Standard solar model uncertainties and their correlations in the analysis of the solar neutrino problem

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Abstract

We present here an analysis of the theoretical uncertainties associated to the predicted solar neutrino rates for the current gallium, chlorine and water-Cherenkov experiments. We estimate the theoretical rates and their error matrices in three independent solar models (proposed by Bahcall and Pinsonneault, Turck-Chièze and Lopes, and Dar and Shaviv), and compare them to the experimental data. We show that the correlations of the theoretical uncertainties are crucial to assess the statistical significance of the solar neutrino deficit. The implications of a parametrization of the errors in terms of the central temperature of the Sun are also discussed.

1 Introduction

One of the most challenging problems in the field of astroparticle physics is the long-standing discrepancy between measured and predicted solar neutrino rates. The most recent results of the GALLEX [1] and SAGE [2] gallium (Ga) experiments, Homestake chlorine (Cl) experiment [3], and Kamiokande (Kam) water-Cherenkov experiment [4] are collected in Table 1. The measured solar neutrino fluxes appear to be systematically lower than theoretically predicted in a variety of standard solar models (SSMs), in particular in the detailed models proposed by Bahcall and Ulrich [5], Bahcall and Pinsonneault [6] and Turck-Chièze and Lopes [7]. The deficit seems to be substantially reduced in the SSM recently proposed by Dar and Shaviv [8]. For the sake of brevity, in the following we will refer to the previous four models as BU, BP, TL and DS respectively.

The critical comparison of the different assumptions underlying the various SSMs is the preferred subject of discussion among model builders, and it will not be addressed in this paper. We prefer instead to investigate accurately the compatibility of the different theoretical predictions with the experimental data. To this aim, in Sec. 2 we consider in detail the analytical prescription required to estimate the theoretical error matrix. In particular, we provide all the tools for a do-it-yourself estimate of the errors. In Sec. 3 we apply this general method to three independent solar models (BP, TL and DS), in order to assess the statistical significance of the neutrino deficit. The implications of charting the different solar models according to their predicted core temperature T_C are discussed in Sec. 4. In Sec. 5 we finally summarize our conclusions about the present status of the solar neutrino problem.

Table 1: Measured neutrino rates and 1σ total experimental errors for the gallium (Ga), chlorine (Cl) and water-Cherenkov (Kam) experiments. The Kam rate is conventionally normalized to the BU standard solar model predictions (BUSSM).

Experiment	Ref.	Rate	Error	Units
GALLEX+SAGE (Ga)	[1, 2]	78	10	SNU
Homestake (Cl)	[3]	2.55	0.25	SNU
Kamiokande (Kam)	[4]	0.50	0.07	Kam/BUSSM

2 Analytical estimate of the theoretical errors

In any solar model, the predicted neutrino rates R_j (in the following $j = 1, 2, 3$ stands for Ga, Cl, Kam) are affected by — in general correlated — theoretical uncertainties. In this section we describe in detail a prescription useful to estimate the full theoretical error matrix analytically¹. As we will see in the following, Tables from 2 to 6 collect all needed input data.

In Table 2 we report the eight relevant neutrino fluxes $\{\phi_i\}_{i=1,\dots,8}$ (pp, pep, hep, Be, B, N, O, F) as estimated in the BU, BP, TL and DS solar models.

The fluxes in Table 2, when multiplied by the energy-averaged detector cross-section factors $\{C_{ij}\}_{j=1,\dots,3}^{i=1,\dots,8}$ reported in Table 3, give the theoretical partial rates R_{ij} ($R_{ij} = C_{ij}\phi_i$) and the total rates R_j ($R_j = \sum_{i=1}^8 R_{ij}$) for the three different types of solar neutrino experiments.

Table 2: Neutrino fluxes ϕ_i ($\text{cm}^{-2}\text{s}^{-1}$) in the BU, BP, TL and DS models.

