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Lattice Space-Time from Poincaré and κ -Poincaré algebras

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Abstract

Introducing a dimensional parameter κ we propose a realization of the Poincaré and κ -Poincaré algebras by non local space-time differential operators. The Klein-Gordon equation takes the form of a finite difference equation. The connection with the deformed Heisenberg algebra is discussed leading to a generalized uncertainty relation.

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1 — Introduction

A deformation of the Poincaré algebra has been obtained by Lukierski, Nowicki, Ruegg and Tolstoj [1, 2] by the contraction procedure for q -deformed Lie algebra firstly introduced by the Firenze Group [3]. The starting point is the q -deformation of the anti de Sitter algebra $U_q[O(3, 2)]$ and then the limit is performed

$$R \rightarrow \infty \quad R \ln q \rightarrow \kappa^{-1}$$

(R is the de Sitter Radius, $0 < \kappa < \infty$), obtaining the κ -deformed Poincaré algebra which is a Hopf algebra with standard real structure. In the present article we consider the κ -Poincaré algebra proposed in ref. [2]. It should be recalled that another version of deformed Poincaré algebra has been proposed from the q -deformation of $D = 4$ conformal algebra [4, 5] or from the realization of q -differential calculus on q -deformed Minkowski space [6]. We recall that the equations defining the κ -Poincaré algebra can be written, after using the nonlinear transformation proposed in ref. [7] as [2]

$$\begin{aligned} [M_i, L_j] &= i \varepsilon_{ijk} L_k \\ [P_0, L_k] &= -i P_k \\ [P_k, L_j] &= -i \kappa \delta_{kj} \sinh(P_0/\kappa) \\ [L_i, L_j] &= -i \varepsilon_{ijk} \left(M_k \cosh(P_0/\kappa) - \frac{1}{4\kappa^2} P_k (\vec{P} \cdot \vec{M}) \right) \\ [P_\mu, P_\nu] &= 0 \\ [M_i, M_j] &= i \varepsilon_{ijk} M_k \\ [M_i, P_0] &= 0 \\ [M_i, P_j] &= i \varepsilon_{ijk} P_k \end{aligned} \tag{1}$$

where P_μ, M_i and L_i are respectively the generators of the translations, rotations and Lorentz boosts (throughout the paper the greek (resp. latin) indices are space-time (resp. space) ones). Let us remark that the above equations differ from those of the Poincaré algebra by factors of order $1/\kappa^2$ in the boost sector. Possible experimentally observable effects of the κ -Poincaré algebra

have been considered in ref. [8], where a lower limit of 10^{12} GeV has been computed from astrophysical observations, and in ref. [9] the consequences of the non additivity of the energy in the κ -Poincaré algebra for cosmic mass defect has been discussed.

The generators can be endowed with the standard Hopf structure and the explicit relations are given in refs. [2, 7]. The realization of eq. (1) in terms of differential operators over the Minkowski space-time takes the form [7] (we consider the spinless case where $\vec{P} \cdot \vec{M} = 0$; see ref. [7] for extension to arbitrary spin)

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ M_i &= \varepsilon_{ijk} x_j P_k \\ L_i &= x_0 P_i - \kappa x_i \sinh(P_0/\kappa) \end{aligned} \quad (2)$$

The κ -deformed Klein-Gordon and Dirac equation has been derived in refs. [2, 10]. The main feature of these equations is to replace the time-derivative by a finite difference equation (FDE)

$$2\kappa \sinh\left(\frac{P_0}{2\kappa}\right) f(t) = D_\kappa^t f(t) = \frac{f(t + \Delta t) - f(t - \Delta t)}{2\Delta t} \quad (3)$$

with the fundamental shift given by $\Delta t = i/2\kappa$. So the κ -deformation introduces a separation between the space and time degrees, which is not unexpected as the contraction procedure does privilege one direction. However we may feel uneasy by the fact that a q-lattice time is introduced while the variations in the space degrees remain continuous. Of course the κ -Poincaré algebra has no more the Lie algebra structure so we cannot any more build a corresponding Poincaré group by exponentiating the algebra neither it does contain the Lorentz algebra as subalgebra. However we still have to consider transformations between "equivalent" observers. Reasonably we can expect that Lorentz-like transformations of the type e.g. standard Lorentz boosts plus correction factors of the order $O(1/\kappa^2)$ still hold, so we may expect that the time evolution in different reference frames be governed by "dilatated" elementary shifts and not by the same elementary shift. The aim of this paper is to propose a scheme in which the q-lattice is generalized to the whole space-time.

