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WITH COMPLEX FRICTION COEFFICIENT**

CALDIROLA-KANAI HAMILTONIAN WITH COMPLEX FRICTION COEFFICIENT^(◇)

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Abstract

In the present note we find the exact solution of the time-dependent Schrödinger equation for the Caldirola-Kanai Hamiltonian with complex friction coefficient by applying the biorthogonal method. The eigenfunction normalization takes place in the extended Hilbert space $L^2\{R \otimes [0, T]\}$ where T is the time. From the biorthonormal solutions we calculate the eigenvalues which are complex and discrete, the matrix elements of the current density and the occupation probability.

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A prototype of time-dependent Hamiltonian for open systems is the famous Caldirola-Kanai (CK) Hamiltonian [1, 2]

$$H(t) = \frac{p^2}{2m} e^{-2\gamma t} + \frac{m}{2} \omega^2 q^2 e^{2\gamma t} \quad (1)$$

on which the relevant literature is very extended [3 – 10].

For $\omega \rightarrow i\omega$ the Hamiltonian (1) goes in the inverted CK Hamiltonian [11]

$$H(t) = \frac{p^2}{2m} e^{-2\gamma t} - \frac{m}{2} \omega^2 q^2 e^{2\gamma t}. \quad (2)$$

It is also possible to consider the more general CK Hamiltonian with complex friction coefficient [12]

$$H(t) = e^{-2(\gamma_1 + i\gamma_2)t} \frac{p^2}{2m} + \frac{m}{2} \omega^2 q^2 e^{2(\gamma_1 + i\gamma_2)t}. \quad (3)$$

Recently Baskoutas et al [13] have studied the time evolution of the above CK Hamiltonians.

In the present note we further investigate the time dependent Schrödinger equation with Hamiltonian (3). Because this Hamiltonian is time-dependent and non Hermitian ($H(t) \neq H^\dagger(t)$), many problems arise for the normalization of the eigenfunctions. In order to overcome such difficulties, we extend the usual Hilbert space to the new space $\tilde{\mathcal{H}}_T \equiv L^2(R \otimes [0, T])$ where T is the time. This extended space has already been used by some authors [14 – 16] for the study of quasienergies, pointwise solutions of time-dependent Schrödinger equations. The main advantage of the use of the space $\tilde{\mathcal{H}}_T$ is that the solutions and the eigenvalues of time-dependent Hamiltonians behave, in many respects, like those of a conservative system [15].

The time-dependent Schrödinger equation for the non-Hermitian CK Hamiltonian (3) has the form:

$$i\hbar \frac{\partial \psi}{\partial t} = (e^{-2\gamma t} \frac{p^2}{2m} + \frac{m}{2} \omega^2 e^{2\gamma t} q^2) \psi(t) \quad (4)$$

where $\gamma = \gamma_1 + i\gamma_2$. We have also to consider the Schrödinger equation for the adjoint operator $H^\dagger$, i.e. [17]

$$i\hbar \frac{\partial \chi}{\partial t} = (e^{-2\gamma^* t} \frac{p^2}{2m} + \frac{m}{2} \omega^2 e^{2\gamma^* t} q^2) \chi(t). \quad (5)$$

We get easily the solutions of equations (4) and (5) which, when normalized in the extended Hilbert space $L^2(R \otimes [0, T])$, take the form:

$$\psi_n(q, t) = \sqrt{\frac{m\Omega}{\pi\hbar}} \cdot \sqrt{\frac{n\gamma T}{\sinh n\gamma T} \frac{1}{2^n n!}} \quad (6)$$

$$\cdot \exp\left\{-it(\Omega + i\gamma)\left(n + \frac{1}{2}\right) - \frac{m}{2\hbar}(\Omega + i\gamma)e^{2\gamma t} q^2\right\} H_n\left(\sqrt{\frac{m\Omega}{\hbar}} e^{\gamma t} q\right);$$

$$\chi_l(q, t) = \sqrt{\frac{m\Omega^*}{\pi\hbar}} \cdot \sqrt{\frac{l\gamma^*T}{\sinh l\gamma^*T} \cdot \frac{1}{2^l l!}} \quad (7)$$

$$\cdot \exp\{-it(\Omega^* + i\gamma^*)(l + \frac{1}{2}) - \frac{m}{2\hbar}(\Omega^* + i\gamma^*)e^{2\gamma^*t}q^2\} H_l(\sqrt{\frac{m\Omega^*}{\hbar}}e^{\gamma^*t}q).$$

