Extended BRST algebra of the Freedman–Townsend model [†]

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ABSTRACT

We construct an explicit nilpotent extended BRST/anti-BRST algebra for the Freedman–Townsend model, through a change of basis of the fields and a corresponding deformation of the algebra of the Abelian 2-form theory. We also discuss the elimination of some auxiliary fields from the gauge-fixed action and make a comparison with the extended Batalin–Vilkovisky formalism.

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1 Introduction

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BRST quantization [1, 2] is one of the most powerful tools of quantum field theory. In recent years, it has been challenged by many models belonging to the class of field theories with *open (reducible)* gauge algebras. Their quantization poses a certain number of difficulties, and, some years ago, Batalin and Vilkovisky [3] proposed a different method (also known as the antifield formalism) to construct a non degenerate quantum action for these theories. Since then a large amount of work has been done on the subject. In particular Batalin, Lavrov and Tyutin [4] furnished an extension of the antifield formalism to the anti-BRST symmetry [5], in order to unify ghost and antighost sectors.

In a previous paper [6] we discussed BRST quantization for some examples of theories with open reducible gauge algebras, such as the Freedman–Townsend model [7] in arbitrary space–time dimensions and its massless (topological) limit.

We obtained the BRST algebra of non-Abelian p-form gauge theory by deforming the algebra of the corresponding Abelian model. The final system of fields required by the algebra contains, besides the classical fields and the ghosts of the geometric sector, also some auxiliary fields, which are necessary to insure nilpotency. The latter can be expressed in terms of objects belonging to the universal enveloping algebra (UEA) of the structure group.

In this paper we show that our previous construction can be generalized to incorporate a nilpotent $s/\bar{s}$ algebra, taking as an example the Freedman–Townsend model in four dimensions. In Sect. 2 we give a brief review of the construction of the pure BRST sector. This allows us to introduce the basic notations and the objects which are relevant to the construction of the extended $s/\bar{s}$ algebra, presented in Sects. 3, 4 and 5. In Sect. 6 we discuss the construction of the quantum action, and the problem of the elimination of auxiliary field. We discuss also the equivalence with the pure BRST quantization. The relation with the extended BV antifield method, introduced in Ref.[4] is presented in Sect. 7. Finally, we draw some conclusions.

2 The BRST algebra of non Abelian p-forms

The non-Abelian p-form theory (in $D=p+2$ space time dimensions) is an example of theory with p-stage reducible gauge symmetry with open gauge algebra [3]. The classical action involves a p-form B_p and a 1-form A both valued in the Lie algebra $Lie(G)$ of a structure group G and is given by

$$S_{cl} = \int_{M_{p+2}} Tr(B_p \wedge F + \frac{m^2}{2} A \wedge *A), \quad (1)$$

where F is the curvature 2-form associated to the connection A :

$$F = dA + \frac{1}{2}[A, A], \quad (2)$$

where $*$ denotes Hodge duality, $\int_M$ integration of forms on the $p+2$ dimensional manifold M_{p+2} , and the brackets denote the graded commutator defined as

$$[A_1, A_2] = A_1 \wedge A_2 - (-)^{deg(A_1)deg(A_2)} A_2 \wedge A_1. \quad (3)$$

The total degree of A , $deg(A)$, is the sum of the form degree and the ghost number. As usual, we will not write explicitly in the following the symbol $\wedge$, in the product of forms.

The action is invariant under the transformations:

$$\delta B_p = D_A \xi_{p-1}, \quad (4)$$

$$\delta A = 0, \quad (5)$$

where ξ_{p-1} is a $p-1$ -form valued in $Lie(G)$ and $D_A = d + [A, \cdot]$ is the covariant derivative.

