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QUANTIZATION OF A HAMILTONIAN SYSTEM
WITH AN INFINITE NUMBER
OF DEGREES OF FREEDOM

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1 Introduction

Quantization of theories with an infinite number of degrees of freedom is the old problem. There are various approaches to it; for example, these are the canonical quantization and the method of Feynman path integrals (see [1-3]). Various aspects of the axiomatic method are presented in the monograph [4]. Among mathematical publications on this subject we indicate books [5,6]. Though a significant progress is achieved in this direction, the problem is not yet solved in the rigorous mathematical sense.

Usually, the Quantum Field Theory is regarded as a quantum theory of a system with a continuous Hamiltonian. In the present paper, we restrict our consideration to a simpler problem. We consider a system with the Hamiltonian:

$$H(p, z) = \sum_{n=-\infty}^{\infty} \{ |p_n|^2 + V(z_n) + \sum_{m=n+1}^{\infty} W_{m-n}(z_n - z_m) \}.$$

Here the functions V and W_{m-n} are real and smooth and $p_n, z_n \in R^d$ where d is a positive integer. We call variables p_n and z_n the momentum and coordinate of a particle with the number n , respectively. It is clear that this model can arise in the approximation of the Hamiltonian of the scalar field theory on a lattice, for example.

Mathematical problems of the classical dynamics of the system with the Hamiltonian H require separate investigations. We do not consider them in the present paper. We want to quantize this system by analogy with the finite-dimensional case when all sums in the Hamiltonian are finite. In fact, we construct an infinite-dimensional equation similar to the usual quantum-mechanical Schrödinger equation. To support our arguments, we prove that it is possible to pass from finite-dimensional quantum problems to the infinite-dimensional one in the limit of unbounded increase of the number of particles. Unfortunately, we are not able to indicate all publications on the subject considered in view of their large quantity. Everywhere we take the Planck's constant $\hbar = 1$. In addition, although we consider particles with equal masses, our construction holds for particles with non-equal masses.

We denote positive constants by $C, C', C_1, C_2, \dots$

Quantization of a Hamiltonian system with an infinite number of degrees of freedom

1) On pages 5-6 the text

"...Taking the space $L_{2,w}$ of complex-valued functions ψ square-integrable over the measure w defined on l_2 with the scalar product $(\phi, \psi)_{2,w} = \int_{l_2} \phi(z) \overline{\psi}(z) w(dz)$, one obtains that the operator D is well-defined on the set $R \subset L_{2,w}$ of all linear combinations of cylindrical functions $f(z_{-N}, \dots, z_N) \in C_0^\infty$ and is symmetric and non-positive on this set.

Lemma 1

The set R is dense in $L_{2,w}$.

Proof. Since the set of functions $f(z_{-N}, \dots, z_N) \in C_0^\infty(R^{d(2N+1)})$ is a dense set in the space

$$L_{2,w}^N = \{f = f(z_{-N}, \dots, z_N) \mid \int_{l_2} |f(z_{-N}, \dots, z_N)|^2 w(dz) < \infty\},$$

it is sufficient to prove that the set $T = \bigcup_{N=1}^{\infty} L_{2,w}^N$ is dense in the space $L_{2,w}$. Furthermore, it is sufficient to prove that any real bounded function $f \in L_{2,w}$ can be approximated arbitrary closely by functions from the set T .

Let a function $f \in L_{2,w}$ be real and $|f| < A$. Let us fix a positive integer M and consider the following sets:

$$L_i = f^{-1}\left(-A + \frac{2A}{M}(i-1), -A + \frac{2A}{M}i\right]$$

where $i = 1, \dots, M$. Obviously, $\bigcup_{i=1}^M L_i = l_2$. By definition of an arbitrary measure, for each i and M there exists a cylindrical set L_i^M such that

$$w((L_i \setminus L_i^M) \cup (L_i^M \setminus L_i)) < M^{-2}.$$

We set $g_M(z) = -A$ for $z \in L_1^M$, $g_M(z) = -A + \frac{2A}{M}(i-1)$ for $z \in L_i^M \setminus (\bigcup_{k=1}^{i-1} L_k^M)$ where $i = 2, \dots, M$ and $g_M(z) = A$ for any $z \in l_2 \setminus (\bigcup_{i=1}^M L_i^M)$. Then, g_M are cylindrical functions. One can simply verify that $g_M \rightarrow f$ in $L_{2,w}$ as $M \rightarrow \infty$. Thus, Lemma 1 is proved."

