The Conserved Charges and Integrability of the Conformal Affine Toda Models

H. Aratyn\(^1\)
Department of Physics
University of Illinois at Chicago
845 W. Taylor St.
Chicago, Illinois 60607-7059

L.A. Ferreira\(^2\), J.F. Gomes\(^2\) and A.H. Zimerman\(^2\)
Instituto de Física Teórica-UNESP
Rua Pamplona 145
01405-900 São Paulo, Brazil

ABSTRACT

We construct infinite sets of local conserved charges for the conformal affine Toda model. The technique involves the abelianization of the two-dimensional gauge potentials satisfying the zero-curvature form of the equations of motion. We find two infinite sets of chiral charges and apart from two lowest spin charges all the remaining ones do not possess chiral densities. Charges of different chiralities Poisson commute among themselves. We discuss the algebraic properties of these charges and use the fundamental Poisson bracket relation to show that the charges conserved in time are in involution. Connections to other Toda models are established by taking particular limits.

\(^1\)Work supported in part by U.S. Department of Energy, contract DE-FG02-84ER40173 and by NSF, grant no. INT-9015799

\(^2\)Work supported in part by CNPq
1 Introduction

The completely integrable field theories are characterized by the existence of infinite number of conserved charges in involution. An important class of integrable theories is provided by the two-dimensional Toda field theories in which integrability is signaled by the zero-curvature form of the equations of motion. It is of interest to reveal the complete integrability of the Toda theories in its basic form constructing directly charges in involution. This has been done in the framework of the affine Toda (AT) model based on the center-less Kac-Moody (KM) algebra. There, the infinite set of local conserved charges was constructed by the special technique of abelianizing the two-dimensional gauge potentials associated to the zero curvature equation [1].

Within the context of Toda field theories one finds that the conformal affine Toda (CAT) model [2, 3] based on the full Kac-Moody algebra occupies a special place due to its conformal invariance, existence of W-infinity symmetry [4, 5], the soliton solutions [6, 7] and the fact that the AT as well as the conformal Toda (CT) model can be obtained from it by taking particular limits. In view of the above one wishes to investigate whether in the CAT model we have a strong form of complete integrability. This paper is devoted to the explicit construction of the local, conserved charges in involution for the CAT model. The construction starts from equations of motion written in the zero-curvature form and proceeds by gauge transforming potentials to their abelian form. We find, in this way, an intriguing structure of charges. The two lowest charges have corresponding chiral charge densities. The remaining are chiral after integration but without having any chiral charge density associated with them. This explains why the local W-symmetry structure of the CAT model could be described by only two spin 1 and spin 2 chiral currents [8, 4, 5]. It is a surprising fact since the W-algebra provided by these two currents is the same for all CAT models irrespectively of the underlying KM algebra. In this paper the missing conservation laws associated with the structure of KM algebra are uncovered. However there are no extra local chiral densities extending the local W symmetry. This fact distinguishes the CAT model among the integrable conformal models. For instance, the charges of the CT model are obtained from chiral densities satisfying a $W_N$ algebra intrinsically related to the Casimir operators of the underlying simple Lie algebra [9].

We prove using the fundamental Poisson bracket relations (FPR) that the charges for CAT model conserved in time are in involution. It is not clear yet whether the charges of given chirality are in involution although it is quite trivial to see that charges of different chiralities mutually Poisson commute.

The densities of the chiral charges have a special form, which allows one to gauge fix the conformal symmetry obtaining the AT charges constructed in [1] from the corresponding CAT charges.

In section 2 we discuss the main properties of the CAT model. We review some basic results and reformulate equations of motion in terms of the special basis of KM algebra [10, 11, 1]. This choice of basis proves to be essential in our construction. The main results are given in section 3 where by successive gauge transformations we cast potentials into an abelian form and find the corresponding conserved charges. Their involution is discussed in section 4 where use is made of the fundamental Poisson bracket relations. Finally in section
5, we give explicit examples for $sl(2)$ and $sl(3)$ constructing the first few charges. We close the paper with section 6 containing discussion.

2 The CAT model

The Conformal affine Toda models (CAT) extend the Affine Toda models (AT) by the introduction of two extra fields which make the model conformally invariant [2, 3]. For each simple Lie algebra $G$ we associate a CAT model with the equations of motion given by [6]:

\[
\begin{align*}
\partial_- \partial_+ \varphi &= \left( q^a e^{K_{ab} \varphi} - q^0 \frac{\partial \omega}{\ partial x} e^{-K_{ab} \varphi} \right) e^\varphi \\
\partial_- \partial_+ \eta &= 0 \\
\partial_- \partial_+ \nu &= 2 \frac{\partial \omega}{\ partial x} e^{-K_{ab} \varphi} \ e^\varphi
\end{align*}
\]  

(2.1)  

(2.2)  

(2.3)

where $K_{ab} = 2 \alpha_a \alpha_b / \alpha_\beta^2$ is the Cartan Matrix of $G$, $a, b = 1, ..., \text{rank} \, G \equiv r$, $\psi$ is the highest root of $G$, $K_{\psi \psi} = 2 \psi, \alpha_a / \alpha_\beta^2$, $l_a^\psi$ are positive integers appearing in the expansion $\frac{\partial \psi}{\ partial x} = l_a^\psi \frac{\partial \omega}{\ partial x \alpha_a}$, where $\alpha_a$ are the simple roots of $G$ and $q^a$ and $q^0$ are coupling constants.

These equations are invariant under conformal transformations:

\[
x_+ \rightarrow \hat{x}_+ = f(x_+) \ , \ x_- \rightarrow \hat{x}_- = g(x_-)
\]  

(2.4)

and

\[
e^{-\varphi}(x_+, x_-) \rightarrow e^{-\hat{\varphi}}(\hat{x}_+, \hat{x}_-) = e^{-\varphi}(x_+, x_-)
\]  

(2.5)

\[
e^{-\nu}(x_+, x_-) \rightarrow e^{-\hat{\nu}}(\hat{x}_+, \hat{x}_-) = \left( \frac{df}{dx_+} \right) B \left( \frac{dg}{dx_-} \right) e^{-\nu}(x_+, x_-)
\]  

(2.6)

\[
e^{-\eta}(x_+, x_-) \rightarrow e^{-\hat{\eta}}(\hat{x}_+, \hat{x}_-) = \left( \frac{df}{dx_+} \right) \left( \frac{dg}{dx_-} \right) e^{-\eta}(x_+, x_-)
\]  

(2.7)

where $f$ and $g$ are analytic functions and $B$ is an arbitrary constant. Therefore $e^{\varphi}$ are scalars under conformal transformations and $e^{-\nu}$ and $e^{-\eta}$ are primary fields of conformal weights $(B, B)$ and $(1, 1)$ respectively [6].

