bar{z}}(dz + \mu_{\bar{z}}^z \cdot d\bar{z})
\]

(12)

\[
\partial_{\bar{z}} = \frac{1}{1 - \mu_{\bar{z}}^z \cdot \mu_{z}^\bar{z}}(\mathcal{D}_z + \mu_{\bar{z}}^z \cdot \mathcal{D}_{\bar{z}})
\]

(13)

\[
dz = \mathcal{D}_z - \mu_{\bar{z}}^z \cdot \mathcal{D}_{\bar{z}}
\]

(14)

These bases are distinguished by their factorized transformation properties under diffeomorphisms:

\[
\mathcal{D}_u = (\mathcal{D}_u z) \mathcal{D}_z \quad \mathcal{D}u = \mathcal{D}_z \mathcal{D}_z u \quad \mathcal{D}_z u = (\mathcal{D}_u z)^{-1}
\]

(15)

thus allowing us to introduce a consistent covariant definition of primary fields:

A field \(\Phi\) over a two-manifold is denoted as primary of weight \((\lambda, \bar{\lambda})\) if its local representations \(\Phi(z, \bar{z})\) transform under a change of coordinates \(z, \bar{z} \rightarrow u, \bar{u}\) according to

\[
\Phi(u, \bar{u}) = \Phi(z, \bar{z}) \cdot (\mathcal{D}_z u)^{-\lambda} \cdot (\mathcal{D}_{\bar{z}} \bar{u})^{-\bar{\lambda}}
\]
In particular any tensor representation of the diffeomorphism group factorizes into appropriate primary fields with integer weights upon expansion with respect to the non-holonomic bases (11,12), but the crucial point is that fractional weights can be defined as well without conformal gauge fixing.

As we remarked before, there is a restriction on the admissible values of the weight \((\lambda, \bar{\lambda})\): In a region of three intersecting patches \(U_I, U_J, U_K\) with coordinates \(z_I, z_J, z_K, z_J = f_{IJ}(z_I, z_J), \ldots\), the product of transition functions for a roundtrip \(z_I \to z_J \to z_K \to z_I\) must yield the identity:

\[
(D_{z_I} f_{JK})^{\lambda} (D_{z_J} f_{IK})^{\lambda} (D_{z_K} f_{IJ})^{\lambda} (D_{z_J} f_{IK})^{\lambda} (D_{z_I} f_{JK})^{\lambda} = 1
\]  

(16)

For integer weights this condition is automatically fulfilled due to \(f_{KJ} = f_{IJ} \circ f_{JI}\) and (15). However, if \(\Delta = \frac{p}{z}, \sigma = \frac{q}{y}\) are representations of \(\Delta\) and \(\sigma\) in terms of integers without common divisors, and if \(q \neq 1\), then it is a non-trivial problem to fix the \(q\)-fold ambiguity in the definition of the transition functions \((D_{z_I} f_{JI})^{\lambda} \cdot (D_{z_J} f_{IK})^{\lambda}\) in the intersections of all patches in such a manner that the condition (16) is fulfilled. To elaborate this further, we split the transition functions into modulus and phase according to

\[
D_{z_I} f_{IJ} = R_{IJ} \exp(i\phi_{IJ})
\]

If we now stick to the convention to choose \(R_{IJ}^{\pm}\) positive real in any intersection \(U_I \cap U_J\), then (16) reduces to

\[
\exp(i\sigma \phi_{IK}) \cdot \exp(i\sigma \phi_{KJ}) \cdot \exp(i\sigma \phi_{JI}) = 1
\]

(17)

and this defines the choice of phases as a sheaf–cohomological problem:

To clarify this define

\[
S_{IJK} \equiv \exp(i\sigma \phi_{IK}) \cdot \exp(i\sigma \phi_{KJ}) \cdot \exp(i\sigma \phi_{JI})
\]

(18)

which is an element of \(Z_q\). Consider the sheaf \(\mathcal{M} \times Z_q\) with base manifold \(\mathcal{M}\) and stalk \(Z_q\). An \(n\)-cochain is a completely antisymmetric functional of intersections of \(n + 1\) patches with values in \(Z_q\):

\[
c(U_{I(0)} \cap U_{I(1)} \cap \ldots \cap U_{I(n)}) = c_{I(0)I(1)\ldots I(n)} = c_{I(1)I(0)\ldots I(n)} \in Z_q
\]

\[
c(\emptyset) = 1
\]