There exists values of ξ_{p-1} for which (4) vanishes *on-shell*, (i.e. when the equations of motion are used); for instance, taking $\xi_{p-1} = D_A \xi_{p-2}$, eq.(4) gives $\delta B_p = [F(A), \xi_{p-2}]$, which vanishes when the equation of motion of B_p , $F(A) = 0$, holds. Actually, this occurs also for lower form degrees, i.e. $\xi_{p-k} = D_A \xi_{p-k-1}$, $k = 1, \dots, p-1$, showing that the theory is $p-1$ *on-shell reducible*.

The $m^2 = 0$ limit corresponds to the case of BF topological gauge theories, which have an additional Yang-Mills gauge invariance $\delta A = D_A \omega$, $\delta B_p =$

$[\omega, B]$. In four dimensions it reduces to the well known Freedman-Townsend model [7]:

$$S_{cl} = \int_{M_4} Tr(B_2 F + \frac{m^2}{2} A * A), \quad (6)$$

In ref. [6] we constructed a nilpotent algebra suitable for the quantization of this kind of theories. We started from the Abelian model, which has an *off-shell* p-stage reducibility and can be quantized by iterating the BRST gauge-fixing procedure, leading to the introduction of *ghosts for ghosts* B_{p-k}^k , $k = 1, \dots, p-1$. These fields and the classical field B_p^0 can be collected in a *non homogeneous* form

$$B = \sum_{l=0}^p B_{p-l}^l, \quad (7)$$

and the nilpotent BRST algebra can be written

$$(d-s)B = 0. \quad (8)$$

To obtain the nilpotent BRST algebra for the non Abelian theory, we define a new basis of fields, which we call the *tilde basis*, through a kind of "rotation" of the original fields

$$\tilde{B} = \Gamma^{-1} B \Gamma, \quad (9)$$

where the Γ are non homogeneous forms valued in the *universal enveloping algebra* (UEA) of $Lie(G)$. Introducing a non homogeneous form of *auxiliary fields*

$$b = \sum_{l=2}^p b_l^{1-l}, \quad (10)$$

given in terms of Γ by

$$b = \Gamma^{-1}(d-s)\Gamma, \quad (11)$$

the BRST algebra of the non Abelian p-form can be written

$$s\tilde{B} = D_A \tilde{B} + [b, \tilde{B}], \quad (12)$$

$$sb = D_A b + b^2 + F(A) \quad (13)$$

(for Freedman-Townsend model $sA = 0$). The nilpotency of the BRST operator s follows from $(d-s)^2 = 0$.

For topological BF models (which are endowed with a Yang-Mills symmetry), the Faddeev-Popov field and the gauge potential appear as the components of lower form degree of b :

$$b = \omega_0^1 + A_1^0 + \sum_{l=2}^p b_l^{1-l}.$$

The resulting BRST algebra reads

$$s\tilde{B} = d\tilde{B} + [b, \tilde{B}] = D_b\tilde{B}, \quad (14)$$

$$sb = db + b^2. \quad (15)$$

The usual transformation laws for the fields B_k and b_l are obtained by projecting (12) and (13) on definite form degrees. Note that the set of fields b forms a closed subalgebra. The expression (11) can be interpreted as the integrated form of the BRST transformation (13); its relation with the unconstrained formulation of BRST is discussed in ref. [8].

A particularly useful expression for Γ is [6]:

$$\Gamma = ge^X \quad (16)$$

where $X = \sum_{k=1}^p c_k$ is a non-homogeneous form of total degree 0. The product of two such elements is given by

$$(g_1 e^{X_1}) \cdot (g_2 e^{X_2}) = (g_1 g_2) e^{H(X_1^g, X_2)} \quad (17)$$

with $X^g = g^{-1} X g$. $H(X_1^g, X_2)$ is the Campbell-Hausdorff series, which, here, reduces to a finite number of terms, since X_1 and X_2 have at least one component of positive form degree, and the higher form degree is equal to the dimension of the space-time i.e. $p+2$.