The equations of motion (2.1)-(2.3) can be written as a zero curvature condition:

\[
\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0
\]  

(2.8)

where the potentials $A_\pm$ lie on the affine Kac-Moody algebra $\hat{G}$ associated to $G$

\[
A_+ = \frac{1}{2} \partial_+ \Phi + e^{\partial \Phi/2} \mathcal{E}_+ \ , \ A_- = -\frac{1}{2} \partial_- \Phi + e^{-\partial \Phi/2} \mathcal{E}_-
\]  

(2.9)

and

\[
\Phi = \sum_{a=1}^r \varphi^a H_a^0 + \eta \hat{\phi} + \nu C
\]  

(2.10)

\[
\mathcal{E}_+ = \sum_{a=1}^r \sqrt{l_a^\psi} \ E_a^0 \varphi^a + E_+^1 \ , \ \mathcal{E}_- = \sum_{a=1}^r \frac{q^a}{\sqrt{l_a^\psi}} E_{\varphi a}^0 + q^0 E_{\varphi}^{-1}
\]  

(2.11)
where
\[ \dot{\rho} = 2\dot{\hat{k}}.H^0 + hD \]  
(2.12)

with \( \dot{\hat{k}} = \frac{1}{2} \sum_{a>0} \dot{\hat{w}}_a \), is the generator used to perform the so-called principal gradation of an affine Kac-Moody algebra.

The generators \( H_\alpha^0, E_\alpha^0, C \) and \( D \) constitute the Chevalley basis of \( \hat{\mathcal{G}} \) and their commutation relations are defined in (5.1). There exists however another basis for \( \hat{\mathcal{G}} \) which will prove very useful in the construction of the conserved charges of the CAT model [10, 12]. In that basis the element \( \mathcal{E}_+ \) appearing in the gauge potentials (2.9) plays a crucial role. One can grade \( \hat{\mathcal{G}} \) using the operator (2.12) as
\[ \hat{\mathcal{G}} = \bigoplus m \hat{\mathcal{G}}_m \]  
(2.13)

where \( m \in \mathbb{Z} \) and such that
\[ [\hat{\rho}, \hat{\mathcal{G}}_m] = m \hat{\mathcal{G}}_m \]  
(2.14)

The subalgebra \( \mathcal{H} \) of \( \hat{\mathcal{G}} \) commuting, modulo the central term \( C \), with \( \mathcal{E}_+ \) is a Heisenberg subalgebra with generators \( E_M \in \hat{\mathcal{G}}_M \), where \( M \) are the exponents of \( \hat{\mathcal{G}} \), satisfying
\[ [E_M, E_N] = \frac{C}{\hbar} Tr(E_M E_N) \delta_{M+N,0} \]  
(2.15)

These exponents have a period equal to the Coxeter number \( h \) of \( \mathcal{G} \) and for \( \hat{\mathcal{G}} \) being an affine Kac-Moody algebra (not twisted) they have the form \( M \equiv m_a + nh \) where \( n \) is an integer and \( m_a, a = 1, 2, \ldots, r \) are the exponents of the simple Lie algebra \( \mathcal{G} \). In addition these exponents possess the symmetry \( M \rightarrow -M \). In particular, unity is always an exponent and it follows that
\[ E_1 = \mathcal{E}_+, \quad E_{-1} = \sum_{a=1}^r \sqrt{t^0_a} E_{-m_a} + E^{-1}_\psi \]  
(2.16)

The complement \( \mathcal{F} \) of \( \mathcal{H} \) in \( \hat{\mathcal{G}} \) is such that the dimension of \( \mathcal{F}_m \subset \hat{\mathcal{G}}_m \) for \( m \neq 0 \) is equal to the rank of \( \mathcal{G} (\equiv r) \). The subspace \( \mathcal{F}_0 \) has dimension \( r + 2 \) and is generated by \( C, D \) and \( H_a^0, a = 1, 2, \ldots, r \). Except for the extra two generators of \( \mathcal{F}_0 \), the basis, \( F_m^a \), of \( \mathcal{F}_m \) can be chosen such that
\[ [E_M, F_n^a] = \gamma^a \cdot v_{[M]} F_{M+n}^a \]  
(2.17)

where \( \gamma^a \) and \( v_{[M]} \) are vectors in the root space of \( \mathcal{G} \) and \( [M] \) means \( M \) modulo \( h \) [11, 12].

### 3 The Construction of the Charges

We now show how to construct the conserved charges for the CAT models. In the case of the AT models this was done by rotating the gauge potentials into an abelian subalgebra such that the commutator term in (2.8) vanishes [1]. One then obtains, by imposing suitable boundary conditions on one of the space-time variables, let us say \( x \), that the quantity \( \int dx A_x \) is constant in the other variable, let us say \( t \). Such procedure proved to be very powerful in the case of the AT models [1] where the underlying algebra is a loop algebra. In the case of
the CAT models the situation is a bit more complicated due to the central term of the Kac-Moody algebra. However, such fact does not prevent us from constructing conserved charges. In fact, it makes things even more interesting. We can rotate the right/left components of the gauge potentials into the negative/positive abelian subalgebras of the Heisenberg algebra $\mathcal{H}$. The structure of those potentials is such that the zero curvature form of the equations of motion leads to the conservation of the densities of the chiral components of the energy-momentum tensor, and also of the spin one current related to the free field $\eta$. This contrasts with the AT model, where only the integrated E-M tensor is conserved [1]. This sheds some light on the connection between the conformal invariance of the model and the central extension of the KM algebra. Another interesting aspect is that the higher chiral conserved charges are integrals of non chiral densities. Therefore the conservation laws of the CAT model are not really described by a $\mathcal{W}$ algebra like in the case of the CT models where one has chiral densities generating a $\mathcal{W}_N$ algebra. However, as we have shown in [8, 4, 5] part of the conservation laws for the CAT model leads to a $\mathcal{W}_\infty$ algebra.