Then there are coboundary operators \(\delta_n\) in the pre–sheaf related to the cover \(\{U_I\}\) mapping \(n\)-cochains to \((n + 1)\)-cochains:

\[
(\delta_n c)_{IJ} = \frac{c_I}{c_J} c_{I J}
\]

\[
(\delta_1 c)_{IK} = c_{IJ} \frac{1}{c_{IK}} c_{JK}
\]

\[
(\delta_2 c)_{IJK} = c_{IJL} c_{IKL} \frac{1}{c_{JKL}} c_{JKL}
\]
and we have
\[ \delta_{n+1} \delta_n = 1 \]
Then \( S \) as defined in (18) is a closed 2-cochain: \( \delta_2 S = 1 \). Unfortunately this does not imply exactness of \( S \), because the phase factors \( \exp(i\sigma_{IJK}) \) generically do not satisfy \( x^q = 1 \). On the other hand exactness is what we are seeking, because in this case we would have
\[
S_{IJK} \equiv \exp(i\sigma_{IJK}) \cdot \exp(i\sigma_{KJI}) \cdot \exp(i\sigma_{JIL}) = (\delta_1 \theta)_{IJK} = \theta_{IJK} \theta_{KLI}
\]
for some 1-cochain \( \theta \) in \( \mathcal{M} \times \mathbb{Z}_q \) and we could rescale the phase factors \( \exp(i\sigma_{IJK}) \rightarrow \exp(i\sigma_{IJK}) \theta_{IJK} \) such that the condition (17) could be fulfilled. Therefore, we may admit only those values for the denominator \( q \) of the spin, which correspond to a trivial cohomology group \( H^2(\mathcal{M}, \mathbb{Z}_q) \). However, it is a classical result on two-manifolds that this cohomology group equals \( \theta \) for every \( \mathcal{M} \) if and only if \( q = 1 \) or \( q = 2 \) [22]. Hence, the spin of primary fields over two-manifolds is restricted to integral or half-integral values. This implies in particular that the fractional values of conformal weights appearing in the conformal grids of minimal models must be combined into the weights \((\lambda, \bar{\lambda})\) of primary fields such that \( \sigma \) is half-integer or integer. This rule seems also justified empirically, because it is in agreement with the weights appearing in explicit realizations of minimal models.

Let us now take a closer look at the correspondence between spinors on the one hand and primary fields of half-integer weights on the other hand:

As remarked before, the isomorphism between tensors and primary fields of integer weights is given by expansion with respect to the anholonomic basis (11,12) [20].

The relation between two-dimensional spinors and covariant half-differentials has been clarified by employing an appropriate zweibein formalism [21]. Therefore, consider complex orthogonal bases in the tangent frames:
\[
\bar{e}_c = \frac{1}{2}(\bar{e}_1 - i\bar{e}_2)
\]
\[
\eta_{\bar{c}\bar{c}} = 0, \quad \eta_{\bar{c}c} = \frac{1}{2}
\]
We stick to the convention that greek indices transform under the symmetry group of the tangent bundle, while latin indices refer to transformations under diffeomorphisms. Remember that in the complex orthogonal bases rotations in the tangent bundle are diagonal:
\[
\Lambda(\phi) = \begin{pmatrix}
\exp(i\phi) & 0 \\
0 & \exp(-i\phi)
\end{pmatrix}
\]
For spinors we choose a Weyl basis \( \gamma_1 = \sigma_1, \gamma_2 = \sigma_2 \) such that the spinor representation of \( SO(2) \) is diagonal as well:
\[
S(\phi) = \begin{pmatrix}
\exp(\frac{i}{2}\phi) & 0 \\
0 & \exp(-\frac{i}{2}\phi)
\end{pmatrix}
\]
In the zweibein formalism the Beltrami parameters appear as ratios of zweibein components: Insertion of
\[ g_{zz} = e_z \zeta e_z \zeta \quad g_{z\bar{z}} = \frac{1}{2}(e_z \zeta e_z \zeta + e_z \zeta e_{\bar{z}} \zeta) \]
into (8) yields
\[ e_z \zeta = \mu_z \bar{z} e_z \zeta \quad (19) \]
Therefore, the primary zweibein which transforms like a primary field of weight \((1,0)\) under diffeomorphisms is
\[ \varepsilon_z \zeta = \varepsilon_z \zeta (1 - \mu_z \bar{z} \mu_z \bar{z}) \]
Equation (19) implies for the inverse zweibein
\[ e^\bar{z} \zeta = -\mu_z \bar{z} e^z \zeta \quad (20) \]
and therefore the diagonal components of the inverse zweibein are primary fields of weight \((-1,0)\) and \((0,-1)\) respectively:
\[ \varepsilon^z \zeta = e^\bar{z} \zeta = \frac{1}{e_z \zeta} \]
Thus \(e^z \zeta\) transforms under factorized representations both under the diffeomorphism group and the tangent space rotations. Therefore the transformation behavior of fractional powers of \(e^z \zeta\) is well behaved. More specifically, \((e^z \zeta)^{-\lambda}(e^z \zeta)^{-\bar{\lambda}}\) is a primary field of weight \((\lambda, \bar{\lambda})\) under diffeomorphisms and a field of spin \(s = \bar{\lambda} - \lambda\) under tangent space rotations, and we know by (17) that \(s\) is restricted to integer and half-integer values. In particular, the sought for isomorphism between covariant half-differentials \(\psi_{\sqrt{s}}\) of weight \((\frac{1}{2}, 0)\) and chiral Weyl spinors \(\psi_{\sqrt{\zeta}}\) is [21]
\[ \psi_{\sqrt{s}} \sqrt{e^z \zeta} = \psi_{\sqrt{\zeta}} \quad (21) \]