The corresponding energy eigenvalues are given by

$$E_n = \hbar(\Omega + i\gamma)(n + \frac{1}{2}); \quad (8)$$

$$\tilde{E}_l = \hbar(\Omega^* + i\gamma^*)(l + \frac{1}{2}), \quad (9)$$

where

$$\Omega = \Omega_1 - i\Omega_2 = \sqrt{\omega^2 - \gamma^2}; \quad (10)$$

$$\Omega_1 = \left[\frac{\sqrt{(\omega^2 + \gamma_2^2 - \gamma_1^2)^2 + \gamma_1^2\gamma_2^2} + (\omega^2 + \gamma_2^2 - \gamma_1^2)}{2} \right]^{1/2}; \quad (11)$$

$$\Omega_2 = \left[\frac{\sqrt{(\omega^2 + \gamma_2^2 - \gamma_1^2)^2 + \gamma_1^2\gamma_2^2} - (\omega^2 + \gamma_2^2 - \gamma_1^2)}{2} \right]^{1/2}. \quad (12)$$

It is very simple to show that the eigenfunctions (6) and (7) satisfy the biorthogonal condition [17]

$$\langle \chi_l(q, t) | \psi_n(q, t) \rangle = \delta_{nl}$$

which now reads explicitly:

$$\begin{aligned} \frac{1}{T} \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} \chi_l^*(q, t) \psi_n(q, t) dq &= \sqrt{\frac{\gamma n T \gamma l T}{\sinh \gamma n T \sinh \gamma l T}} \\ \cdot \sqrt{\frac{1}{2^{l+n} l! n!}} \cdot \frac{1}{T} \int_{-T/2}^{T/2} e^{\gamma(n+l)t} dt \int_{-\infty}^{\infty} e^{-\frac{m\Omega}{\hbar} e^{2\gamma t} q^2} H_n(\sqrt{\frac{m\Omega}{\hbar}} e^{\gamma t} q) H_l(\sqrt{\frac{m\Omega}{\hbar}} e^{\gamma t} q) \\ \cdot \sqrt{\frac{m\Omega}{\hbar}} e^{\gamma t} dq &= \\ = \frac{\gamma n}{\sinh \gamma n T} \cdot \int_{-T/2}^{T/2} e^{2\gamma n t} dt \delta_{nl} &= \delta_{nl} \end{aligned} \quad (13)$$

with Real $\Omega > 0$.

On the other hand, it is also easy to check that the eigenfunctions $\psi_n(q, t)$ and $\psi_l^*(q, t)$ aren't orthogonal, i.e.

$$\langle \psi_l | \psi_n \rangle = \frac{1}{T} \int dt \int \psi_l^*(q, t) \psi_n(q, t) dq \neq \delta_{nl}. \quad (14)$$

Let us define the density current in the biorthogonal representation as

$$\mathcal{I} = \frac{i\hbar}{2m} \left(\chi_l(q, t) \frac{\partial \psi_n^*}{\partial q} - \chi_l^*(q, t) \frac{\partial \psi_n}{\partial q} \right). \quad (15)$$

Then its matrix elements can be evaluated, after some algebra, and their expressions are:

$$\mathcal{I}_{nl} = \frac{i\hbar}{2m} \sqrt{\frac{m|\Omega|}{2\hbar}} \left\{ \sqrt{n+1} (A_{n+1} - A_{n+1}^*) \delta_{n+1,l} - \sqrt{n} (A_n - A_n^*) \delta_{n-1,l} \right\} \quad (16)$$

where:

$$\begin{aligned} A_{n+1} &= (1 + i\frac{\gamma}{\Omega}) \sqrt{\frac{n(n+1)}{\sinh\gamma^*(n+1)T \cdot \sinh\gamma nT}} \frac{\sinh[(n+1)\gamma + \frac{1}{2}\Omega]T}{[(n+1)\frac{\gamma}{|\gamma|} + \frac{i}{2}\frac{\Omega}{|\gamma|}]} = \\ &= (1 + i\frac{\gamma}{\Omega}) \sqrt{\frac{(n+1)\gamma^*T}{\sinh\gamma^*(n+1)T} \frac{\gamma nT}{\sinh\gamma nT}} \frac{\sinh[(n+1)\gamma + \frac{i}{2}\Omega]T}{[(n+1)\gamma + \frac{i}{2}\Omega]T} \end{aligned} \quad (17)$$