We now show that there are two infinite sets of chiral conserved charges. Each right (left) chiral charge is associated with a generator $E_M$ for $M < 0$ ($M > 0$). In fact, the charges will be conserved in time $t$, $x_+$ or $x_-$ depending upon the boundary conditions one imposes on the fields. In this section we will work out the conservation laws using light cone variables. In section 4 we show that the charges conserved in time are in involution.

Let us start with the right charges. We first perform a gauge transformation

$$A_\pm \rightarrow g^{-1}A_\pm g + g^{-1}\partial_\pm g$$

with $g = e^{\frac{\Phi}{2}}$. We then get

$$A_+ \rightarrow A^R_+ = \partial_+ \Phi + E_1$$
$$A_- \rightarrow A^R_- = e^{\Phi}e^{-\Phi}$$

The idea now is to perform a second gauge transformation to rotate the potential $A^R_+$ as

$$A^R_+ \rightarrow 
A^R_+ = g_R^{-1}A^R_+g_R + g_R^{-1}\partial_+ g_R = E_1 + \partial_+ \eta \hat{\rho} + \sum_{M > 0} A^{R,(M)}_+ E_{-M}$$

where $g_R$ is an exponentiation of the generators of $\hat{\mathcal{G}}$ with negative eigenvalues of $\hat{\rho}$ given in (2.12) [13]. In fact, since $E_{-M}$ for $M > 1$ commutes with $E_1$ we can take $g_R$ to be an exponentiation of $E_{-1}$ and the generators of $\mathcal{F}_{-m}$ for $m > 0$. To show that (3.3) is possible, let us write

$$A^R_+ = \sum_{m = 1}^{\infty} A^{R,(m)}_+ , \text{ with } A^{R,(m)}_+ \in \hat{\mathcal{G}}_m$$

$$g_R = e^{(\chi E_{-1} + \sum_{m > 0} \zeta_m )} , \text{ where } \zeta_m \in \mathcal{F}_{-m}$$

Then

$$A^{R,(1)}_+ = E_1$$
\[ A_+^{R(0)} = [E_1, \chi E_{-1} + \zeta_{-1}] + \partial_+ \Phi \]
\[ A_+^{R(-1)} = [E_1, \zeta_{-2} + [\partial_+ \Phi, \chi E_{-1} + \zeta_{-1}] + \frac{1}{2!} [\chi E_{-1} + \zeta_{-1}, [\chi E_{-1} + \zeta_{-1}, E_1]] \]
\[ + \partial_+ \chi E_{-1} + \partial_+ \zeta_{-1} \]
\[ \vdots \]
\[ A_+^{R(-m)} = [E_1, \zeta_{-m-1}] + X_{-m} \] (3.6)

where \( X_{-m} \) depends on \( \zeta_m \) for \( n \leq m \) only.

One can observe that the \( r \) parameters in \( \zeta_{-m-1} \) can be chosen such a way as to cancel the \( r \) components of \( A_+^{R(-m)} \) lying in \( F_{-m} \). The exception is on the level zero where we have also to fix \( \chi \), in addition to \( \zeta_{-1} \), in order to remove the \( r + 1 \) components of \( A_+^{R(0)} \) lying in the direction of \( H^a_a \) \( (a = 1, 2, \ldots r) \) and \( C \). Then \( A_+^{R(0)} \) will be just \( \partial_+ \eta \dot{\rho} \). Therefore it is possible to gauge transform \( A_+^R \) into the form (3.3).

For the potential \( A_+^R \) we get
\[ A_+^R \to A_+^R = g^{-1}_R A_+^R g_R + g^{-1}_R \partial_+ g_R \]
\[ = \sum_{M > 0} A_+^{R(M)} F_{-M} + \sum_{a=1}^{r} \sum_{m > 0} b_{-1}^{R(a,m)} F_{-m}^a \] (3.7)

The components of \( A_+^R \) in the direction of \( F_{-m}^a \), as we show below, vanish as a consequence of the equations of motion. The zero curvature condition (2.8) for the gauge transformed potentials can be split into three parts, one in the direction of \( \dot{\rho} \), other lying on the Heisenberg subalgebra and another on its complement. So one gets
\[ 0 = \partial_+ \partial_- \eta \dot{\rho} \] (3.8)
\[ 0 = \sum_{M > 0} \left( \partial_+ A_{-}^{R(M)} - \partial_- A_+^{R(M)} \right) F_{-M} \]
\[ + \left[ E_1 + \partial_+ \eta \dot{\rho} + \sum_{M > 0} A_+^{R(M)} F_{-M} + \sum_{M > 0} A_{-}^{R(M)} F_{-M} \right] \] (3.9)
\[ 0 = \sum_{m > 0} \partial_+ b_{-1}^{R(a,m)} F_{-m}^a + \left[ E_1 + \partial_+ \eta \dot{\rho} + \sum_{M > 0} A_+^{R(M)} F_{-M} + \sum_{a=1}^{r} \sum_{m > 0} b_{-1}^{R(a,m)} F_{-m}^a \right] \] (3.10)

In eq. (3.10) the term with highest eigenvalue of \( \dot{\rho} \) is
\[ [E_1, \sum_{a=1}^{r} b_{-1}^{R(a,1)} F_{-1}^a] = 0 \] (3.11)

Since there are no generators of \( \mathcal{F} \) commuting with \( E_1 \) (see (2.17)) one concludes that \( b_{-1}^{R(a,1)} = 0 \). Now, the term with highest eigenvalue of \( \dot{\rho} \) in (3.10) becomes \([E_1, \sum_{a=1}^{r} b_{-2}^{R(a,2)} F_{-2}^a] = 0\) and for the same reason \( b_{-2}^{R(a,2)} = 0 \). So, following this reasoning one concludes that the second term on the right hand side of (3.7) vanishes. Therefore we have rotated the two components of the gauge potential into the Heisenberg subalgebra (except for the term \( \partial_+ \eta \dot{\rho} \) in \( A_+^R \)). One observes that the only non-vanishing term, involving \( E_1 \), in the commutator in (3.9) is given by
\[ [E_1, A_+^{R(1)} F_{-1}^a] \sim C \] (3.12)
Since this is the only term in the direction of $C$ in (3.9) one concludes that $\mathcal{A}_{-}^{R,(1)} = 0$. Hence we have proved that

$$\mathcal{A}_{-}^{R} = \sum_{M > 1} \mathcal{A}_{-}^{R,(M)} E_{-M}$$

(3.13)