Having established equivalence between tensors and spinors on the one hand and covariant primary fields on the other hand, it is also desirable to develop a covariant primary differential calculus. Therefore, we introduce a covariant primary derivative \(D_z\) which maps primary fields of weight \((\lambda, \bar{\lambda})\) and spin \(s\) into primary fields of the same spin and weight \((\lambda + 1, \bar{\lambda})\):
\[ D_z \Phi = D_z \Phi - \lambda \Gamma^\zeta_{zz} \Phi - \bar{\lambda} \Gamma^{\bar{z}}_{zz} \Phi - is\Omega_{z} \Phi \quad (22) \]
Covariance of this construction with respect to diffeomorphisms \(z \rightarrow u(z, \bar{z})\) and rotations \(\tilde{\zeta} \rightarrow \tilde{\zeta} \exp(-i\phi)\) implies
\[ \Gamma^u_{uu} = (D_u u)^{-1} \Gamma^z_{zz} - (D_z u)^{-2} D_z D_u u \quad (23) \]
\[ \Gamma^{\bar{\mu}}_{\bar{u}z} = (D_{\bar{u}} u)^{-1} \Gamma^{\bar{z}}_{zz} - (D_z u)^{-1}(D_{\bar{z}} \bar{u})^{-1} D_z D_{\bar{u}} \bar{u} \quad (24) \]
\[ \Omega_u = (D_u u)^{-1}(\Omega_z + D_z \phi) \quad (25) \]
In applications of this formalism in two-dimensional field theory there frequently appear the anholonomy coefficients of the primary bases (11,12), because these coefficients automatically appear as connection coefficients, if conformally gauge fixed actions like the Ising model or the bosonic string are covariantized in this formalism [20]:

\[
[D_{\bar{z}}, D_z] = C^z_{\bar{z} z} D_{\bar{z}} - C^z_{z \bar{z}} D_z \\
\frac{1}{1 - \mu_z \bar{\mu}_{\bar{z}}} (D_{\bar{z}} \mu_{\bar{z}} - \mu_z D_z \mu_{\bar{z}})
\]

The commutator of the covariant primary derivatives is then

\[
[D_{\bar{z}}, D_z] \Phi = (C^z_{\bar{z} z} - \Gamma^z_{z \bar{z}}) D_{\bar{z}} \Phi - (C^z_{z \bar{z}} - \Gamma^z_{\bar{z} z}) D_z \Phi - \lambda \mathcal{R}_{z \bar{z}} \Phi + \bar{\lambda} \mathcal{R}_{z \bar{z}} \Phi - i \mathcal{F}_{z \bar{z}} \Phi
\]

(26)
with curvature and field strength

\[
\mathcal{R}_{z \bar{z}} = D_{\bar{z}} \Gamma^z_{z \bar{z}} - D_z \Gamma^z_{z \bar{z}} - C^z_{\bar{z} z} \Gamma^z_{\bar{z} z} + C^z_{z \bar{z}} \Gamma^z_{z \bar{z}} \\
\mathcal{F}_{z \bar{z}} = D_{\bar{z}} \Omega_{\bar{z}} - D_z \Omega_{\bar{z}} - C^z_{\bar{z} z} \Omega_{\bar{z}} + C^z_{z \bar{z}} \Omega_{\bar{z}}
\]

Thus curvatures consist of primary fields of weight (1,1) in this formalism. However, due to the absence of second order terms in the connection coefficients, \( \mathcal{R} \) is not a mere translation of the ordinary curvature tensor into the primary basis.