Reference solar model	pp $\times 10^{10}$	pep $\times 10^8$	hep $\times 10^3$	Be $\times 10^9$	B $\times 10^6$	N $\times 10^8$	O $\times 10^8$	F $\times 10^6$
BU [5]	6.00	1.40	7.58	4.69	5.76	6.09	5.22	5.16
BP [6]	6.00	1.43	1.23	4.89	5.69	4.92	4.26	5.39
TL [7]	6.03	1.39	—	4.34	4.43	3.83	3.18	—
DS [8]	6.04	1.40	—	4.30	2.77	0.747	0.217	5.21

Table 3: Energy-averaged detection cross sections C_{ij} , taken from the BU paper [5]. The same matrix is used by TL. The elements (C_{32} , C_{52}) are updated in BP from the old BU values (3.90, 1.06) to the new values (4.26, 1.09). Units are such that the expected rates $C_{ij}\phi_i$ are directly given in SNU for the Ga and Cl experiments, and as a ratio to the BU predicted rate for Kamiokande (Kam/BUSSM).

Experiment	pp $\times 10^{-9}$	pep $\times 10^{-8}$	hep $\times 10^{-6}$	Be $\times 10^{-9}$	B $\times 10^{-6}$	N $\times 10^{-9}$	O $\times 10^{-8}$	F $\times 10^{-8}$
Ga	1.18	2.15	7.30	7.32	2.43	6.18	1.16	1.17
Cl	0	0.16	3.90	4.89	1.06	0.17	0.07	0.07
Kam	0	0	0.66	0	0.173	0	0	0

¹We used this method in previous works [9, 10]; an explicit presentation of it (not as complete as here) has been given in Ref. [11].

The accuracy in the theoretical estimate of the three neutrino rates R_j is limited by:

- 1) The uncertainties in the capture cross-section of each detector. These are completely specified by the uncorrelated relative errors $\Delta \ln C_{ij}$ affecting the cross-section factors C_{ij} , as given in Table 4.
- 2) The uncertainties in the astrophysical parameters used as input in building the different solar models. Their relevant components can be traced back to the nuclear S -factors S_{11} , S_{33} , S_{34} , $S_{1,14}$, S_{17} , the metallicity (Z/X), the Sun age (Age) and the opacity (Opa). We denote these input parameters² as $\{X_k\}_{k=1,\dots,9}$. The corresponding relative errors $\Delta \ln X_k$, as adopted by BU, BP, TL and DS, are reported in Table 5. Notice that the estimate of $\Delta \ln X_k$ is more conservative in TL than in BU and BP. We will assume the TL values for the missing entries in the fourth row of Table 5 (DS model).

Table 4: 1σ relative errors $\Delta \ln C_{ij}$ of the energy-averaged cross-sections, taken from the BU, BP papers.

Experiment	pp	pep	hep	Be	B	N	O	F
Ga	0.025	0.070	0.310	0.033	0.300	0.035	0.065	0.065
Cl	0	0.020	0.033	0.020	0.033	0.020	0.020	0.020
Kam*	0	0	0	0	0	0	0	0

*Radiative correction uncertainties of the scattering cross section are negligible.

Table 5: 1σ relative errors $\Delta \ln X_k$ of the nine relevant input parameters X_k , adopted in the BU, BP, TL and DS solar models.

SSM	S_{11}	S_{33}	S_{34}	$S_{1,14}$	S_{17}	Lum	Z/X	Age	Opa
BU	0.017	0.057	0.020	0.120	0.073	0.002	0.063	0.007	0.010
BP	0.015	0.060	0.032	0.120	0.090	0.002	0.063	0.007	0.020
TL	0.040	0.050	0.040	—	0.130	0.005	0.100	0.010	0.050
DS	—	—	0.067	—	0.120	—	—	—	—

²Actually, the opacity is itself a whole set of input parameters (opacity tables). However, changes in the opacity tables produce highly correlated shifts in the fluxes ϕ_i , as they were due to a single parameter (Opa).