must be replaced by

"We set

$$(\phi, \psi)_{2,w} = \overline{(\psi, \phi)}_{2,w} = \int_{l_2} \rho_M^{-\frac{1}{2}} \rho_N^{\frac{1}{2}} \phi(z) \overline{\psi(z)} w(dz)$$

for $\phi \in C_0^\infty(R^{d(2N+1)})$ and $\psi \in C_0^\infty(R^{d(2M+1)})$ where $N \leq M$. Then, we extend this function $(\cdot, \cdot)_{2,w}$ to the linear space R consisting of all linear combinations of cylindrical functions of the class C_0^∞ using the bilinearity of this form. One can simply verify that $(\cdot, \cdot)_{2,w}$ is the well-defined scalar product on the space R . By $L_{2,w}$ we denote the completion of the space R with the norm $\|g\|_{2,w} = (g, g)_{2,w}^{\frac{1}{2}}$. Let also $L_{2,w}^N$ be the completion of the space $R_N = \{f(z_{-N}, \dots, z_N) \in C_0^\infty(R^{d(2N+1)})\}$ with the norm $\|\cdot\|_{2,w}$. Then, straightforward calculations show that the operator D is symmetric and non-positive on the space R . By construction, the following result also takes place.

Lemma 1

The set $\bigcup_N R_N$ is dense in $L_{2,w}$."

2) In Lemma 4, the phrase

"The operator B_N defined on the domain $R \cap L_{2,w}^N$..."

must be replaced by

"The operator B_N defined on the domain R_N ..."

3) On page 9, line 5 from below, the expression " $h \in L_{2,w}^N$ " must be replaced by " $h \in R_{N_0}$ "

2 Finite-dimensional quantum mechanics with weighted spaces of states

Consider a system with the Hamiltonian

$$H_N(\bar{p}_N, \bar{z}_N) = \sum_{n=-N}^N \{ |p_n|^2 + V(z_n) + \sum_{m=n+1}^N W_{m-n}(z_n - z_m) \}.$$

Here $\bar{p}_N = (p_{-N}, \dots, p_N)$, $\bar{z}_N = (z_{-N}, \dots, z_N) \in R^{d(2N+1)}$. As usual, to quantize this system, one has to substitute the operators of the n th momentum $P_n = \frac{1}{i} \frac{\partial}{\partial z_n}$ and of the n th coordinate $Q_n = z_n$ for the n th momentum and coordinate (see [3]). Therefore, the quantum Schrödinger equation for this system is the following:

$$i\phi_t^N = -\Delta\phi^N + \left\{ \sum_{n=-N}^N [V(z_n) + \sum_{m=n+1}^N W_{m-n}(z_n - z_m)] \right\} \phi^N \quad (1)$$

where $\phi^N = \phi^N(t, \bar{z}_N)$ is the complex-valued wave function, $z_n = (z_{n,1}, \dots, z_{n,d}) \in R^d$ and $\Delta = \sum_{i=-N}^N \sum_{j=1}^d \frac{\partial^2}{\partial z_{i,j}^2}$. In this case, the standard space of states is $L_2(R^{d(2N+1)})$ with the norm

$$\|g\|_2^2 = \int_{R^{d(2N+1)}} |g(x)|^2 dx.$$

On the other hand, one can modify this approach taking the weighted Lebesgue space $L_{2,\rho}$ with the norm

$$\|h\|_{2,\rho}^2 = \int_{R^{d(2N+1)}} \rho(x) |h(x)|^2 dx$$

instead of the space L_2 (here $\rho(x)$ is a measurable positive function). Taking $\psi^N = \rho^{-\frac{1}{2}} \phi^N$ and assuming for ϕ^N Eq. (1), we obtain the following Schrödinger equation

$$i\psi_t^N = -\rho^{-\frac{1}{2}} \Delta \rho^{\frac{1}{2}} \psi^N + U_N(\bar{z}_N) \psi^N \quad (2)$$

with the space of states $L_{2,\rho}$ (here $U_N = \sum_{n=-N}^N \{ V(z_n) + \sum_{m=n+1}^N W_{m-n}(z_n - z_m) \}$). Because for Eq. (1) an observable is a self-adjoint operator $B = B^*$ on the space L_2 with the mean value

$$\langle B \rangle = (B\phi^N, \phi^N)_2,$$

one has with the new notation that the observable B transforms into $B' = \rho^{-\frac{1}{2}} B \rho^{\frac{1}{2}}$ with the mean value

$$\langle B' \rangle = (B' \psi^N, \psi^N)_{2, \rho}.$$

Of course, all these considerations are only formal.