3 The construction of the extended BRST/anti-BRST algebra.

The algebra of the BRST and anti-BRST transformations of the free Abelian 2-form theory, is generated by two graded operators s and $\bar{s}$, which, together with the external derivative operator, satisfy the anticommutation relations:

$$s^2 = \bar{s}^2 = d^2 = 0, \quad \{s, \bar{s}\} = \{d, s\} = \{d, \bar{s}\} = 0 \quad (18)$$

In order to deal with non homogeneous forms involving also antighost fields, it is convenient to introduce a *bigrading*. Each field is written as $\phi_p^{(g, \bar{g})}$, where p is the form degree, with the convention that the action of the operators s and $\bar{s}$ on a field raises of one unit the value of g and $\bar{g}$ respectively, i.e. s has degree $(1, 0)$ and $\bar{s}$ $(0, 1)$. The usual ghost number is given by $gh(\phi) = g - \bar{g}$. A non homogeneous form field ϕ has the following expansion

$$\phi = \sum_{p, g, \bar{g}} \phi_p^{(g, \bar{g})} \quad (19)$$

where the total degree $deg = p + g + \bar{g}$ is the same for any component field $\phi_p^{(g, \bar{g})}$.

The fields entering in the quantization of the Abelian two form, i.e. classical field, ghosts, antighosts and Stueckelberg fields, are shown in Fig 1, with their bigrading explicited.

$$\begin{array}{ccccc} & & (0,0) & & \\ & & B_2 & & \\ & (0,1) & (1,1) & (1,0) & \\ & B_1 & \pi_1 & B_1 & \\ (0,2) & (1,2) & (1,1) & (2,1) & (2,0) \\ B_0 & \pi_0 & B_0 & \pi_0 & B_0 \end{array}$$

Fig. 1.

They can be collected in two non homogeneous forms, B and π , of total degree equal to 2 and 3 respectively:

$$B = \sum_{p=0}^2 \sum_{\substack{g \geq 0, \bar{g} \geq 0 \\ p+g+\bar{g}=2}} \phi_p^{(g, \bar{g})}, \quad \pi = \sum_{p=0}^1 \sum_{\substack{g \geq 1, \bar{g} \geq 1 \\ p+g+\bar{g}=3}} \phi_p^{(g, \bar{g})} \quad (20)$$

Note that the components of B and π in Fig.1 or, equivalently, in the expressions (20) and which are the *physical fields* entering in the gauge-fixing, have a restricted set of values of p , g and $\bar{g}$, with respect to the general expansion (19). Nevertheless, we will often ignore such a restriction in the following, since (i) this allows us to write simpler expressions with non homogeneous forms, and (ii) the *non-physical* components so introduced will not be independent fields, since it will always be possible to express them in terms of the physical ones.

Introducing the two nilpotent operators

$$\sigma = s + \bar{s}, \quad \delta = s - \bar{s}; \quad (21)$$

it is immediate to show that the algebra of BRST/anti-BRST transformations of the abelian form can be obtained by projecting in components of definite form degree the equations:

$$(d - \sigma)B = 0, \quad (22)$$

$$\delta B = \pi; \quad (23)$$

from which follows also:

$$(d - \sigma)\pi = 0. \quad (24)$$

Making, for future convenience, the field redefinitions $\frac{1}{2} \begin{smallmatrix} (1,1) \\ B_0 \end{smallmatrix} \rightarrow \begin{smallmatrix} (1,1) \\ B_0 \end{smallmatrix}$ and $-\frac{1}{2}\pi \rightarrow \pi$ the transformations of the physical fields, obtained from (22), (23) and (24) are:

$$\begin{aligned} s \begin{smallmatrix} (0,0) \\ B_2 \end{smallmatrix} &= d \begin{smallmatrix} (1,0) \\ B_1 \end{smallmatrix}, \\ s \begin{smallmatrix} (1,0) \\ B_1 \end{smallmatrix} &= d \begin{smallmatrix} (2,0) \\ B_0 \end{smallmatrix}, \quad s \begin{smallmatrix} (0,1) \\ B_1 \end{smallmatrix} = d \begin{smallmatrix} (1,1) \\ B_0 \end{smallmatrix} - \begin{smallmatrix} (1,1) \\ \pi_1 \end{smallmatrix}, \\ s \begin{smallmatrix} (2,0) \\ B_0 \end{smallmatrix} &= 0, \quad s \begin{smallmatrix} (1,1) \\ B_0 \end{smallmatrix} = -\frac{1}{2} \begin{smallmatrix} (2,1) \\ \pi_0 \end{smallmatrix}, \quad s \begin{smallmatrix} (0,2) \\ B_0 \end{smallmatrix} = -\begin{smallmatrix} (1,2) \\ \pi_0 \end{smallmatrix}, \\ \bar{s} \begin{smallmatrix} (0,0) \\ B_2 \end{smallmatrix} &= d \begin{smallmatrix} (0,1) \\ B_1 \end{smallmatrix}, \\ \bar{s} \begin{smallmatrix} (0,1) \\ B_1 \end{smallmatrix} &= d \begin{smallmatrix} (0,2) \\ B_0 \end{smallmatrix}, \quad \bar{s} \begin{smallmatrix} (1,0) \\ B_1 \end{smallmatrix} = d \begin{smallmatrix} (1,1) \\ B_0 \end{smallmatrix} + \begin{smallmatrix} (1,1) \\ \pi_1 \end{smallmatrix}, \\ \bar{s} \begin{smallmatrix} (0,2) \\ B_0 \end{smallmatrix} &= 0, \quad \bar{s} \begin{smallmatrix} (1,1) \\ B_0 \end{smallmatrix} = \frac{1}{2} \begin{smallmatrix} (1,2) \\ \pi_0 \end{smallmatrix}, \quad \bar{s} \begin{smallmatrix} (2,0) \\ B_0 \end{smallmatrix} = \begin{smallmatrix} (2,1) \\ \pi_0 \end{smallmatrix}, \\ s \begin{smallmatrix} (1,1) \\ \pi_1 \end{smallmatrix} &= \frac{1}{2} d \begin{smallmatrix} (2,1) \\ \pi_0 \end{smallmatrix}, \quad s \begin{smallmatrix} (2,1) \\ \pi_0 \end{smallmatrix} = 0, \quad s \begin{smallmatrix} (1,2) \\ \pi_0 \end{smallmatrix} = 0, \\ \bar{s} \begin{smallmatrix} (1,1) \\ \pi_1 \end{smallmatrix} &= \frac{1}{2} d \begin{smallmatrix} (1,2) \\ \pi_0 \end{smallmatrix}, \quad \bar{s} \begin{smallmatrix} (2,1) \\ \pi_0 \end{smallmatrix} = 0, \quad \bar{s} \begin{smallmatrix} (1,2) \\ \pi_0 \end{smallmatrix} = 0. \end{aligned} \quad (25)$$

Eqs. (22), (23) and (24) contain also the set of relations which express the non physical components as derivatives of physical ones; for instance: $\begin{smallmatrix} (1,0) \\ \pi_2 \end{smallmatrix} = \begin{smallmatrix} (0,0) \\ s B_2 \end{smallmatrix} = d \begin{smallmatrix} (1,0) \\ B_1 \end{smallmatrix}$.

The extension of our previous construction to the BRST/anti-BRST algebra is immediate through the introduction of bigraded Γ objects, having the following expansion

$$\Gamma = \sum_{\substack{p \geq 0, q, \bar{q} \leq 0 \\ p+q+\bar{q}=0}} \begin{smallmatrix} (g, \bar{g}) \\ \Gamma_p \end{smallmatrix}. \quad (26)$$

The simplest parametrization of Γ is given in terms of an element g of G and two 1-forms $\begin{smallmatrix} (-1,0) \\ c_1 \end{smallmatrix}$ and $\begin{smallmatrix} (0,-1) \\ c_1 \end{smallmatrix}$:

$$\Gamma = g e^Y \quad (27)$$

where $Y = \frac{1}{2} \left(\begin{smallmatrix} (-1,0) \\ c_1 \end{smallmatrix} + \begin{smallmatrix} (0,-1) \\ c_1 \end{smallmatrix} \right)$ and $sg = \bar{s}g = 0$.