The error propagation from X_k to ϕ_i is governed by the matrix $\{\alpha_{ik}\}$ of the logarithmic derivatives $\alpha_{ik} = \partial \ln \phi_i / \partial \ln X_k$, whose numerical estimate, as performed by Bahcall and collaborators, is reported in Table 6. The last row is obtained by comparing the output fluxes relative to models with different opacity input tables [5, 6]. Several entries of this matrix have been independently estimated by the authors of Refs. [7, 12], which quote a good agreement with the BU, BP values. The only element of $\{\alpha_{ik}\}$ quoted in the Kovetz and Shaviv model [13] (on which the DS model is based) is $\alpha_{56} = 7.1$, close to the value 6.76 of Table 6. At this point, it seems reasonable to use the same matrix $\{\alpha_{ik}\}$ as that in Table 6 for all models.

Table 6: Logarithmic derivatives $\alpha_{ik} = \partial \ln \phi_i / \partial \ln X_k$, taken from [5, 6].

$X_k \setminus \phi_i$	pp	pep	hep	Be	B	N	O	F
S_{11}	+0.14	-0.17	-0.08	-0.97	-2.59	-2.53	-2.93	-2.94
S_{33}	+0.03	+0.05	-0.45	-0.43	-0.40	+0.02	+0.02	+0.02
S_{34}	-0.06	-0.09	-0.08	+0.86	+0.81	-0.05	-0.05	-0.05
$S_{1,14}$	-0.02	-0.02	-0.01	+0.00	+0.01	+0.85	+1.00	+0.01
S_{17}	+0.00	+0.00	+0.00	+0.00	+1.00	+0.00	+0.00	+0.00
Lum	+0.73	+0.87	+0.12	+3.40	+6.76	+5.16	+5.94	+6.25
Z/X	-0.08	-0.17	-0.22	+0.58	+1.27	+1.86	+2.03	+2.09
Age	-0.07	+0.00	-0.11	+0.69	+1.28	+1.01	+1.27	+1.29
Opa	+0.14	+0.24	+0.54	-1.38	-2.60	-1.68	-2.05	-2.15

The entries of the matrix $\{\alpha_{ik}\}$ are not linearly independent. In fact, the fluxes are constrained by the equation of energy production (see, e.g., Ref. [14])

$$\sum_{i=1}^8 (Q - 2\overline{E}_i) \phi_i = 2K \quad , \quad (1)$$

where $Q = 26.73$ MeV, $\overline{E}_i$ is the i -th flux average energy (in MeV), and K is the solar constant ($K = 0.849 \times 10^{12}$ MeV cm⁻² s⁻¹), proportional to the luminosity ($Lum = X_6$). By considering the flux variations induced by the X_k uncertainties, the above equation leads to the following interesting sum rule

$$\frac{\sum_{i=1}^8 (Q - 2\overline{E}_i) \phi_i \alpha_{ik}}{\sum_{i=1}^8 (Q - 2\overline{E}_i) \phi_i} = \delta_{k6} \quad (k = 1, \dots, 9) \quad , \quad (2)$$

which is satisfied at the 2% level by using BU or BP fluxes. This means that energy conservation is correctly implemented: a perturbation of X_k with $k = 6$ ($k \neq 6$) induces a redistribution of the fluxes while varying (keeping constant) K , as it should be.

Now we calculate the relevant error matrices. As said before, the matrix $\sigma_{j_1 j_2}^2(\text{TH})$ of the total theoretical errors affecting the rates R_j is the sum of two distinct matrices, one with the errors due to the detector cross sections (CS errors) and the other constructed with the uncertainties concerning the astrophysical parameters (AP errors) :

$$\sigma_{j_1 j_2}^2(\text{TH}) = \sigma_{j_1 j_2}^2(\text{CS}) + \sigma_{j_1 j_2}^2(\text{AP}) \quad (j_1, j_2 = 1, \dots, 3) \quad . \quad (3)$$