Similar to the tensor formalism one may impose constraints on the connection: The requirement of invariance of the metric under parallel translations implies

\[
\Gamma^z_{z \bar{z}} = D_{\bar{z}} \ln(g_{z \bar{z}}) - \Gamma^z_{z \bar{z}}
\]

(27)
while the requirement of vanishing torsion implies

\[
\Gamma^z_{z \bar{z}} = C^z_{\bar{z} z}
\]

(28)
The consistency of the torsion constraint with (24) follows easily from the transformation behavior of the Lie bracket.

On the other hand, one may also impose a zweibein postulate:

\[
D_{\bar{z}} e^z_\zeta = 0 \\
D_z e^z_\zeta = 0
\]

implying

\[
i \Omega_z = \Gamma^z_{z \bar{z}} + D_{\bar{z}} \ln(e^\zeta_\zeta) \\
= -\Gamma^z_{\bar{z} z} - D_z \ln(e^\zeta_\zeta)
\]

The zweibein postulate implies in particular invariance of the metric under parallel translations (27).

In the sequel we will concentrate on primary fields of weights \((\frac{1}{2},0)\) or \((0,\frac{1}{2})\). In 2D field theory these half-differentials appear as fermionic degrees of freedom of superstrings or in the fermionic formulation of the Ising model. However, as we will see shortly, they were immanent in the physics literature since a long time in a different guise.
3 Massless Fermions and Half–Differentials

For convenience, I employ the Weyl representation for Dirac matrices. To clarify the relation between spinors in 3+1 dimensions and half–differentials, we introduce stereographic coordinates in momentum space:

\[ z = \frac{p^+}{|\vec{p}| - p_3} \]

\[ \tilde{z} = -\frac{1}{z} \]  

According to (1) the relation between local representations \( \psi(z, \tilde{z}, |\vec{p}|) \) and \( \psi(\tilde{z}, \tilde{z}, |\vec{p}|) \) of a half–differential of weight \( (\frac{1}{2}, 0) \) is

\[ \psi(\tilde{z}, \tilde{z}, |\vec{p}|) = -z\psi(z, \tilde{z}, |\vec{p}|) \]  

However, insertion of (29) demonstrates that this is the Weyl equation for a massless spinor with opposite signs of chirality and energy:

\[ (|\vec{p}| + \vec{p} \cdot \vec{\sigma}) \left( \frac{\psi(z, \tilde{z}, |\vec{p}|)}{\psi(\tilde{z}, \tilde{z}, |\vec{p}|)} \right) = 0 \]  

Similarly, the relation between local representations of a primary field of weight \( (0, \frac{1}{2}) \)

\[ \phi(\tilde{z}, \tilde{z}, |\vec{p}|) = z\phi(z, \tilde{z}, |\vec{p}|) \]

is the Weyl equation for a massless spinor of equal signs of energy and chirality:

\[ (|\vec{p}| - \vec{p} \cdot \vec{\sigma}) \left( \frac{\phi(\tilde{z}, \tilde{z}, |\vec{p}|)}{\phi(z, \tilde{z}, |\vec{p}|)} \right) = 0 \]

To complete the proof that local representations of half–differentials create Weyl spinors as indicated in (31, 33), it remains to demonstrate that these objects exhibit a spinorial transformation behavior under the full Lorentz group: Under parity or time reversal \( z(\vec{p}) \) goes to \(-\tilde{z}(\vec{p})\) and thus half–differentials of weight \( (\frac{1}{2}, 0) \) become half–differentials of weight \( (0, \frac{1}{2}) \) and vice versa. Under proper orthochronous Lorentz transformations \( \Lambda(\omega) = \exp(\frac{i}{2} \omega^{\mu \nu} L_{\mu \nu}) \), with \( \omega \) the usual set of rotation and boost parameters, \( z(\vec{p}) \) goes to

\[ z'(\vec{p}') = U \circ z(\vec{p}) = \frac{a\vec{z} + b}{c\vec{z} + d} \]

if \( E = |\vec{p}| \), and to

\[ z'(\vec{p}') = U^{-1}T \circ z(\vec{p}) = \frac{dz - c}{a - bz} \]

\[ 11 \]
if $E = -|\vec{p}|$.  
Here $U$ is the positive chirality spin representation of $\Lambda$:

$$U(\omega) = \exp(\frac{1}{2} \omega^{\mu\nu} \sigma_{\mu\nu}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$$

The transformation laws (34,35) are most easily verified through the usual decomposition of $\Lambda$ into rotations and a boost$^4$.