3 Preliminaries

Our first main assumption consists of the locality of interactions between particles. It means the equality

$$W = \sum_{n=-\infty}^{\infty} \sum_{m=n+1}^{\infty} W_{m-n}(z_n - z_m) = \sum_{n=-\infty}^{\infty} \sum_{m=n+1}^{n+K} W_{m-n}(z_n - z_m) \quad (3)$$

with some K finite and independent of n . Further, we take the real space

$$l_2 = \{u = (\dots, u_{-n}, \dots, u_n, \dots) : u_n \in R^d, \|u\|_{l_2}^2 = \sum_{n=-\infty}^{\infty} |u_n|^2\}$$

for variables z_n . Of course, l_2 is a Hilbert space with the scalar product $(u, v)_{l_2} = \sum_{-\infty}^{\infty} (u_k, v_k)_{R^d}$. Our hypothesis (3) leads us to the well-definiteness

of the potential energy of the exterior fields $\sum_{n=-\infty}^{\infty} V(u_n)$ and the energy of interactions W under suitable assumptions on this functions.

We take the standard orthonormal basis $\{e_{k,r}\}$, $r = \overline{1, d}$, $k = 0, \pm 1, \pm 2, \dots$, in l_2 where all components of the vector $e_{k,r}$ except the component with the number (k, r) are equal to zero and the component with the number (k, r) is equal to 1. Then, let positive numbers $\lambda_{k,r}$, $r = \overline{1, d}$, $k = 0, \pm 1, \pm 2, \dots$ be such that $\sum_{k=-\infty}^{\infty} \sum_{r=1}^d \lambda_{k,r}^{-1} < \infty$. We take the self-adjoint operator S defined onto l_2 by the rule: $S e_{k,r} = \lambda_{k,r}^{-1} e_{k,r}$. It is obvious that S is a positive operator of trace class. Therefore, one has the (countably additive) Gaussian measure w on l_2 with the zero mean value and the correlation operator S . It is known that w is a Borel measure satisfying $w(l_2) = 1$.

Let us briefly remind the method of construction of a Gaussian measure. We consider our concrete case. So, a set $M \subset l_2$ is called the cylindrical set iff

$$M = \{u \in l_2 : [(u, e_{-N,1})_{l_2}, \dots, (u, e_{-N,d})_{l_2}, \dots, (u, e_{N,1})_{l_2}, \dots, (u, e_{N,d})_{l_2}] \in F\}$$

for some integer positive N and a Borel set $F \subset R^{d(2N+1)}$. We put for a cylindrical set

$$w(M) = (2\pi)^{-\frac{d(2N+1)}{2}} \prod_{i=-N}^N \prod_{r=1}^d \lambda_{i,r}^{\frac{1}{2}} \int_F e^{-\frac{1}{2} \sum_{i=-N}^N \sum_{r=1}^d \lambda_{i,r} x_{i,r}^2} dz$$

where dz is the Lebesgue measure in $R^{d(2N+1)}$. Then, since S is the operator of trace class, one can extend the measure constructed to the minimal sigma-algebra containing the algebra of cylindrical sets. It can be proved that it is the Borel sigma-algebra (see [7,8] for details).

We set $\rho_N(z) = (2\pi)^{-\frac{d(2N+1)}{2}} \prod_{i=-N}^N \prod_{r=1}^d \lambda_{i,r}^{\frac{1}{2}} e^{-\frac{1}{2} \sum_{i=-N}^N \sum_{r=1}^d \lambda_{i,r} x_{i,r}^2}$ where $z \in l_2$. We call a function $f(z)$ defined on l_2 the cylindrical function iff $f = f(z_{-N}, \dots, z_N)$ for some N .