In consequence the terms proportional to $E_{-1}$ in (3.9) lead to

$$\partial_{-} \mathcal{A}_{+}^{R,(1)} = 0$$

(3.14)

$\mathcal{A}_{+}^{R,(1)}$ corresponds, in fact, to a chiral component of the CAT model stress tensor. Notice that the central term played a crucial role in making such local density chiral. In addition (3.8) leads to

$$\partial_{-} J^{R} = 0 \ , \text{ with } J^{R} \equiv \partial_{+} \eta$$

(3.15)

The chiral densities $J^{R}$ and $\mathcal{A}_{+}^{R,(1)}$ are respectively the spin 1 and 2 fields shown, in [8], to be the only remaining Kac-Moody currents in the reduction of the two-loop WZNW model [3, 14], and used to construct an $\mathcal{W}_{\infty}$ for the CAT model [4, 5].

After those considerations what is left of (3.9) is

$$(\partial_{+} - MJ^{R})\mathcal{A}_{-}^{R,(M)} - \partial_{-}\mathcal{A}_{+}^{R,(M)} = 0 \ , \text{ for } M > 1$$

(3.16)

These equations are in fact the conservation laws for the CAT model. They can be rewritten as

$$\partial_{+} a_{-}^{R,(M)} = \partial_{-} a_{+}^{R,(M)}$$

(3.17)

where we have introduced

$$a_{+}^{R,(M)} = \epsilon^{-M} \int^{x_{+}} d\nu_{+} J^{R} \mathcal{A}_{+}^{R,(M)}$$

$$a_{-}^{R,(M)} = \epsilon^{-M} \int^{x_{+}} d\nu_{+} J^{R} \mathcal{A}_{-}^{R,(M)}$$

(3.18)

Therefore by imposing suitable boundary conditions on the fields it follows from (3.15) and (3.17) that the charges

$$Q_{R}^{M} = \int d\nu_{+} \epsilon^{-M} \int^{x_{+}} d\nu_{+} J^{R} \mathcal{A}_{+}^{R,(M)}$$

(3.19)

are chiral

$$\partial_{-} Q_{R}^{M} = 0$$

(3.20)

Notice that $\epsilon^{-M} \int^{x_{+}} d\nu_{+} J^{R}$ is $x_{-}$ independent, but the quantities $\mathcal{A}_{+}^{R,(M)}$ are not. Therefore the densities of the chiral charges (3.19) are not chiral themselves. Hence we have here a situation different from that of conformal field theories with extended symmetry. There, the symmetries are described by, in general a $\mathcal{W}$-algebra and the densities are chiral (like in the CT models for instance). It would be interesting to study the algebra of these non chiral densities.

The gauge potentials $\mathcal{A}_{\pm}^{R}$ can in fact be rotated into the abelian subalgebra of $\mathcal{H}$ generated by $E_{-M}$ for $M < 0$. This is done by two gauge transformations. Denoting

$$g_{1}(x_{+}) = \epsilon^{-\phi} \int^{x_{+}} J^{R} d\nu_{+} \quad \text{and} \quad g_{2}(x_{+}) = \exp(-E_{1} \int^{x_{+}} \epsilon^{-\nu_{+}} J^{R} d\nu_{+} d\nu_{+})$$

(3.21)
we get

$$\mathcal{A}_+^R \rightarrow \mathcal{A}_+^R = g_1^{-1} \mathcal{A}_+^R g_1 + g_1^{-1} \partial_+ g_1$$

$$e^{\int^{x_+} J^R dz_+} E_1 + \sum_{M>0} a_+^{R,(M)} E_{-M}$$

(3.22)

where $a_+^{R,(M)}$ was defined in (3.18). In addition

$$\mathcal{A}_+^R \rightarrow a_+^R = g_2^{-1} \mathcal{A}_+^R g_2 + g_2^{-1} \partial_+ g_2$$

$$a_+^{R(0)} C + \sum_{M>0} a_+^{R,(M)} E_{-M}$$

(3.23)

where

$$a_+^{R(0)} = \frac{1}{\hbar} Tr(E_1 E_{-1}) \mathcal{A}_+^{R,(1)} e^{\int^{x_+} J^R dz_+} \int^{x_+} e^{\int^{y_+} J^R dy_+} dy_+$$

(3.24)

The other component of the gauge potential transforms as

$$\mathcal{A}_-^R \rightarrow \mathcal{A}_-^R = g_1^{-1} \mathcal{A}_-^R g_1 + g_1^{-1} \partial_- g_1$$

$$\sum_{M>0} a_-^{R,(M)} E_{-M}$$

(3.25)

where $a_-^{R,(M)}$ was also defined in (3.18). Finally

$$\mathcal{A}_-^R \rightarrow a_-^R = g_2^{-1} \mathcal{A}_-^R g_2 + g_2^{-1} \partial_+ g_2$$

$$\mathcal{A}_-^R$$

(3.26)

since $E_1$ commutes with all $E_{-M}$ for $M < 0$ and $J^R$ is $x_-$ independent.

Therefore the gauge potentials $a_+^R$ are abelian and satisfy the zero curvature condition:

$$\partial_+ a_+^R - \partial_- a_+^R = 0$$

(3.27)

Notice that the only non-local quantity in these potentials is $a_+^{R(0)}$, but it is harmless due to the chiral character of the quantities involved. Since $\int^{x_+} J^R dy_+ = \eta_+(x_+)$, where $\eta_+(x_+, x_-) = \eta_+(x_+) + \eta_-(x_-)$ is a solution of the equations of motion, the exponential terms appearing in $a_+^R$ are local quantities and depend only on the free field $\eta$. That is an interesting fact and it is a consequence of the conformal symmetry. As we have shown in [6] the dynamics of the CAT model on a space-time $(x_+, x_-)$ is the same as that of the AT model (except for the passive field $\nu$) on a space time $(\hat{x}_+, \hat{x}_-)$ where $x$'s and $\hat{x}$'s are related by the conformal transformation

$$\hat{x}_+ = \int^{x_+} dy_+ e^{\nu(x_+)}$$

$$\hat{x}_- = \int^{x_-} dy_- e^{\nu(x_-)}$$

(3.28)

So, for every solution of the $\eta$ field the CAT model is an AT model in a different space-time.