The occupation probability of the l^{th} energy level is given by:

$$P_{n \rightarrow l}^{(\gamma)} = \left| \frac{1}{T} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} \chi_l^*(q, 0) \psi_n(q, t) dq dt \right|^2, \quad (18)$$

where

$$\begin{aligned} &\frac{1}{T} \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} \chi_l^*(q, 0) \psi_n(q, t) dq = \\ &= \sqrt{\frac{m\Omega}{\pi\hbar}} \sqrt{\frac{l\gamma T}{\sinh l\gamma T} \frac{n\gamma T}{\sinh n\gamma T} \frac{1}{2^{l+n} l! n!}} \cdot \\ &\cdot \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-it(\Omega+i\gamma)(n+\frac{1}{2})} \int_{-\infty}^{\infty} e^{\frac{m\Omega}{2\hbar} \cdot [(\Omega+i\gamma)e^{2\gamma t} + (\Omega-i\gamma)]} \cdot \\ &\cdot H_l\left(\sqrt{\frac{m\Omega}{\hbar}} q\right) H_n\left(\sqrt{\frac{m\Omega}{\hbar}} q e^{\gamma t}\right) dq. \end{aligned} \quad (19)$$

For $\sqrt{\frac{m\Omega}{\hbar}} q = x$ the above integral takes the form:

$$\frac{1}{\sqrt{\pi}} \sqrt{\frac{l\gamma T}{\sinh l\gamma T} \frac{n\gamma T}{\sinh n\gamma T} \frac{1}{2^{l+n} l! n!}} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{-it(\Omega+i\gamma)(n+\frac{1}{2})}.$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} [e^{2\gamma t}(1 + \frac{i\gamma}{\Omega}) + (1 - \frac{i\gamma}{\Omega})] H_l(x) H_n(e^{\gamma t} x) dx. \quad (20)$$

The integral

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} [e^{2\gamma t}(1 + i\frac{\gamma}{\Omega}) + (1 - i\frac{\gamma}{\Omega})] H_l(x) H_n(e^{\gamma t} x) dx \quad (21)$$

where

$$a = \frac{1}{2} [e^{2\gamma t}(1 + i\frac{\gamma}{\Omega}) + (1 - \frac{\gamma}{\Omega})] \text{ with } \text{Re} a > 0$$

is different from zero only when the numbers l and n are of the same parity. We have

$$\int_{-\infty}^{\infty} e^{px^2} \left\{ \frac{H_{2l+1}(bx) H_{2n+1}(cx)}{H_{2l}(bx) H_{2n}(cx)} \right\} dx = (2l + \delta)! (2n + \delta)! \sqrt{\frac{\pi}{p}} \sum_{k=0}^{\min(l,n)} \frac{1}{(l-k)!(n-k)!(2k+\delta)!} \left(\frac{b^2}{p} - 1\right)^{l-k} \left(\frac{c^2}{p} - 1\right)^{n-k} \left(\frac{2bc}{p}\right)^{2k+\delta} \quad (22)$$

where $\text{Re} p > 0$ and $\delta = \begin{cases} 1 \\ 0 \end{cases}$.

The integral (21) has the same form of (22) for

$$l \rightarrow \begin{cases} 2l+1 \\ 2l \end{cases}, n \rightarrow \begin{cases} 2n+1 \\ 2n \end{cases}, p = \frac{1}{2} [e^{2\gamma t}(1 + i\frac{\gamma}{\Omega}) + (1 - i\frac{\gamma}{\Omega})],$$

$b = 1$ and $c = e^{\gamma t}$.

For $n = 0$ and $l \rightarrow 2l$ we obtain therefore

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} [e^{2\gamma t}(1 + i\frac{\gamma}{\Omega}) + (1 - i\frac{\gamma}{\Omega})] H_{2l}(x) dx = \\ &= \frac{(2l)!}{l!} \sqrt{\frac{2\pi}{e^{2\gamma t}(1 + i\frac{\gamma}{\Omega}) + (1 - i\frac{\gamma}{\Omega})}} \left(\frac{2}{e^{2\gamma t}(1 + i\frac{\gamma}{\Omega}) + (1 - i\frac{\gamma}{\Omega})} - 1 \right)^l = \\ &= \frac{(2l)!}{l!} \frac{(-1)^l (1 + e^{\frac{\gamma}{\Omega}})^l e^{-\frac{\gamma}{2}t} (\sinh \gamma t)^l}{(\cosh \gamma t + i\frac{\gamma}{\Omega} \sinh \gamma t)^{l+\frac{1}{2}}} \end{aligned} \quad (23)$$