The contributions to $\sigma_{j_1 j_2}^2(\text{CS})$, coming from the uncertainties $\Delta \ln C_{ij}$ in the detector cross sections, are uncorrelated in both the indices (i, j) , and give rise to the diagonal matrix

$$\begin{aligned} \sigma_{j_1 j_2}^2(\text{CS}) &= \delta_{j_1 j_2} \sum_{i_1=1}^8 \sum_{i_2=1}^8 \delta_{i_1 i_2} \frac{\partial R_{j_1}}{\partial \ln C_{i_1 j_1}} \frac{\partial R_{j_2}}{\partial \ln C_{i_2 j_2}} \Delta \ln C_{i_1 j_1} \Delta \ln C_{i_2 j_2} \\ &= \delta_{j_1 j_2} \sum_{i=1}^8 R_{i j_1}^2 (\Delta \ln C_{i j_1})^2 \quad . \end{aligned} \quad (4)$$

The contributions to $\sigma_{j_1 j_2}^2(\text{AP})$, coming from the uncertainties in the input parameters X_k , affect all the fluxes ϕ_i systematically and give rise to off-diagonal elements in the error matrix

$$\begin{aligned} \sigma_{j_1 j_2}^2(\text{AP}) &= \sum_{k=1}^9 \frac{\partial R_{j_1}}{\partial \ln X_k} \frac{\partial R_{j_2}}{\partial \ln X_k} (\Delta \ln X_k)^2 \\ &= \sum_{i_1=1}^8 \sum_{i_2=1}^8 R_{i_1 j_1} R_{i_2 j_2} \sum_{k=1}^9 \alpha_{i_1 k} \alpha_{i_2 k} (\Delta \ln X_k)^2 \quad . \end{aligned} \quad (5)$$

The sum of the theoretical and experimental error matrices gives the total error matrix: denoting by $\sigma_{j_1}^{\text{exp}}$ the experimental errors given in Table 1,

$$\begin{aligned} \sigma_{j_1 j_2}^2(\text{TOT}) &= \sigma_{j_1 j_2}^2(\text{TH}) + \sigma_{j_1 j_2}^2(\text{EXP}) \\ &= \sigma_{j_1 j_2}^2(\text{TH}) + \delta_{j_1 j_2} \sigma_{j_1}^{\text{exp}} \sigma_{j_2}^{\text{exp}} \quad . \end{aligned} \quad (6)$$

At this point, the statistical significance of the solar neutrino deficit is measured by the global χ^2 function

$$\chi^2 = \sum_{j_1, j_2=1}^3 (R_{j_1} - R_{j_1}^{exp}) [\sigma_{j_1 j_2}^2(\text{TOT})]^{-1} (R_{j_2} - R_{j_2}^{exp}) \quad . \quad (7)$$

Finally, it is useful to define also the 8×8 flux error matrix $\sigma_{i_1 i_2}^2$, which receives only “astrophysical” contributions

$$\begin{aligned} \sigma_{i_1 i_2}^2 &= \sum_{k=1}^9 \frac{\partial \phi_{i_1}}{\partial \ln X_k} \frac{\partial \phi_{i_2}}{\partial \ln X_k} (\Delta \ln X_k)^2 \\ &= \phi_{i_1} \phi_{i_2} \sum_{k=1}^9 \alpha_{i_1 k} \alpha_{i_2 k} (\Delta \ln X_k)^2 \quad . \end{aligned} \quad (8)$$

The analytical formulae for the error matrices reported above can be compared with some numerical estimates performed in the literature. In particular, values of $\sqrt{\sigma_{ii}}$ and of some of their single astrophysical (AP) components have been published in Table XIII of Ref. [5] and Table VII of Ref. [6]. Values of $\sqrt{\sigma_{jj}(\text{TH})}$ and of some of their astrophysical (AP) and detector cross-section (CS) components are given in Table XVI of Ref. [5], Table X of Ref. [6] and Table VI of Ref. [7]. Our results agree with all these published values with great accuracy when the appropriate input for each reference model is used.