One can easily convince oneself, by studying (3.6), that the quantity $\mathcal{A}_+^{R,(M)}$ is a polynomial in the derivatives of the CAT model fields and each term in it contains exactly $M + 1$ derivatives w.r.t. $x_+$ and no derivatives w.r.t. $x_-$. As we have seen in (2.5)-(2.7) the exponentials $e^{-\phi_+}$ and $e^{-\phi_+}$ are primary fields of conformal weights $(0,0)$ and $(1,1)$. By choosing
$B = 0$ we also get that $e^{-\nu}$ is also a $(0,0)$ primary field. Therefore, under the conformal transformation (3.28) we have

$$e^{-\tilde{\omega}_0(\tilde{x}_+,\tilde{x}_-)} \rightarrow e^{-\tilde{\omega}_0(x_+,x_-)}$$

where we are working with a fixed solution of the field $\eta$, and from (3.19)

$$\int \left( e^{J^+ \cdot \frac{d}{dx} + J^- R} dx^+ \right) e^{-\frac{1}{2} \mathcal{L}_\nu} \mathcal{A}_{\nu}^{R,(M)}(x_+, x_-) \rightarrow \int d\tilde{x}_+ \mathcal{A}_{\nu}^{R,(M)}(\tilde{x}_+, \tilde{x}_-)$$

It is a remarkable fact that the term $e^{-M(x_+,\tilde{x}_-)}$ comes exactly with the right form to make such a transformation to the AT charges possible. The density $\mathcal{A}_{\nu}^{R,(M)}(\tilde{x}_+, \tilde{x}_-)$ is now $\eta$-independent, although it may be $\nu$-dependent. They are the densities for the AT model (plus the passive field $\nu$) and they correspond to the densities constructed in [1].

We now discuss the construction of the left charges for the CAT models. We first perform a global gauge transformation on the gauge potentials (2.9) with the group element $\Omega = \exp \sum \omega_b 2 \lambda_b \cdot H^0 / \alpha_1^2 + \omega_b D$, where $\lambda_a$ are the fundamental weights of $G$, $\omega_a = \log (q_a / l_\psi^a)$ and $\omega_b = \sum_\gamma n_\gamma^b \omega_b + \log q_0$ with $n_\gamma^b$ being the integers in the expansion $\psi = n_\gamma^b \alpha_1$. We then get

$$A_+ \rightarrow \tilde{A}_+ = \Omega A_+ \Omega^{-1} = \frac{1}{2} \partial_+ \Phi + e^{\frac{1}{2} \Phi} \mathcal{E}_+$$

$$A_- \rightarrow \tilde{A}_- = \Omega A_- \Omega^{-1} = -\frac{1}{2} \partial_- \Phi + e^{-\frac{1}{2} \Phi} E_-$$

where

$$\mathcal{E}_+ = \sum_\gamma \frac{q_0}{l_\psi^a} E_0^a + q_0 E_-$$

Next, we perform the gauge transformation (3.1) with $g = e^{-\Phi/2}$. We get

$$\tilde{A}_+ \rightarrow A_+ = \mathcal{E}_+ e^{-\Phi}$$

$$\tilde{A}_- \rightarrow A_- = -\partial_- \Phi + E_-$$

We now rotate these potentials into the Heisenberg subalgebra with a third gauge transformation

$$A_\pm \rightarrow \tilde{A}_\pm = g_\pm^{-1} A_\pm g_\pm + g_\pm^{-1} \partial_\pm g_\pm$$

with $g_\pm = e^{(E_1 + \sum_\gamma \zeta_\gamma)}$ and $\zeta_\gamma \in \mathcal{F}_\gamma$. Following the same procedure we used in the case of the right charges we obtain that

$$A_- = E_{-1} - \partial_- \eta \hat{h} + \sum_{M > 0} \mathcal{A}_{-}^{L,(M)} E_M$$

$$A_+ = \sum_{M > 0} \mathcal{A}_{+}^{L,(M)} E_M$$

(3.36)
Then by performing two consecutive gauge transformations with $\hat{g}_1 = \exp(\hat{\rho} \int x^{-1} J^L dy_{-})$ and then $\hat{g}_2 = \exp(-E_{-1} \int x^{-1} \exp(\int x^\prime J^L dz_{-}) dy_{-})$, with $J^L$ defined in (3.41), we get

$$\mathcal{A}^L_{-} \rightarrow a^L_{-} = a^{L(0)}_{-} C + \sum_{M>0} a^{L(M)}_{-} E_M$$

$$\mathcal{A}^L_{+} \rightarrow a^L_{+} = \sum_{M>1} a^{L(M)}_{+} E_M$$

with

$$a^{L(0)}_{-} = -\frac{1}{\hbar} Tr(E_1 E_{-1}) \mathcal{A}^{L(1)}_{-} e^{-\int x^{-1} J^L dz_{-}} \int x^{-1} e^{\int x^\prime J^L dz_{-}} dy_{-}$$

$$a^{L}_{\pm} = \exp(-M \int x^{-1} J^L dy_{-}) \mathcal{A}^L_{\pm}$$

(3.37)

So, the gauge potentials are abelian and satisfy

$$\partial_{+} a^L_{-} - \partial_{-} a^L_{+} = 0$$

(3.39)

and lead to the conservation laws

$$\partial_{+} a^{(1)}_{-} = 0$$

(3.40)

$$\partial_{+} J^L = 0 \ , \text{ with } J^L \equiv \partial_{-} \eta$$

(3.41)

$$\partial_{+} Q^L_{M} = 0 \ , \text{ for } M > 1$$

(3.42)

where

$$Q^L_{M} = \int dx_{-} a^{L(M)}_{-}$$

(3.43)

We recognize in the above the similar structure as the one we previously constructed for the right charges.