Further, for a cylindrical function f of the class $C_0^\infty(R^{d(2N+1)})$ we set

$$Df = \rho_N^{-\frac{1}{2}} \Delta_N \rho_N^{\frac{1}{2}} f(z_{-N}, \dots, z_N)$$

where $\Delta_N = \sum_{i=-N}^N \sum_{r=1}^d \frac{\partial^2}{\partial x_{i,r}^2}$. Since for any cylindrical function f there exists at most one positive integer N such that $f \in C_0^\infty(R^{d(2N+1)})$, it is correct. Further, we extend our operator D to the set of all linear combinations of cylindrical functions of the class C_0^∞ using the linearity of the operator D . It is possible because if one has $0 = \sum_{k=1}^m f_k(z_{-N_k}, \dots, z_{N_k})$ where $N_1 < N_2 < \dots < N_m$ and $f_k \in C_0^\infty(R^{d(2N_k+1)})$, then $f_k \equiv 0$ for all k , and therefore the representation $f(z) = \sum_{k=1}^m f_k(z_{-N_k}, \dots, z_{N_k})$ where $N_1 < N_2 < \dots < N_m$ and $f_k \in C_0^\infty(R^{d(2N_k+1)})$ is unique for any function f . Taking the space $L_{2,w}$ of complex-valued functions ψ square-integrable over the measure w defined on l_2 with the scalar product $(\phi, \psi)_{2,w} = \int_{l_2} \phi(z) \overline{\psi(z)} w(dz)$, one obtains that the operator D is well-defined on the set $R \subset L_{2,w}$ of all linear combinations of cylindrical functions $f(z_{-N}, \dots, z_N) \in C_0^\infty$ and is symmetric and non-positive on this set.

Lemma 1

The set R is dense in $L_{2,w}$.

Proof. Since the set of functions $f(z_{-N}, \dots, z_N) \in C_0^\infty(R^{d(2N+1)})$ is a

dense set in the space

$$L_{2,w}^N = \{f = f(z_{-N}, \dots, z_N) \mid \int_{l_2} |f(z_{-N}, \dots, z_N)|^2 w(dz) < \infty\},$$

it is sufficient to prove that the set $T = \bigcup_{N=1}^{\infty} L_{2,w}^N$ is dense in the space $L_{2,w}$. Furthermore, it is sufficient to prove that any real bounded function $f \in L_{2,w}$ can be approximated arbitrary closely by functions from the set T .

Let a function $f \in L_{2,w}$ be real and $|f| < A$. Let us fix a positive integer M and consider the following sets:

$$L_i = f^{-1}\left(-A + \frac{2A}{M}(i-1), -A + \frac{2A}{M}i\right]$$

where $i = 1, \dots, M$. Obviously, $\bigcup_{i=1}^M L_i = l_2$. By definition of an arbitrary measure, for each i and M there exists a cylindrical set L_i^M such that

$$w((L_i \setminus L_i^M) \cup (L_i^M \setminus L_i)) < M^{-2}.$$

We set $g_M(z) = -A$ for $z \in L_1^M$, $g_M(z) = -A + \frac{2A}{M}(i-1)$ for $z \in L_i^M \setminus (\bigcup_{k=1}^{i-1} L_k^M)$ where $i = 2, \dots, M$ and $g_M(z) = A$ for any $z \in l_2 \setminus (\bigcup_{i=1}^M L_i^M)$. Then, g_M are cylindrical functions. One can simply verify that $g_M \rightarrow f$ in $L_{2,w}$ as $M \rightarrow \infty$. Thus, Lemma 1 is proved.

4 Quantization of the infinite-dimensional system

Now we present our infinite-dimensional Schrödinger equation corresponding to the Hamiltonian H . It is

$$i \frac{\partial}{\partial t} \psi = -D\psi + U(z)\psi, \quad t \in R, \quad z \in l_2 \quad (4)$$

with the initial data

$$\psi|_{t=0} = \psi_0(z) \quad (5)$$

where $\psi = \psi(t, z)$ is an unknown complex-valued function of $t \in R$ and $z \in l_2$ and $U(z) = \sum_{n=-\infty}^{\infty} V(z_n) + W$ is the potential. Of course, the problem (4),(5) is considered for $\psi(t, \cdot) \in L_{2,w}$. We have to impose additional

restrictions on the functions V and W_n for the operator $-D + U$ having self-adjoint extensions. They are the following.

(H1) Let V and W_n be real C^1 - functions.

(H2) Let there exist constants $C > 0$ and $\gamma \in (0, 2)$ such that $|V(s)| \leq C|s|^2 e^{|s|^{2-\gamma}}$ for all s .