It is very important to test now the off-diagonal elements of the error matrix, generated by the astrophysical (AP) uncertainties. To this aim we use the set of 8-plets of fluxes ϕ_i obtained by Bahcall in a Monte Carlo evaluation of “1000 standard solar models”³. These 1000 different models are generated by randomly varying (according to Gaussian distributions) the five input parameters S_{11} , S_{33} , S_{34} , S_{17} , Z/X $\{X_k, k = 1, 2, 3, 5, 7\}$. If we now restrict the k -summation in Eq. (8) to $\{k = 1, 2, 3, 5, 7\}$ and calculate analytically the flux correlation matrix $\rho_{i_1 i_2} = \sigma_{i_1 i_2}^2 / \sqrt{\sigma_{i_1 i_1}^2 \sigma_{i_2 i_2}^2}$, we obtain the values reported in the upper half of Table 7. In the same table (lower half) we report the numerical correlations of the 1000 SSM fluxes. The non-trivial agreement of all the off-diagonal entries completes the desired check of Eqs. (3–6), which will be at the basis of the analysis presented in the next section.

³We thank J.N. Bahcall for kindly providing us with the results of his 1000 SSM Monte Carlo.

It is interesting to observe that as inputs of our analytical method we require the outputs of $\mathcal{O}(N)$ solar model replicas (for example, to estimate the α_{ik} 's), where N is the number of astrophysical parameters used as variables (with $N = 9$ in our most general case, $N = 5$ in Table 7). Conversely, a full Monte Carlo evaluation of the errors requires $\mathcal{O}(N^m)$ solar code runs, where m is the number of sampling points in the Gaussian distributions of the N variables: the Monte Carlo method is thus intrinsically limited by computing time if the effects of a large number of parameters are to be studied.

Table 7: Off-diagonal elements of the flux error matrix. Upper half: our estimate of the flux correlation matrix $\rho_{i_1 i_2}$, using BU input data and taking into account only the errors induced by S_{11} , S_{33} , S_{34} , S_{17} , Z/X . Lower half: correlation matrix extracted from the 1000 SSM Monte Carlo simulation of Bahcall.

$\phi_{i_1} \setminus \phi_{i_2}$	pp	pep	hep	Be	B	N	O	F
pp	1							
pep	0.784	1						
hep	0.142	0.242	1					
Be	-0.960	-0.766	0.080	1				
B	-0.792	-0.591	-0.139	0.746	1			
N	-0.928	-0.778	-0.437	0.792	0.748	1		
O	-0.930	-0.767	-0.432	0.793	0.751	1.000	1	
F	-0.930	-0.772	-0.433	0.793	0.750	1.000	1.000	1
pp	1							
pep	0.770	1						
hep	0.077	0.284	1					
Be	-0.916	-0.693	0.046	1				
B	-0.727	-0.503	-0.157	0.742	1			
N	-0.885	-0.735	-0.461	0.796	0.729	1		
O	-0.886	-0.718	-0.453	0.800	0.734	1.000	1	
F	-0.886	-0.721	-0.454	0.800	0.734	1.000	1.000	1
ϕ_{i_1} / ϕ_{i_2}	pp	pep	hep	Be	B	N	O	F

3 An analysis in three different solar models

The analytic prescription of the previous section is now applied to the BP, TL and DS solar models. They can be thought of as representative of models yielding “high” (BP), “intermediate” (TL) and “low” (DS) neutrino rates, as a result of different — although “standard” — physical assumptions.

The main results are collected in Table 8, where we report, for each of the three reference models, the predicted rates R_j , the total theoretical errors $\sqrt{\sigma_{jj}(\text{TH})}$ with their correlations, and the global χ^2 (including or not the correlation effects).

Let us stress that the correlations of the theoretical errors in Table 8 are sizeable, and it would be a gross approximation, in principle, to neglect them in the χ^2 estimate, as can be seen by comparing the last two columns of the table. The impact of the correlation matrix is often ignored when the statistical significance of the solar neutrino problem is criticized [15]. The “uncorrelated” approximation is particularly dangerous in the DS model, where it leads to a much lower χ^2 (13.5 instead of 20.5). The effect is opposite in the BP model, and happens to be negligible in the TL model. We will turn again to this point.