Now assume $E = |\vec{p}|$: A half-differential $\phi$ of weight $(0, \frac{1}{2})$ then transforms according to (1) into

$$\phi'(z', \bar{z}', |\vec{p}'|) = (c\bar{z} + d)\phi(z, \bar{z}, |\vec{p}|)$$

implying

$$\begin{pmatrix} \phi'(z', \bar{z}', |\vec{p}'|) \\ \phi'(z', \bar{z}', |\vec{p}'|) \end{pmatrix} = U \cdot \begin{pmatrix} \phi(z, \bar{z}, |\vec{p}|) \\ \phi(z, \bar{z}, |\vec{p}|) \end{pmatrix}$$

Thus a half-differential of weight $(0, \frac{1}{2})$ is equivalent to a spin--(\frac{1}{2}, 0)--representation of the proper orthochronous Lorentz group $L^1_+$. Similarly, it is proved that a half-differential of weight $(\frac{1}{2}, 0)$ is equivalent to a spin--(0, \frac{1}{2})--representation of the proper orthochronous Lorentz group if $E = |\vec{p}|$:

$$\begin{pmatrix} \psi'(z', \bar{z}', |\vec{p}'|) \\ \psi'(z', \bar{z}', |\vec{p}'|) \end{pmatrix} = U^{-1}\cdot \begin{pmatrix} \psi(z, \bar{z}, |\vec{p}|) \\ \psi(z, \bar{z}, |\vec{p}|) \end{pmatrix}$$

On the other hand, if $E = -|\vec{p}|$, then this corresponds to $U \leftrightarrow U^{-1}$ in the equations above, and the assignment of the half-differentials $\phi$ and $\psi$ to representations of $L^1_+$ is changed.

Note that his construction works in both directions: Half-differentials thus yield Weyl spinors in Minkowski space and vice versa. In this framework the half-differentials span projective representations of the Lorentz group, which are true representations due to the dependence of the half-differentials on $|\vec{p}|$.

4 Propagators

According to Sec. 3 the expansion of a massless spinor in terms of helicity states can be written

$$\Psi(p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \phi(z, \bar{z}, |\vec{p}|) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -\bar{z} \end{pmatrix} \psi(z, \bar{z}, |\vec{p}|)$$

$^4$The analog of Eq. (34) in configuration space is known since a long time. However, it seems to have escaped attention that this implies a covariant notion of analyticity in momentum space, and that expressing $z$ as a ratio actually means to introduce half-differentials.
which is the usual expansion expressed in terms of half-differentials. The massless spinor in space-time contains positive and negative frequency contributions of this kind:

\[
\Psi(x) = \frac{1}{\sqrt{2\pi}} \int \frac{d^3 \vec{p}}{2|\vec{p}|} \left( \Psi_+(\vec{p}) \exp(i\vec{p} \cdot x) + \Psi_-(\vec{p}) \exp(-i\vec{p} \cdot x) \right)
\]

Eq. (36) yields representations of the corresponding correlation functions in terms of primary fields:

\[
\langle \Psi(\vec{p}) \overline{\Psi}(\vec{p}') \rangle = \sum_{i_1, i_2 = +, -} \int \frac{d^3 \vec{p}_1}{2|\vec{p}_1|} \int \frac{d^3 \vec{p}_2}{2|\vec{p}_2|} \left( \langle \phi(\vec{p}_1) \phi^+(\vec{p}_2) \rangle \delta_{zz} \right) 
\]

Therefore, the 2-point functions on the right hand side transform under a factorized representation of the Lorentz group. This makes this representation very convenient for the investigation of all correlations \( \langle \Psi(p) \overline{\Psi}(p') \rangle \) which comply with Lorentz covariance.