(H3) Let $|V'(s)| \leq C|s|$ and $|W'_n(s)| \leq C|s|$ for some positive numbers C and s_0 where $|s| < s_0$ and $n = \overline{1, K}$.

(H4) Let $V(s) \geq 0$ for all s .

(H5) Let there exist $C < \frac{1}{2(K+1)}$ such that

$$|W_n(z_1 - z_2)| < C(V(z_1) + V(z_2))$$

for all z_1, z_2 and $n = \overline{1, K}$.

Remark 1

Let us present examples of functions V and W satisfying the hypothesis (H5). First, let $d = 1$. Then, they may be $V(s) = m|s|^2 + V_1(s)$ where $m > 0$ and $V_1(s) \geq 0$ and $W_0 = 0$, $W_1(s) = as^2$ (here $|a| < \frac{m}{4}$ and $K = 1$). Obviously, this example may arise in the case of the scalar field theory when one considers its approximation on a lattice. Second, for the same function V let functions W_n satisfy the following condition: $|W_n(s)| \leq C(|s|^p + |s|^2)$ for some $p \geq 2$ and a small enough constant $C > 0$. Since $\sum_{n=-\infty}^{\infty} |z_n|^p \leq \sum_{n=-\infty}^{\infty} |z_n|^2$ for all $z \in l_2$, the condition (H5) is valid in this case, too. In addition, in each case it is simple to find conditions for V_1 to satisfy hypothesis (H1)–(H4).

Lemma 2

Under the hypotheses (H2) and (H5) the estimate

$$|U(z)| \leq C e^{\|z\|_{l_2}^{2-\gamma}}$$

is valid with some $C > 0$ and $\gamma_1 \in (0, 2)$ independent of $z \in l_2$.

Proof. Let $z \in l_2$. Then, the sequence $\{z_n\}$ is bounded by the constant $\|z\|_{l_2}$. Hence, one has:

$$|W(z)| \leq \left| \sum_{n=-\infty}^{\infty} V(z_n) \right| \leq C e^{\|z\|_{l_2}^{2-\gamma}} \sum_{n=-\infty}^{\infty} |z_n|^2 = C \|z\|_{l_2}^2 e^{\|z\|_{l_2}^{2-\gamma}} \leq C' e^{\|z\|_{l_2}^{2-\gamma}}$$

where $C, C' > 0$ and $\gamma_1 \in (0, 2)$ and Lemma 2 is proved.

Lemma 3

Under the hypotheses (H1)–(H5) the operator $U : \psi(z) \rightarrow U(z)\psi(z)$ maps any bounded function $\psi \in L_{2,w}$ into $L_{2,w}$. In addition, $\|U\|_{2,w} < \infty$ (here the norm of the function U is taken).

Proof. First of all, by Lemma 2 one has that $U(z)$ is defined for any $z \in l_2$. Let us prove that $U(z)$ is a continuous function from l_2 into R . Indeed, one has by the Hölder inequality and the boundedness of sequences $\{z_n^{(1)}\}$ and $\{z_n^{(2)}\}$ where $z^{(1)}$ and $z^{(2)}$ are elements of the space l_2 :

$$\begin{aligned} & |U(z^{(1)}) - U(z^{(2)})| = \\ &= \sum_{n=-\infty}^{\infty} \left\{ \nabla V(\bar{z}_n)(z_n^{(1)} - z_n^{(2)}) + \sum_{m=n+1}^{n+K} \nabla W_{m-n}(\bar{z}_n)(z_n^{(1)} - z_n^{(2)} - z_m^{(1)} + z_m^{(2)}) \right\} \leq \\ &\leq C(\|z^{(1)}\|_{l_2}, \|z^{(2)}\|_{l_2})\|z^{(1)} - z^{(2)}\|_{l_2}. \end{aligned}$$

Let $\psi \in L_{2,w}$ be a bounded function. Then, since the function $U(z)$ is measurable with respect to the measure w , the Hölder inequality implies:

$$\left\{ \int_{l_2} |U(z)\psi(z)|^2 w(dz) \right\}^{\frac{1}{2}} \leq C\|U\|_{2,w}.$$

Hence, to prove Lemma 3, it is sufficient to prove that $\|U\|_{2,w} < \infty$. For this aim, we use Lemma 2 and the following well-known result: for any Gaussian measure w on a real separable Hilbert space X one has

$$\int_X e^{\alpha\|x\|_X^2} w(dx) < \infty$$

for sufficiently small $\alpha > 0$ (see [8]). By this result and Lemma 2 one has: $\|U\|_{2,w} < \infty$ and Lemma 3 is proved.