Among the three models, BP has the highest χ^2 , as a result of higher predicted rates and relatively smaller theoretical input errors. However, even in the DS model (lower rates, conservative errors) the value assumed by the global χ^2_{corr} is certainly still large enough to elevate the solar neutrino *deficit* to the status of a *problem*.

Table 8: Theoretical rates, with their errors and correlations, and global χ^2 [with (corr) and without (uncorr) correlations] in the BP, TL and DS models. Units as in Table 1.

SSM	EXP	Rate	1σ err.	Correlation matrix			χ^2_{corr}	χ^2_{uncorr}
BP	Ga	131.55	6.41	1			44.0	58.0
	Cl	7.98	1.00	0.628	1			
	Kam	0.99	0.14	0.620	0.975	1		
TL	Ga	122.66	7.43	1			23.3	21.8
	Cl	6.24	1.34	0.820	1			
	Kam	0.77	0.19	0.808	0.991	1		
DS	Ga	113.26	5.33	1			20.5	13.5
	Cl	4.31	0.87	0.792	1			
	Kam	0.48	0.12	0.776	0.990	1		

We must note at this point that our Ga and Cl rates for the DS model in Table 8 are slightly larger than those quoted by the authors (109 and 4.2 SNU respectively, see Ref. [8]). This disturbing difference implies that Dar and Shaviv use an alternative, but unspecified, set of cross-section factors C_{ij} . However, even using their Ga and Cl quoted rates, we get $\chi^2_{corr} = 17.2$, which still implies a very low confidence level.

We now give a graphical representation of the information contained in Tables 1 and 8, in order to make more evident the “distance” between theory and experiment. In Figs. 1, 2 and 3, using as coordinates the total rates R_j ($j = 1, 2, 3 = \text{Ga, Cl, Kam}$), we show the planar projections of the $\Delta\chi^2 = 9$ theoretical three-dimensional regions for the BU, TL and DS models, respectively. Also shown are the projections of the $\Delta\chi^2 = 9$ experimental ellipsoids (dashed ellipses). A further projection onto the coordinate axes would yield the 3σ theoretical and experimental errors.

The strong incompatibility between theory and experiment in the BP model is evident in Fig. 1. The theoretical errors are here larger than the experimental ones, but the distance between the two central values keeps the two $\Delta\chi^2 = 9$ ellipsoids in the (Ga, Cl, Kam) space widely separated. From Fig. 1 it is easy to convince oneself that to neglect the theoretical correlations would be a gross approximation. In this case the theory-experiment compatibility would get worse.

As can be seen in Fig. 2, the TL model is also in a very poor agreement with the experimental data (even if better than BP). It is clear that even stretching the theoretical uncertainties to their 3σ limits, this model “misses” the bulk of the experimental allowed region. The mismatch is particularly evident in the (Cl, Kam) plane.

In Fig. 3 we finally consider the DS model, characterized by relatively low neutrino rates. Here the correlation effects appear particularly important in the (Ga, Kam) and (Cl, Kam) planes: by neglecting them, one would obtain an artificially large superposition between the theoretical and experimental confidence regions. Again we see that in the (Cl, Kam) plane the ellipse describing the DS model does not point towards the data. Since the systematic errors of the Homestake experiment are subject to an endless debate, it is important to note that in the DS model we find only a marginal agreement with the data, even in the (Ga, Kam) plane.

The results reported in Table 8 and the discussion of Figs. 1-3 demonstrate that the present experimental neutrino rates cannot be convincingly reconciled with the standard solar model predictions, even stretching the theoretical uncertainties of low-rate models to their 3σ limits. The correlation of the theoretical errors has proved to be crucial in this respect.

4 The “central temperature” parametrization

As we have seen in the previous section, the theoretically allowed domain in the rate space could be substantially enlarged if the correlations were — incorrectly — ignored. Conversely, the popular parametrization of the neutrino flux uncertainties in terms of the central temperature T_C of the Sun (see, e.g., Refs. [16, 17]) has the effect of reducing the theoretical domain and overestimating the correlations.