2 — Space-time lattice

a) Poincaré algebra.

Let us consider for the moment the non deformed Poincaré algebra. If we have in our theory a constant κ' with the dimension of the inverse of time and of the inverse of length (in natural units), a possible realization for the translation sector P_μ in terms of non local differential operators is

$$\hat{P}_\mu = \kappa' \sinh \left(\frac{-i\partial_\mu}{\kappa'} \right) \quad (4)$$

The above choice is arbitrary, but it has the nice property to give the standard differential operator in the limit $\kappa' \rightarrow \infty$. If we compute the invariant $I(x)$

$$\hat{P}_\mu I(x) = 0 \quad (5)$$

we find

$$I(x_\mu) = F(x_\mu) \quad (6)$$

where $F(x_\mu)$ is an arbitrary function of the space-time which satisfies

$$F(x_\mu + \Delta x_\mu) = F(x_\mu - \Delta x_\mu) \quad (7)$$

for the elementary shift $\Delta x_\mu = i/\kappa'$. Let us remark that the kernel of the operator

$$S_{\kappa'}^\mu = \cosh \left(\frac{-i\partial_\mu}{\kappa'} \right) \quad (8)$$

is formed by the function $G(x_\mu)$ such that

$$G(x_\mu + \Delta x_\mu) = -G(x_\mu - \Delta x_\mu) \quad (9)$$

Therefore the operators defined by eq. (8) are invertibles on the subspace of the translational invariant functions over the Minkowski space which satisfy eq. (7). So on this subspace (physical space) we can introduce the following operators (no summation over μ)

$$\hat{x}_\mu = \frac{1}{2} \left(x_\mu (S_{\kappa'}^\mu)^{-1} + (S_{\kappa'}^\mu)^{-1} x_\mu \right) \quad (10)$$

and we can check that the generators given by the following equations

$$\begin{aligned}
\hat{P}_i &= \kappa' \sinh \left(\frac{-i\partial_i}{\kappa'} \right) \\
P_0 &= -i\partial_t \\
\widehat{M}_k &= i \varepsilon_{ijk} \hat{x}_j \hat{P}_i \\
\hat{L}_i &= x_0 \hat{P}_i - \kappa' \hat{x}_i \sinh(P_0/\kappa')
\end{aligned} \tag{11}$$

are a realization of the Poincaré algebra. However in this approach we have loosened any relation between the Poincaré group (as the standard transformation group over the Minkowski space) and the corresponding algebra and we have introduced a constant with suitable dimension, but otherwise completely arbitrary.

Let us remark that we still have

$$[\hat{P}_\mu, x_\nu] = -i\delta_{\mu\nu} S_{\kappa'}^\mu \tag{12}$$

$$[\hat{x}_\mu, \hat{x}_\nu] = 0 \tag{13}$$

$$[\hat{P}_\mu, \hat{x}_\nu] = -i\delta_{\mu\nu} \tag{14}$$

but now $\hat{P}_\mu$ are no more the generators of the translation and $\hat{x}_\mu$ is a hermitian not multiplicative operator. The transformation properties of $\hat{x}_\mu$ and $\hat{P}_\mu$ with respect to time and space inversion are the same as those of x_μ and P_μ .

b) κ -Poincaré algebra

Let us consider now the case of the κ -Poincaré algebra. No connection is now present with a space-time transformation group as we are no more dealing with a Lie algebra. Moreover the construction procedure has given a constant κ with the good dimension. If we restrict our function space to the functions which satisfy eq. (13) for the space coordinates we can replace in eq. (2) x_i by $\hat{x}_i$ given by eq. (10) and P_i by $\hat{P}_i$ given by eq. (4) and eqs. (1) are still satisfied in an unmodified way (we consider $\vec{P} \cdot \vec{M} = 0$). As the commutation relations are unchanged the Hopf structure is left invariant as it does not depend on the particular realization of the generators. If we compute now the invariants for the rotations, i.e. for the generators $\widehat{M}_i$, we find