Our first aim now is to prove that the operator $-D + U$ defined on the domain R is essentially self-adjoint. Let E be the identical operator. Let $V_N(z) = \sum_{n=-N}^N V(z_n)$ and let $B_N = E - D_N + V_N$ be the operator with the domain $R \cap L_{2,w}^N$ where $D_N \psi = \rho_N^{-\frac{1}{2}} \sum_{i=-N}^N \sum_{j=1}^d \frac{\partial^2}{\partial x_{i,j}^2} \rho_N^{\frac{1}{2}} \psi$.

Lemma 4

The operator B_N defined on the domain $R \cap L_{2,w}^N$ is essentially self-adjoint on the space $L_{2,w}^N$ for any N .

Proof. It is clear that the operator D_N is well-defined on $R \cap L_{2,w}^N$. Consider on the space $L_2(R^{d(2N+1)})$ the operator $G_N = -\Delta + V_N + E$ with the domain $C_0^\infty(R^{d(2N+1)})$. It is known that it is essentially self-adjoint (see [9], theorem X.28). Let $\overline{G}_N$ be its closure with the domain H . We want to prove that the operator B_N^* adjoint to B_N on the space $L_{2,w}^N$ has the domain $\rho_N^{-\frac{1}{2}}H$ and is of the kind $B_N^* = \rho_N^{-\frac{1}{2}}\overline{G}_N\rho_N^{\frac{1}{2}}$. Indeed, for any $\phi \in C_0^\infty(R^{d(2N+1)})$ and $\psi \in \rho_N^{-\frac{1}{2}}H$ the equality

$$(B_N\phi, \psi)_{2,w} = (\phi, \rho_N^{-\frac{1}{2}}\overline{G}_N\rho_N^{\frac{1}{2}}\psi)_{2,w}$$

is obvious.

Let $\psi \notin \rho_N^{-\frac{1}{2}}H$ but the adjoint operator B_N^* is defined: $B_N^*\psi \in L_{2,w}^N$. Taking $u = \rho_N^{\frac{1}{2}}\phi$, $v = \rho_N^{\frac{1}{2}}\psi$, we obtain:

$$(G_N u, v)_2 = (u, \xi)_2$$

for some $\xi \in L_2(R^{d(2N+1)})$ and for all $u \in C_0^\infty(R^{d(2N+1)})$. But since $v \notin H$, it contradicts the properties of the operator G_N . So, $B_N^* = \rho_N^{-\frac{1}{2}}\overline{G}_N\rho_N^{\frac{1}{2}}$.

Further, for the operator B_N^* with the domain $\rho_N^{-\frac{1}{2}}H$ the equality

$$(B_N^*\phi, \psi)_{2,w} = (\phi, B_N^*\psi)_{2,w}$$

is obvious for arbitrary ϕ and ψ from this domain. Hence, the operator B_N^* is self-adjoint. Thus, Lemma 4 is proved.

Lemma 5

Under hypotheses (H1)–(H5) the operator $A = E - D + V$ with the domain R is essentially self-adjoint.

Proof. Let $V(z) = \sum_{n=-\infty}^{\infty} V(z_n)$. According to theorem X.26 from [9], to prove Lemma 5, it is sufficient to prove that $A(R)$ is a dense set in $L_{2,w}$. Fix arbitrary $\epsilon > 0$ and a cylindrical function $h \in R$. Let $h \in L_{2,w}^N$. Then, one has that $\|V_N - V\|_{2,w} \rightarrow 0$ as $N \rightarrow \infty$ because $V_N(z) \rightarrow V(z)$ as $N \rightarrow \infty$ monotonously for any $z \in l_2$ and by Lemma 3. Hence, there exists $N_1 \geq N$ such that

$$\|V_{N_1} - V\|_{2,w} < \frac{\epsilon}{2(\|h\|_{2,w} + \epsilon)}.$$

Further, since by Lemma 4 and theorem X.26 from [9] $B_{N_1}(R \cap L_{2,w}^{N_1})$ is a dense set in $L_{2,w}^{N_1}$, there exists $\phi \in R \cap L_{2,w}^{N_1}$ such that

$$\|B_{N_1}\phi - h\|_{2,w} < \frac{\epsilon}{2}.$$

So, we obtain

$$\|A\phi - h\|_{2,w} \leq \frac{\epsilon}{2} + \|(V - V_{N_1})\phi\|_{2,w} \leq \frac{\epsilon}{2}(1 + \|\phi\|_{2,w}[\|h\|_{2,w} + \epsilon]^{-1}).$$