In its extreme form, this parametrization relates all possible neutrino flux variations to the single “variable” T_C through a power law dependence

$$\phi_i(t) = \phi_i t^{\beta_i} \quad (t = 1 + \Delta T_C / T_C) \quad , \quad (9)$$

where the temperature exponents β_i can be estimated through a Monte Carlo procedure, as in Refs. [16, 12]. We point out that in a linear approximation the values of β_i are not independent, being constrained by the analogue of Eq. (2) at constant luminosity

$$\frac{\sum_{i=1}^8 (Q - 2\bar{E}_i) \phi_i \beta_i}{\sum_{i=1}^8 (Q - 2\bar{E}_i) \phi_i} = 0 \quad . \quad (10)$$

A possible set is $\{\beta_i\}_{i=1,\dots,8} = \{-1.1, 3, 5, 8, 18, 27, 32, 32\}$, which satisfies the above equation at the 4% level using BU or BP fluxes.

If large (a few per cent) T_C variations are allowed, then the linear expansion $\phi_i \simeq \phi_i (1 + \beta_i \Delta T_C / T_C)$ is not adequate and the full power law must be used. To guarantee the validity of the energy production equation (1) at any temperature, we prefer to rescale all fluxes with a normalization factor $\mathcal{N}(t)$

$$\phi_i(t) = \mathcal{N}(t) \phi_i t^{\beta_i} \quad , \quad \mathcal{N}(t) = \frac{\sum_{i=1}^8 (Q - 2\bar{E}_i) \phi_i}{\sum_{i=1}^8 (Q - 2\bar{E}_i) \phi_i t^{\beta_i}} \quad . \quad (11)$$

Notice that, due to Eq. (10), $\mathcal{N}(t) - 1$ is second order in $\Delta T_C / T_C$.

Now we discuss pros and cons of the T_C parametrization. At first order, it implies that the matrix α_{ik} is a tensor product of two vectors

$$\alpha_{ik} = \frac{\partial \ln \phi_i}{\partial \ln X_k} = \beta_i \left(\frac{\partial \ln T_C}{\partial \ln X_k} \right) \quad . \quad (12)$$

However, it turns out that the above factorization can be approximately obtained only for submatrices of $\{\alpha_{ik}\}$. This simply means that T_C variations cannot simulate the effects of all astrophysical uncertainties at the same time⁴, and at least a mixed (T_C + some X_k) approach is required [18] [12]. In particular, the errors induced by nuclear cross-section factors like S_{17} and S_{34} have sizeable components “orthogonal” to the T_C uncertainties.

We now continue the discussion with the help of Fig. 4. In this figure we show, in the same planes as in Figs. 1-3, the one-dimensional theoretical domains allowed by T_C variations (ΔT_C lines), together with the central values of three standard (BP, TL, DS) and some non-standard solar model predictions. The T_C -dependent rates were calculated with the help of Eq. (11). The (almost straight) ΔT_C lines are conventionally normalized to give $\Delta T_C = 0$ at the BP model, and have approximately the same slope as the main axes of the slanted ellipses of Figs. 1-3.

Four representative neutrino rate predictions involving non-standard (i.e. unjustified) assumptions are shown in Fig. 4. They correspond to models with artificially high-helium (High- Y) or low-metal (Low- Z) primordial abundance, zeroed S_{34} -factor ($S_{34}=0$) or B-flux (No ^8B), as considered in Ref. [5]. The first three such models lie approximately along the ΔT_C lines (like BP, TL and DS), and this evidences the usefulness of the T_C -parametrization in charting both standard and non-standard solar models. The fourth one (No ^8B) is however considerably far from the ΔT_C lines, showing that changes in the nuclear reaction network can induce shifts orthogonal to ΔT_C . This indicates the limits of the simple T_C parametrization: too strong correlations, too narrow theoretical allowed regions. It would be better to think of the ΔT_C lines as of bands, with as large a (3σ) width as needed to cover the BP, TC and DS ellipses in Figs. 1-3.