While the behavior of the parameters \((z, \bar{z}, |\vec{p}|)\) under rotations is completely specified by (34) in the positive energy case, for boosts we also have to specify the behavior of \(|\vec{p}|\). For a boost \(\exp(-uL_{03})\) we have

\[
z' = \exp(-u)z \\
|\vec{p}'| = \frac{|\vec{p}|}{z \bar{z} + 1} (\exp(-u)z \bar{z} + \exp(u))
\]

We may now determine the 3-dimensional correlators of primary fields appearing on the right hand side of (37) by methods similar to the methods employed to fix 2- and 3-point functions of primary fields in 2D conformal field theory: Choose a generating set of the symmetry group, write down the covariance conditions in infinitesimal form and solve the resulting differential equations. Lorentz covariance then fixes the \((1, 0) \otimes (0, 1)\)-differential \(\langle \psi(\vec{p}_1) \phi^+(\vec{p}_2) \rangle\) up to a function \(f_1(|\vec{p}_1|/|\vec{p}_2|)\):

\[
\langle \psi(\vec{p}_1) \phi^+(\vec{p}_2) \rangle = f_1 \left( \frac{|\vec{p}_1|}{|\vec{p}_2|} \right) \frac{1 + z_1 \bar{z}_2}{\sqrt{|\vec{p}_1||\vec{p}_2|}} \delta_{zz}(z_1 - z_2)
\]

Similarly, the \((1, 0) \otimes (1, 0)\)-differential \(\langle \phi(\vec{p}_1) \phi^+(\vec{p}_2) \rangle\) takes the form

\[
\langle \phi(\vec{p}_1) \phi^+(\vec{p}_2) \rangle = \frac{1}{z_1 - z_2} f_2 \left( \frac{|\vec{p}_1||\vec{p}_2|}{1 + z_1 \bar{z}_2} \right) \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + z_1 \bar{z}_2^2)(1 + z_2 \bar{z}_1^2)}
\]

while invariance under C, P or T implies

\[
\langle \psi(\vec{p}_1) \phi^+(\vec{p}_2) \rangle = \langle \phi(\vec{p}_2) \psi^+(\vec{p}_1) \rangle \\
\langle \psi(\vec{p}_1) \psi^+(\vec{p}_2) \rangle = \langle \phi(\vec{p}_2) \phi^+(\vec{p}_1) \rangle
\]
thus establishing the result we were seeking.

As expected, Lorentz symmetry alone complies with a bilocal propagator in momentum space, and it restricts the chiral symmetry preserving part to parallel momenta. On the other hand, chiral symmetry breaking terms must account for breaking of translational invariance, in agreement with (39).

The unperturbed result for the on-shell correlation

$$\langle \psi(\vec{p}) \overline{\psi}(\vec{p}') \rangle = -2p \cdot \gamma |\vec{p}| \delta(\vec{p} - \vec{p}')$$

is recovered from Eqs. (37–39) for $f_1(x) = \delta(x-1)$, $f_2 = 0$, and thus asymptotic freedom implies that $f_1(x)$ contains $\delta(x-1)$ as a summand, while $f_2$ is bound to vanish for $|\vec{p}_1| |\vec{p}_2| \rightarrow \infty$.

Off-shell extensions of (37) can be inferred from the requirement to yield the same propagator in configuration space:

$$S(x, x') = \frac{\Theta(t-t')}{{(2\pi)^3}} \int \frac{d^3\vec{p}}{2|\vec{p}|} \int \frac{d^3\vec{p}'}{2|\vec{p}'|} \exp(ip \cdot x)i(\Psi(\vec{p})\overline{\Psi}(\vec{p}')) \exp(-ip' \cdot x')$$

$$- \frac{\Theta(t'-t)}{{(2\pi)^3}} \int \frac{d^3\vec{p}}{2|\vec{p}|} \int \frac{d^3\vec{p}'}{2|\vec{p}'|} \exp(-ip \cdot x)i(\Psi(\vec{p})\overline{\Psi}(\vec{p}')) \exp(ip' \cdot x')$$

$$= \frac{1}{{(2\pi)^4}} \int d^4p \int d^4p' \exp(ip \cdot x)S(p, p') \exp(-ip' \cdot x')$$

(40)

thus fixing the structure up to the 2 functions $f_1, f_2$. From this point of view chiral symmetry breaking in the massless limit of QCD amounts to the question which $f_2$ terms result from confinement, and how they eventually translate into master terms of effective theories.

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References