and the occupation probability (18) takes the form:

$$\begin{aligned} P_{0 \rightarrow 2l}^{(\gamma)}(T) &= \left| \frac{(2l)! (-1)^l (1 + i\frac{\gamma}{\Omega})^l}{l!} \sqrt{\frac{2l\gamma T}{\sinh 2l\gamma T}} \right. \\ &\quad \left. \frac{1}{\gamma T} \int_{-\frac{\gamma T}{2}}^{\frac{\gamma T}{2}} \frac{e^{i\frac{\Omega}{2}u} \sinh^l u}{(\cosh u + i\frac{\gamma}{\Omega} \sinh u)^{l+1/2}} du \right|^2 \end{aligned} \quad (24)$$

where $u = \gamma t$. Eq. (24) describes the Landau-Zener-like transitions through a dynamical barrier for an oscillator with a complex CK damping, where the damping itself acts as a perturbation [18 – 20]. Notice that, in absence of an external driving term, only transitions which conserve parity are allowed.

Analogous results are obtained for the case of real γ .

References

1. P. Caldirola: *Nuovo Cimento* **18**, 393 (1941); *Lett. Nuovo Cimento* **20**, 589 (1977); *Nuovo Cimento* **B77**, 241 (1983); *Hadronic J.* **6**, 1400 (1983).
2. E. Kanai: *Progr. Theor. Phys.* **3**, 440 (1948).
3. E. Kerner: *Can. J. Phys.* **36**, 371 (1938).
4. V. Dodonov and M. Manko: *Nuovo Cimento* **B44**, 265 (1978); *Phys. Rev.* **A20**, 550 (1979); *Invariants and the Evolution of Nonstationary Quantum Systems*, ed. by V. Markov (Nova Science, N. York 1989).
5. R. Hasse: *J. Phys.* **A11**, 1245 (1978); *Rep. Progr. Phys.* **41**, 1627 (1978).
6. J. Messer: *Act. Phys. Austriaca* **50**, 1 (1979).
7. H. Dekker: *Phys. Rep.* **80**, 1 (1981).
8. P. Exner: *J. Math. Phys.* **24**, 1129 (1983).
9. A. Jannussis et al: *Lett. Nuovo Cimento* **38**, 181 (1983); *Hadronic J.* **6**, 1434 (1983).
10. S. Baskoutas, A. Jannussis: *Nuovo Cimento* **B107**, 255 (1992).
11. S. Baskoutas, A. Jannussis and R. Mignani: *Phys. Lett.* **A164**, 17 (1992).
12. B. Veljanoski and A. Jannussis: *Hadronic J* **10**, 193 (1987).
13. S. Baskoutas, A. Jannussis and R. Mignani : *Nuovo Cimento* **B108**, 953 (1993).
14. A. Jannussis, G. Brodimas, A. Leodakis and V. Papatheou: *Lett. Nuovo Cimento* **30**, 289 (1981).
15. H. Sambe: *Phys. Rev.* **A7**, 2203 (1973).
16. J. Okuniewicz: *Ph. D. Dissertation* (University of Minnesota, March 1973).
17. H.C.Baker: *Phys. Rev.* **30**, 773 (1984); H.C. Baker and R.L. Singleton, jr.: *Phys. Rev.* **A42**, 10 (1990); G. Dattoli, A. Torre and R. Mignani: *Phys. Rev.* **A42**, 1467 (1990).
18. P. Croxson: *Phys. Rev.* **A49**, 588 (1994). R.K. Colegrave, P. Croxson, U. Ramjit and E. Kheyraadi: *Physica* **A161**, 118 (1989).
19. S. Baskoutas, A. Jannussis and R. Mignani: "Landau-Zener-like transitions of a driven CK oscillator" I.N.F.N. preprint nr. 992, Rome, 1994, submitted for publication.
20. P. Angelopoulou, S. Baskoutas, A. Jannussis, R. Mignani and V. Papatheou: "Non-Hermitian tunnelling of open quantum systems", I.N.F.N. preprint nr. .. (Rome, 1994), submitted for publication.