$$I(x_i, t) = \vec{x}^2 + F(x_i, t) \tag{15}$$

and the invariants for all the the generators $\hat{P}_\mu$, $\hat{M}_i$ and $\hat{L}_i$ take the form

$$J(x_i, t) = \vec{x}^2 - t^2 + F(x_i, t) \quad (16)$$

where $F(x_i, t)$ satisfies eq. (7). The Casimir operator can now be written

$$\left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right)\right)^2 - \sum_{i=1}^3 \kappa^2 \sinh^2\left(\frac{P_i}{\kappa}\right) = m^2 \quad (17)$$

If we derive from eqs. (1) and (2) the κ -Klein-Gordon equation or the κ -Dirac equation then a FDE for all the space-time coordinates will appear. So the κ -Klein-Gordon equation can be written

$$\left[\sum_{i=1}^3 \left(\kappa^2 \sin\left(\frac{\partial_i}{\kappa}\right)\right)^2 - \left(2\kappa \sin\left(\frac{\partial_t}{2\kappa}\right)\right)^2\right] \phi(\vec{x}, t) = m^2 \phi(\vec{x}, t) \quad (18)$$

Notice that the space elementary shift is twice the time shift. Let us emphasize that the κ introduced in the equation of $\hat{P}_i$ may be different for any space coordinate and may also be different from κ parameter introduced by the contraction procedure for the time.

3 — Discussion

Let us summarize the main points of our scheme. We make a realization of the generators of Poincaré or κ -Poincaré algebra as non local differential operators on a subspace of the functions over the Minkowski space satisfying eq. (7) where $\Delta x_\mu = i/\kappa$ and κ is a parameter with dimension whose introduction is necessary in order to define the non local operators. We consider all the space-time shifts equal. In this scheme even the position operator is a non local differential operator on the space coordinates x_i .

Such a realization, while it may appear somewhat arbitrary in the framework of the Poincaré algebra, looks more natural in the framework of the κ -Poincaré algebra as it allows to treat all the space-time variables on the same way. However in this case the generator P_0 is still a time translation generator realized by a local differential operator, but the structure of the algebra requires the appearance of the non local operator $\sinh(P_0/\kappa)$. We emphasize once more that in the scheme of κ -Poincaré algebra the parameter κ appears naturally in

the boost sector of the commutation relations of eq. (1), while it is introduced by hand in the structure of the $\hat{P}_i$. We have been considering the form of the κ -Poincaré algebra of ref. [2], but our scheme can be applied changing the hyperbolic trigonometric functions in trigonometric functions. i.e. replacing κ by $i\kappa$, for the realization of the κ -Poincaré algebra of refs. [1, 7].

The choice we have made for $\hat{P}_\mu$, $\hat{M}_i$ and $\hat{L}_i$ raises new subtle question. Is really $\hat{x}_i$ the position operator ? It appears that x_i is no more a position operator. Indeed it has the following commutation relation with $\hat{P}_i$

$$[x_i, \hat{P}_j] = i\delta_{ij} S_\kappa^i = i\delta_{ij} \sqrt{1 + \hat{P}^2/\kappa^2} \simeq i\delta_{ij} + O(1/\kappa^2) \quad (19)$$

and it does not transform as a vector under the action of $\hat{M}_i$. In fact using the following equation

$$[x_i, (S_\kappa^j)^{-1}] = \frac{-i\delta_{ij}}{\kappa^2} (S_\kappa^i)^{-1} \hat{P}_i (S_\kappa^i)^{-1} \quad (20)$$

we derive

$$[\hat{M}_i, x_j] = i \varepsilon_{ijk} \hat{x}_k S_\kappa^j + \varepsilon_{ilk} [\hat{x}_l, x_j] \hat{P}_k \quad (21)$$

where

$$[\hat{x}_i, x_j] = \frac{i\delta_{ij}}{2\kappa^2} \left(x_i (S_\kappa^i)^{-1} \hat{P}_i (S_\kappa^i)^{-1} + (S_\kappa^i)^{-1} \hat{P}_i (S_\kappa^i)^{-1} x_i \right) \quad (22)$$