Notice that the densities $a^{R(M)}_{+}$ ($a^{L(M)}_{-}$) are functions of $x_{+}$ ($x_{-}$) derivatives of the fields only. For Lagrangeans which are quadratic in time derivatives of the fields, it follows that the Poisson bracket between $x_{+}$ and $x_{-}$ derivatives of the fields vanishes. Therefore the right and left charges Poisson commute. So, the algebra of the chiral charges split into two commuting isomorphic subalgebras.

4 Involution of Charges

In this section we will use the fundamental Poisson bracket relation (FPR) and abelianization to verify that the $x$-component charges obtained above are indeed in involution. We will follow closely approach of references [15] and [1]. Let us recall some main steps of this formalism. The basic role is played by the FPR relation constructed in [3] for the CAT model:

$$\{ A_{x}(x) \otimes A_{y}(y) \}_{PB} = -\frac{1}{2} \delta(x-y) \left[ \mathcal{I} P , 1 \otimes A_{x}(x) + A_{x}(x) \otimes 1 \right]$$

(4.1)

where $A_{x} = A_{+} + A_{-}$, with $A_{\pm}$ given in (2.9) and

$$\mathcal{I} P = \Phi_{+} - \Phi_{-}$$

(4.2)
and $\Phi_\pm$ are given by
\begin{equation}
\Phi_+ = \sum_{m=1}^{\infty} \sum_{\alpha=1}^{r} \eta_{ab} H_a^m \otimes H_b^{-m} + \sum_{\alpha>0} \frac{\alpha^2}{2} F_\alpha^0 \otimes E_\alpha^0 \\
+ \sum_{m=1}^{\infty} \sum_{\alpha>0} \frac{\alpha^2}{2} \left( E_\alpha^m \otimes E_{-\alpha}^{-m} + E_{-\alpha}^{-m} \otimes E_\alpha^m \right)
\end{equation}
and $\Phi_-$ is obtained from $\Phi_+$ by the interchange of left and right entries, and $\eta_{ab}$ is the inverse of $\eta_{ab} = \frac{2}{a^2} K_{ab}$.

Due to the commutator term in (2.8), the quantities $\int_{-l}^{l} A_x dx$ are not conserved in time. However denoting $a_x^{R/L} = a_+^{R/L} + a_-^{R/L} = \sum_{M>0} a_x^{R/L,M} E_{\pm M}$, with $a_x^{R/L}$ constructed in section 3, one obtains that the quantities
\begin{equation}
P_m^R/L \equiv \int_{-l}^{l} a_x^{R/L,M} dx
\end{equation}
are conserved in time when periodic boundary conditions are imposed on the fields at $x = \pm l$.

However the bracket involving $a_x^{R/L}$ is not easy to evaluate directly. Since they are related to $A_x$ via gauge transformations as shown in section 3, we will make use of the gauge invariant quantities $\text{Tr} U^n$ with $U$ being the path ordered exponential
\begin{equation}
U = T \exp[- \int_{-l}^{l} A_x(x) dx]
\end{equation}
Indeed, using the zero curvature condition (2.8) and the non abelian version of Stoke's theorem [1] one gets that $\text{Tr} U^n$ are conserved in time.

From FPR (4.1) we can prove that
\begin{equation}
\{ A_x(x) \otimes U \}_PB = -[A_x(x) \otimes 1 , Q(x)] - \frac{\partial Q(x)}{\partial x}
\end{equation}
where
\begin{equation}
Q(x) = 1 \otimes T e^{- \int_{-l}^{l} A_x(y) dy} \frac{1}{2} \text{P} 1 \otimes T e^{- \int_{-l}^{l} A_x(y) dy}
\end{equation}
Relation (4.6) can be used to prove another useful identity
\begin{equation}
\{ A_x(x) , \text{Tr} U^n \}_PB = -[\partial_x + A_x(x) , \hat{Q}_m(x)]
\end{equation}
with
\begin{equation}
\hat{Q}_m(x) = m \text{Tr} \left( Q(x) 1 \otimes U^{m-1} \right)
\end{equation}
One can now show that the Poisson bracket $\{ T \exp \left( - \int_{-l}^{l} A_x(y) dy \right) , \text{Tr} U^n \}_PB$ is equal to an integration of a total derivative and therefore vanishes when we assume the appropriate periodic boundary conditions for the interval $[-l,l]$. Hence by this standard argument it follows that the quantities $\text{Tr} U^n$ are in involution for all integers $m$
\begin{equation}
\{ \text{Tr} U^n , \text{Tr} U^m \}_PB = 0
\end{equation}
Since $\text{Tr} U^n$ are gauge invariant, they are equal to $\text{Tr} (u^{R/L})^n$, where $u^{R/L} = T \exp[ - \int_{-l}^{l} a_x^{R/L}(x) dx]$. But as we have shown in section 3 the potentials $a_x^{R/L}$ are abelian and so path ordering is not essential. Therefore the conserved quantities $\text{Tr} U^n$ are local. In fact, they are functionally related to the charges (4.4). From this we conclude that the integrals of $x$-components of $a_x^{R/L,M}$, i.e. the charges (4.4), must also be in involution as a consequence of (4.10).
5 Examples

The commutation relations for an affine Kac-Moody algebra in the Chevalley basis are given by

\[
\begin{align*}
[H^m_a, H^n_b] &= C \frac{2}{\alpha_a^2} K_{ab} m \delta_{m+n,0} \\
[H^m_a, E^{m}_{\pm\alpha}] &= \pm K_{ab} E^{m+n}_{\pm\alpha} \\
[E^m_a, E^n_{-\alpha}] &= \ell^a \ H^{m+n} + C \frac{2}{\alpha^2} m \delta_{m+n,0} \\
[E^m_a, E^n_{\beta}] &= \epsilon(\alpha, \beta) E^{m+n}_{\alpha+\beta} \text{ if } \alpha + \beta \text{ is a root of } \mathcal{G} \\
[D, H^m_b] &= m H^m_b \\
[D, E^m_a] &= m E^m_a
\end{align*}
\]

(5.1)

where \(K_{ab} = 2\alpha_a \alpha_a / \alpha_a^2 = n_a^0 K_{ba}\), with \(n_a^0\) and \(\ell^a\) being the integers in the expansions \(\alpha = n_a^0 \alpha_a\) and \(\alpha / \alpha_a^2 = \ell^a \alpha_a / \alpha_a^2\), and \(\epsilon(\alpha, \beta)\) being structure constants. We now give explicit expressions for the first two non trivial charges for the case of the \(SL(2)\) and \(SL(3)\) CAT models.