Furthermore, multiplying the equality $B_{N_1}\phi = h + h_1$ where $h_1 \in L_{2,w}^{N_1}$ and $\|h_1\|_{2,w} < \epsilon$ by ϕ and integrating over $L_{2,w}$, we find:

$$\|\phi\|_{2,w} < \|h\|_{2,w} + \epsilon.$$

The latest two inequalities imply:

$$\|A\phi - h\|_{2,w} < \epsilon$$

and Lemma 5 is proved.

Corollary

Under the hypotheses (H1)–(H5) the operator $-D + U$ with the domain R is essentially self-adjoint on $L_{2,w}$.

Proof. Let $T = A + W$. Then, by the hypothesis (H5) one has for any $\phi \in R$:

$$|(W\phi, \phi)_{2,w}| \leq C(V\phi, \phi)_{2,w} \leq C(A\phi, \phi)_{2,w}$$

with some $C \in (0, 1)$. Obviously, this inequality is valid also for any ϕ from the domain of the operator $\bar{A}$. Thus, by theorem X.17 from [9] the operator T with the domain R is essentially self-adjoint and Corollary is proved.

Remark 2

By analogy one can simply prove that operators $-D_N + U_N$ with the domain R are essentially self-adjoint on $L_{2,w}$.

Our interpretation of Eq. (4) is usual. It means that the function $|\psi(t, \cdot)|^2$ is the density of probability of finding our particles in a domain Ω of the space l_2 . So, we have to prove the following result.

Proposition

Let hypotheses (H1)–(H5) be valid and let $\psi(t, z)$ be a solution of the problem (4), (5). Then,

$$\int_{I_2} |\psi(t, z)|^2 w(dz) = 1$$

for any $t \in R$ if

$$\int_{I_2} |\psi_0(z)|^2 w(dz) = 1.$$

To prove this statement, we proceed as in the finite-dimensional quantum mechanics. We denote the closure of the operator $-D + U$ by $-D + U$, again. Then, the (generalized) solution of the problem (4), (5) is given by the formula

$$\psi(t, \cdot) = e^{-it(-D+U)}\psi_0.$$

Since $e^{-it(-D+U)}$ is a unitary operator on $L_{2,w}$, Proposition is proved.

Further, we have to introduce observables. As usual, we call a self-adjoint operator B on the space $L_{2,w}$ the observable. The whole spectral theory of self-adjoint operators in Hilbert spaces is applicable in our case. In particular, spectral theorems are valid. We define the mean value of an observable B by the formula

$$\langle B(t) \rangle = (B\psi(t, \cdot), \psi(t, \cdot))_{2,w}.$$

There are no problems with formal definitions of operators of the momentum and coordinate of the particle with number n . For a cylindrical C_0^∞ -function from $L_{2,w}^N$ with $N > |n|$ we introduce these operators by the rule:

$$P_n \psi(z) = \frac{1}{i} \rho_N^{-\frac{1}{2}} \frac{\partial}{\partial z_n} \rho_N^{\frac{1}{2}} \psi(z) \quad \text{and} \quad Q_n \psi(z) = z_n \psi(z),$$

respectively. Then, we extend these definitions to all linear combinations of cylindrical functions from C_0^∞ .

Remark 3

The existence of self-adjoint extensions of the operators P_n and Q_n is an open question. Of course, these are symmetric operators.

5 Passing from a finite-dimensional quantum mechanics to the infinite-dimensional one

We want to demonstrate the possibility of passing to the limit $N \rightarrow \infty$ from the Schrödinger equation (2) to Eq. (4). We take $\rho(z) = \rho_N$ where ρ is the function from Section 2. The space of states for the Schrödinger equation (2) is $L_{2,w}^N$. Equation (2) completed by initial data takes the following form:

$$i \frac{\partial}{\partial t} \psi^N = (-D_N + U_N) \psi^N, \quad (6)$$

$$\psi^N|_{t=0} = \psi_0^N(x) \in L_{2,w}^N. \quad (7)$$

Theorem

Let the hypotheses (H1)-(H5) be valid. Consider problems (4), (5) and (6), (7). Let $\psi_0^N \rightarrow \psi_0$ as $N \rightarrow \infty$ in the sense of the space $L_{2,w}$. Then, for any $t > 0$ one has

$$\psi^N(t, \cdot) \rightarrow \psi(t, \cdot) \quad \text{as } N \rightarrow \infty \text{ in } L_{2,w}.$$