In order to convince that the choice of $\hat{x}_i$ as position operator is consistent we have to compute its time derivative and to see if we get a quantity which can be reasonably interpreted as the velocity. In order to perform this calculation in the Heisenberg representation we have to write a realization of the algebra in the p -representation. We find

$$\begin{aligned} \hat{P}_i &= \kappa \sinh(p_i/\kappa) \\ \hat{P}_0 &= p_0 \\ \hat{M}_i &= \varepsilon_{ijk} \hat{q}_j \hat{P}_k \\ \hat{L}_i &= \hat{q}_i \kappa \sinh(p_0/\kappa) + i \frac{\partial}{\partial p_0} \hat{P}_i \\ \hat{q}_i &= \frac{i}{2} \left[(\cosh(p_i/\kappa))^{-1} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial p_i} (\cosh(p_i/\kappa))^{-1} \right] \end{aligned} \quad (23)$$

Then we can compute, using eq. (23),

$$\frac{d}{dt} \hat{q}_i = i[\hat{P}_0, \hat{q}_i] = \frac{\sinh(p_i/\kappa)}{\sinh(p_0/\kappa)} \simeq \frac{p_i}{p_0} + \frac{1}{6\kappa^2} \frac{p_i}{p_0} (p_i^2 - p_0^2) + O(1/\kappa^4) \quad (24)$$

which looks like a natural generalization of the standard expression for the velocity p_i/p_0 . The same expression is derived in the case of the Poincaré algebra. As the commutation relation between the $\hat{x}_i$ and $\hat{P}_i$ are unchanged, see eq. (19), with respect to the undeformed case the Heisenberg uncertainty relations are unchanged. If we define as position operator the following one

$$Q_i = \frac{1}{2} \left[\hat{q}_i \cosh\left(\frac{p_0}{2\kappa}\right) + \cosh\left(\frac{p_0}{2\kappa}\right) \hat{q}_i \right] \quad (25)$$

which still transforms under rotation as a vector

$$[\widehat{M}_i, Q_j] = i\epsilon_{ijk} Q_k \quad (26)$$

then we have

$$[Q_i, \hat{P}_j] = i\delta_{ij} \cosh(p_0/2\kappa) \quad (27)$$

and

$$[Q_i, Q_j] = -\frac{i}{4\kappa^2} \epsilon_{ijk} \widehat{M}_k \quad (28)$$

and we find the deformed Heisenberg algebra introduced by Maggiore in ref. [11]. The time derivative of the position operator eq. (25) is

$$\frac{d}{dt} Q_i = i[\hat{P}_0, Q_i] = \frac{\sinh(p_i/\kappa)}{2 \sinh(p_0/2\kappa)} \simeq \frac{p_i}{p_0} + \frac{1}{6\kappa^2} \frac{p_i}{p_0} \left(p_i^2 - \frac{p_0^2}{4} \right) \quad (29)$$

But now the momentum-position uncertainty relations are modified and we find from eqs. (27) and (17)

$$\Delta Q_i \Delta \hat{P}_j \geq \frac{1}{2} \delta_{ij} \left(1 + \frac{p^2 + m^2}{4\kappa^2} + O(1/\kappa^4) \right)^{1/2} \quad (30)$$

so we find a generalized uncertainty principle, from which we can recover the relation introduced by Veneziano ref. [12] in the framework of string theory and whose algebraic explanation has been given in ref. [11].

A remark: it would have been more nice to define the position operator in terms of the generators of the κ -Poincaré algebra generalizing the Newton-Wigner position operator. We have not followed this more consistent approach as the generalization of the Newton-Wigner definition to the deformed case has a large amount of arbitrariness. e.g. the definition of ref. [11] is different from that of ref. [13].

In conclusion as in the deformed case the starting point is not a transformation group over the Minkowski space, but a purely algebraic structure (not of the Lie type) we can realize the generators as non local (by means of the parameter κ) or local differential operators in different not equivalent ways which all give, in the limit $\kappa \rightarrow \infty$, the standard Poincaré algebra but whose correction factors of the order $O(1/\kappa^2)$ are different.

It seems extremely interesting to investigate algebraic structure deformed or not deformed for a space-time lattice, but the condition that in the limit $\kappa \rightarrow \infty$ we find again the standard algebraic structure is a very weak one and one can find different structures with different physical consequences.

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