5.1 The Example of \(SL(2)\)

The simple algebra \(SL(2)\) has just one positive (simple) root and its Cartan matrix is \(K = 2\). The equations of motion are those given in (2.1)-(2.3) with \(K_{ab} = K_{\psi b} = 2\), \(l^a = 1\) and we shall normalize \(\psi^2 = 2\). The basis discussed in section 2 in this case is given by

\[
\begin{align*}
E_{2m+1} &= E^m_a + E^{m+1}_a \\
F_{2m+1} &= E^m_a - E^{m+1}_a \\
F_{2m} &= H^m - \frac{1}{2} C \delta_{m,0}
\end{align*}
\]

(5.2) (5.3) (5.4)

in addition to the generators \(C\) (central term) and \(\hat{\rho} = \frac{1}{2} H^0 + 2D\). The commutation relations are

\[
\begin{align*}
[E_{2m+1}, E_{2n+1}] &= C(2m + 1) \delta_{m+n,0} \\
[E_{2m+1}, F_{2n+1}] &= -2F_{2(m+n+1)} \\
[E_{2m+1}, F_{2n}] &= -2F_{2(m+n)+1} \\
[F_{2m+1}, F_{2n+1}] &= -C(2m + 1) \delta_{m+n+1,0} \\
[F_{2m+1}, F_{2n}] &= -2E_{2(m+n)+1} \\
[F_{2m}, F_{2n}] &= C 2m \delta_{m+n,0}
\end{align*}
\]

(5.5) (5.6) (5.7) (5.8) (5.9) (5.10)

and they are all eigenvectors of \(\hat{\rho}\)

\[
\begin{align*}
[\hat{\rho}, E_{2m+1}] &= (2m + 1) E_{2m+1} \\
[\hat{\rho}, F_{2m+1}] &= (2m + 1) F_{2m+1} \\
[\hat{\rho}, F_{2m}] &= 2m F_{2m}
\end{align*}
\]

(5.11) (5.12) (5.13)
Using the procedure of section 3 one can calculate the charges. The first two right charge densities are given by

\[
a^R_{+,1} = e^{-f^{(1)} J_R} \left( \frac{1}{2} (\partial_+ \varphi)^2 - \frac{1}{2} \partial_+^2 \varphi \right) + \frac{1}{2} \partial_+ \varphi \partial_+ \eta + \partial_+ \eta \partial_+ \nu - \partial_+^2 \nu \quad (5.14)
\]

\[
a^R_{-,1} = 0 \quad (5.15)
\]

and

\[
a^R_{+,3} = \frac{1}{8} e^{-3 f^{(3)} J_R} \left( - (\partial_+ \varphi)^4 + \partial_+ \varphi \partial_+^3 \varphi + \frac{1}{2} \partial_+^2 \eta (\partial_+ \varphi)^2 - \frac{5}{2} (\partial_+ \eta \partial_+ \varphi)^2 \\
+ (\partial_+ - 3 \partial_+ \eta) \left( - \frac{1}{3} (\partial_+ \varphi)^3 - \frac{2}{3} \partial_+ \nu (\partial_+ \varphi)^2 - \frac{3}{2} \partial_+^2 \nu \partial_+ \varphi + \partial_+^2 \varphi \right) \right) \quad (5.16)
\]

\[
a^R_{-,3} = \frac{1}{8} e^{-3 f^{(3)} J_R} \left( q_0 e^{-2 \varphi^2 + \eta} \left( \frac{1}{3} (\partial_+ \varphi)^2 + \partial_+ \eta \partial_+ \varphi + \frac{4}{3} \partial_+ \nu \partial_+ \varphi - \partial_+^2 \varphi \right) \\
+ q_1 e^{2 \varphi^2 + \eta} \left( - (\partial_+ \varphi)^2 - \partial_+ \eta \partial_+ \varphi - \frac{4}{3} \partial_+^2 \varphi \right) \right) \quad (5.17)
\]

As explicit calculation has shown the integral of charge densities \(a^R_{k} \quad k = 1, 3, 5 \) becomes \( \nu \)-independent under appropriate boundary conditions. We suspect that this intriguing feature will remain valid for the other charges.

5.2 The Example of \( SL(3) \)

The Cartan matrix for \( SL(3) \) is given by \( K_{11} = K_{22} = 2 \) and \( K_{12} = K_{21} = -1 \) and \( t^a \psi = 1 \), \( a = 1, 2 \). We shall denote by \( \alpha_1 \) and \( \alpha_2 \) the simple roots and the highest is \( \psi = \alpha_1 + \alpha_2 \) which we normalize as \( \psi^2 = 2 \). We also have \( K_{\psi 1} = K_{\psi 2} = 1 \). The structure constants \( \epsilon(\alpha, \beta) \), appearing in (5.1), can be chosen such that \( \epsilon(\alpha, \beta) = -\epsilon(-\alpha, -\beta), \epsilon(\alpha_1, \alpha_2) = 1, \epsilon(-\alpha_1, \psi) = 1 \) and \( \epsilon(-\alpha_2, \psi) = -1 \). The basis discussed in section 2 is given by

\[
E_{3n+1} = E_1 + E_2 + E_{n+1} \ \\
E_{3n-1} = E_1 - E_2 + E_{n-1} \ \\
F_{3n+1}^1 = E_1 + \gamma E_2 + \gamma^2 E_{n+1} \ \\
F_{3n}^1 = H_1 + \gamma H_2 + \gamma^2 H_6 \ \\
F_{3n-1}^1 = E_1 + \gamma E_2 + \gamma^2 E_{n-1} \ \\
F_{3n+1}^2 = E_1 + \gamma^2 E_2 + \gamma E_{n+1} \ \\
F_{3n}^2 = H_1 + \gamma^2 H_2 + \gamma H_6 \ \\
F_{3n-1}^2 = E_1 + \gamma^2 E_2 + \gamma^2 E_{n-1} \quad (5.18)
\]

where \( H_6 = - H_1 - H_2 + \delta_{n,0} C \), and where all generators are defined in the Chevalley basis and satisfy the commutation relations (5.1). The parameter \( \gamma \) is a cubic root of unity, i.e. \( \gamma^3 = 1 \).