Proof. The conditions of Theorem imply

$$\begin{aligned} & \|\psi(t) - \psi^N(t)\|_{2,w} \leq \\ & \| [e^{-it(-D+U)} - e^{-it(-D_N+U_N)}] \psi_0 \|_{2,w} + \|\psi_0 - \psi_0^N\|_{2,w}. \end{aligned} \quad (8)$$

The last term in the right-hand side of the inequality (8) tends to zero. Consider the first term. One has that $(-D_N + U_N)\psi \rightarrow (-D + U)\psi$ as $N \rightarrow \infty$ for any $\psi \in R$ (because $\|(U_N - U)\psi\|_{2,w} \leq C\|U_N - U\|_{2,w} \rightarrow 0$ as in the proof of Lemma 5). In addition, R is the essential domain for all operators $(-D + U)$ and $(-D_N + U_N)$ where $N = 1, 2, 3, \dots$ Therefore, by Theorems VIII.20 and VIII.25 from [9]

$$\| [e^{-it(-D+U)} - e^{-it(-D_N+U_N)}] \psi_0 \|_{2,w} \rightarrow 0$$

as $N \rightarrow \infty$. Thus, Theorem is proved.

Remark 4

It is known that it may be difficult enough even in the finite-dimensio-

nal case to answer the question if a Schrödinger operator is essentially self-adjoint. In our infinite-dimensional case difficulties increase because there are no such powerful methods as created for the finite-dimensional problem. We need this property for our operators $-D+U$ and $-D_N+U_N$ to prove Theorem from this Section. So, we present some sufficient conditions (see the hypotheses (H1)–(H5)) for the essential self-adjointness of these operators. It is very probable that they are far from necessary conditions.

Remark 5

Our system with the Hamiltonian H admits a simple physical interpretation. Let h be a positive number, let $x_n = nh$ where $n = 0, \pm 1, \pm 2, \dots$ and let in each position x_n we have a particle with the coordinate z_n and the momentum p_n interacting with nearest particles with numbers $m : |m - n| \leq K$. In addition, our particles are in the field of external forces described by the potential V . Obviously, this model can be described by the Hamiltonian H .

Remark 6

One can repeat our one-dimensional construction in the multi-dimensional case when the Hamiltonian takes the following form:

$$H = \sum_{i_1, \dots, i_m = -\infty}^{\infty} \left\{ |p_{i_1, \dots, i_m}|^2 + V(z_{i_1, \dots, i_m}) + \right. \\ \left. + \sum_{k_1=0, \dots, k_m=0}^K W_{k_1, \dots, k_m}(z_{i_1, \dots, i_m} - z_{i_1+k_1, \dots, i_m+k_m}) \right\}$$

where $p_{i_1, \dots, i_m} \in R^d$ and $z_{i_1, \dots, i_m} \in R^d$ are the momentum and coordinate of the particle with number $(i_1, \dots, i_m)$.

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Квантование некоторой гамильтоновой системы
с бесконечным числом степеней свободы

Рассматривается квантование некоторой дискретной гамильтоновой системы с бесконечным числом степеней свободы. Предлагаемый подход основан на аналогии с обычной конечномерной квантовой механикой. Таким образом строится некоторое бесконечномерное уравнение Шредингера. В поддержку предлагаемого подхода доказан предельный переход от конечномерной квантовой механики к предлагаемой конструкции в пределе, когда число частиц стремится к бесконечности. В работе используются строгие математические методы.

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Zhidkov P.E.

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Quantization of a Hamiltonian System
with an Infinite Number of Degrees of Freedom

We propose a method of quantization of a discrete Hamiltonian system with an infinite number of degrees of freedom. Our approach is analogous to the usual finite-dimensional quantum mechanics. We construct an infinite-dimensional Schrödinger equation. We show that it is possible to pass from the finite-dimensional quantum mechanics to our construction in the limit when the number of particles tends to infinity. In the paper, rigorous mathematical methods are used.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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