Such bands are limited by solid slanted lines in Fig. 5. They represent the envelopes of the ellipses due to the T_C -dependent theoretical errors, obtained by applying our analytical prescriptions to the fluxes of Eq. (11) at any T_C . Concerning $\Delta \ln X_k$, we used the most conservative estimates. In the (Ga, Cl, Kam) space, the resulting theoretical allowed region is cone-like, and intersects the experimental ellipsoid only marginally. Its closest approach is at $\Delta T_C = -6.5\%$, with $\chi^2 = 14.7$, still a very high value. Thus, to fictitiously introduce a new degree of freedom (T_C) is of no real help. Note that, even allowing for *both* low T_C and a larger experimental Cl rate, one can improve the fit in the (Cl, Kam) plane only at the price of getting a worse fit in the (Ga, Cl) plane. The rôle of the gallium data in excluding astrophysical solutions of the solar neutrino problem will be hopefully strengthened by the scheduled (summer 1994) GALLEX calibration by means of a radioactive source.

⁴Obviously, detector cross section uncertainties are *a priori* independent of T_C .

5 Conclusions

In this paper we have faced the problem of an analytical calculation in matrix form of the theoretical errors affecting the solar neutrino fluxes. After an explicit description of the approach, we have thoroughly compared our output with the numerical results of Monte Carlo simulations. It is evident that the approach can be easily extended to cover new neutrino experiments and/or the uncertainties induced by a more refined parametrization.

We have shown that the correlations among theoretical errors play an important rôle in determining the statistical significance of the solar neutrino deficit. It is misleading to neglect the off-diagonal errors, the relevance of these correlation effects being more significant when lower theoretical rates are provided by the solar model. We have studied the predictions of three independent standard solar models, and charted standard and non-standard solar models with the help of the T_C -parametrization. Even in the most favourable cases, the theory–experiment agreement is only marginal.

Theoretical solutions to this discrepancy (if one wants to avoid new neutrino physics) are forced to involve also effects orthogonal to ΔT_C . However, we do not know effects able to shift considerably the theoretical predictions towards the experimental data. Moreover, even unexpectedly large systematic errors in the chlorine experiment cannot be considered as decisive as could be naïvely thought.

Finally, let us underline how clearly our representation in the space (Ga, Cl, Kam) shows all these features, and of what considerable help it is in understanding the interplay between possible shifts of the theoretical and of the experimental allowed regions.

Acknowledgements

One of us (E.L.) wishes to thank the CERN Theory Division for kind hospitality during all stages of this work.

Note added in proof

While completing this work, we become aware of a recent paper by E. Gates, L.M. Krauss and M. White [19], which also addresses the problem of analytically estimating the correlation matrices, in particular to get improved MSW fits. In the present paper we have instead focused our interest on standard and non-standard solar models, having already applied the full error matrix estimate in our MSW analyses [9, 10, 11].

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Figure captions

Fig. 1: Projections of the $\Delta\chi^2 = 9$ experimental and theoretical confidence ellipsoids onto the neutrino rate planes (Ga, Cl), (Ga, Kam) and (Cl, Kam). The theoretical estimate refers to the Bahcall and Pinsonneault model (BP).

Fig. 2: As in Fig. 1, but for the solar model of Turck-Chièze and Lopes (TL).

Fig. 3: As in Fig. 1, but for the solar model of Dar and Shaviv (DS).

Fig. 4: T_C parametrization: one-dimensional domains allowed in the (Ga, Cl), (Ga, Kam) and (Cl, Kam) planes by $-10\% < \Delta T_C < +3\%$ (oblique lines). Also shown are the rates (central value only) predicted by three standard (BP, TL, DS) and four non-standard (High- Y , Low- Z , $S_{34}=0$, No ^8B) solar models. See text for details.

Fig. 5: Total theoretical 3σ range (solid lines), allowed by all known solar model uncertainties at any T_C . Also shown are the projections of the $\Delta\chi^2 = 9$ experimental confidence ellipsoid.

Fig. 1

Fig. 2

Fig. 3

Fig.4

Fig. 5