Using the procedure of section 3 one can check that the first two right charge densities are given by

\[
a^R_{+,1} = \frac{1}{3} e^{-f^{(1)} J_R} \left( (\partial_+ \varphi_1)^2 + (\partial_+ \varphi_2)^2 - \partial_+ \varphi_1 \partial_+ \varphi_2 + \partial_+ \eta \partial_+ \varphi_1 + \partial_+ \eta \partial_+ \varphi_2 \right)
\]
\[ a^{-1}_{R,(1)} = 0 \] (5.19)

and
\[ a^{R,(2)}_+ = \frac{1}{3} e^{-2 f^2 + J_R} \bigg( (\partial_+ \varphi_1)^2 \partial_+ \varphi_2 - (\partial_+ \varphi_2)^2 \partial_+ \varphi_1 + \frac{1}{2} \partial_+ \varphi_1 \partial_+^2 \varphi_2 - \frac{1}{2} \partial_+ \varphi_2 \partial_+^2 \varphi_1 \bigg) \]

\[ a^{R,(2)}_- = \frac{1}{6} e^{-2 f^2 + J_R} \left( q_1 \partial_+ \varphi_2 e^{2 \varphi_1 - \varphi_2 - \eta} - q_2 \partial_+ \varphi_1 e^{2 \varphi_2 - \varphi_1 + \eta} + q_0 (\partial_+ \varphi_1 - \partial_+ \varphi_2) e^{-\varphi_1 - \varphi_2 + \eta} \right) \] (5.20)

Notice that \( a^{R,(2)}_+ \) are \( \nu \) independent.

### 6 Discussion

We constructed an infinite number of conserved charges for the Conformal Affine Toda models. These charges are intrinsically related to the Heisenberg subalgebra \( \mathbf{H} \) of the underlying KM algebra. As we have pointed out an interesting aspect of these conservation laws is that, although the CAT model is a conformally invariant model, its symmetries are not described by chiral currents only. The chiral charges we constructed do not possess chiral densities, except of course for the stress tensor components and the spin 1 currents \( \partial_\pm \eta \). Therefore not all the symmetries give rise to a \( W \)-algebra [8, 4, 5].

In ref. [8] we have shown how to obtain the chiral densities of the CAT model via the Hamiltonian reduction as the remaining KM currents of the two-loop WZNW model [3]. The densities were obtained, for each chiral sector, by constraining the KM currents and then by choosing appropriate gauge fixing conditions such that the so called remaining currents depend on the CAT model fields only. However, we have not succeeded in obtaining the whole spectrum of CAT model charges via Hamiltonian reduction. We now comment that this is possible at the level of the conservation laws by considering the constraints on both chiral sectors. Let us consider for instance the right currents. After the constraints are imposed they are given by [8]

\[ J_{\text{red}} = k(\hat{g}^{-1} \partial_+ \hat{g})_{\text{red}} = \mathcal{M}^{-1} (\partial_+ \Phi + E_1) \mathcal{M} + \mathcal{M}^{-1} \partial_+ \mathcal{M} \]

\[ = E_1 + \sum_{M > 0} j_M E_{-M} + \sum_{m > 0} \sum_{n=1}^r j^m_n F^a_{-m} \] (6.1)

where \( \mathcal{M} \) is an exponentiation of the negative step operators of the KM algebra and appear in the Gauss decomposition \( \hat{g} = \mathcal{N} \mathcal{A} \mathcal{M} \) [8]. Notice that (6.1) looks very similar to (3.3). However the quantities entering \( g_R \) in (3.3) are arbitrary parameters whilst those in \( \mathcal{M} \) are WZNW fields. In [8] we have shown that one has to gauge fix the currents \( j^m_n \) and \( j_M \), for \( M > 1 \) in order to eliminate the unwanted WZNW fields in \( \mathcal{M} \). We were left then with only two remaining currents. If one does not gauge fix all the \( j_M \) currents, they will be remaining chiral currents depending on some unwanted fields. However, using the constraints on the left sector one can eliminate such fields when those currents are inserted into the equations.
describing the conservation laws. Such mixture of constraints on both chiral sectors accounts for the non chiral character of the charge densities. Therefore, the Hamiltonian reduction provides the same conservation laws we obtained in this paper, although it does not give directly the expression for the charge densities.

We now make some comments on the relation among the conserved charges for the different hierarchies of Toda models. One notices that by performing global gauge transformations of the type (3.32) one can eliminate the coupling constants from one of the components of the gauge potential. In particular, we have chosen the gauge such that the charges (3.19) and (3.43) are independent of the coupling constants. By taking the limit $q_0 \to 0$ one observes from (2.3) that the $\nu$ field becomes free and the second exponential on the r.h.s. of (2.1) is dropped. Since the charges (3.19) and (3.43) are unchanged under such limit they are also conserved charges of the model obtained by that limit. In particular, by taking the special case where $\eta = 0$, one observes that the models under consideration are the Affine Toda ($q_0 \neq 0$) and Conformal Toda ($q_0 = 0$). Of course, they both carry the extra passive field $\nu$, which does not affect the $\varphi$'s dynamics. Let us remark however that although the charge densities $a_\nu^{R(M)}$ depend explicitly on the field $\nu$ the corresponding charges have been checked to be independent of $\nu$ fields for $M = 1, 2, 3$. It would be interesting to prove whether this is a general feature as suggested by the explicit verifcation. Therefore, if the field $\nu$ really drops from the integrated charges, we have shown that the chiral charges of the AT and CT models are the same. In the case of the CT model we already know that not only the charges but also the densities are chiral. This would then generalize the result already known in the literature about the relation between the AT and CT charges for the case of $Sl(N)$ [16, 17].

Finally, we would like to mention that the corresponding FPR (4.1) for the chiral components of the gauge potentials contains $\delta'$ terms. The involution of the chiral charges therefore is not as clear as that for the charges conserved in time. It would be interesting to verify whether all the light-cone charges share that property.

Acknowledgements We gratefully acknowledge support within CNPq/NSF Cooperative Science Program. One of us (HA) thanks Instituto de Física Teórica-UNESP for kind hospitality.

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