As for the BRST case [6], we introduce a new basis of “rotated” fields, $\tilde{B}$ and $\tilde{\pi}$ given by

$$\tilde{B} = \Gamma^{-1} B \Gamma, \quad \tilde{\pi} = \Gamma^{-1} \pi \Gamma. \quad (28)$$

Substituting into eqs.(22) and (23), one obtains

$$(d - \sigma)\tilde{B} + [\Gamma^{-1}(d - \sigma)\Gamma, \tilde{B}] = 0, \quad (29)$$

$$\delta \tilde{B} + [\Gamma^{-1}\delta\Gamma, \tilde{B}] = \tilde{\pi}. \quad (30)$$

The definition of the non homogeneous form built with the *auxiliary fields* is similar to that of the BRST case [6]:

$$A = g^{-1} dg - \sigma Y, \quad (31)$$

$$\beta = \Gamma^{-1}(d - \sigma)\Gamma - A.$$

Note that $\beta_1 = 0$.

One recovers the spectrum of the auxiliary fields b of the pure BRST algebra by taking $\begin{smallmatrix} (0,-1) \\ c_1 \end{smallmatrix} = 0$ and $\bar{s} \begin{smallmatrix} (-1,0) \\ c_1 \end{smallmatrix} = 0$. Symmetrically, the pure anti-BRST algebra is obtained in the limit $\begin{smallmatrix} (-1,0) \\ c_1 \end{smallmatrix} = 0$, if $s \begin{smallmatrix} (0,-1) \\ c_1 \end{smallmatrix} = 0$. We must then have

$$\delta Y = 0, \quad (32)$$

which enforces $\delta\Gamma = 0$ and $\delta\beta = 0$.

From the nilpotency of $(d - \sigma)$ and from (31), the following transformation for β results

$$\sigma\beta = D_A\beta + \beta^2 + F(A). \quad (33)$$

The transformations of $\tilde{B}$ can then be written

$$\sigma \tilde{B} = D_A \tilde{B} + [\beta, \tilde{B}], \quad \delta \tilde{B} = \tilde{\pi}. \quad (34)$$

Applying σ and δ respectively to the second of these equations, one gets

$$\sigma \tilde{\pi} = D_A \tilde{\pi} + [\beta, \tilde{\pi}], \quad \delta \tilde{\pi} = 0. \quad (35)$$

The relation $\delta \beta = 0$ is solved by

$$\beta = \delta \hat{\beta}. \quad (36)$$

where $\hat{\beta}$ is some non homogeneous form of zero degree. In terms of the explicit parametrization (27), eq.(36) means to solve $\delta Y = 0$ through $Y = -\frac{1}{2}\delta X$, i.e. $\begin{pmatrix} -1,0 \\ c_1 \end{pmatrix} = \bar{s} \begin{pmatrix} -1,-1 \\ X_1 \end{pmatrix}, \quad \begin{pmatrix} 0,-1 \\ c_1 \end{pmatrix} = -s \begin{pmatrix} -1,-1 \\ X_1 \end{pmatrix}.$

4 The transformations of the $\hat{\beta}$ fields.

It is somewhat involved to express $\hat{\beta}$ in terms of X ; we make use of a transgression formula. For this purpose, we introduce a family of Γ objects, depending on one parameter $t \in [0, 1]$, in the following way:

$$\Gamma(t) = ge^{tY}. \quad (37)$$

From this expression and using eq. (31) one obtains as the component of form degree 1:

$$A(t) = g^{-1}dg - t\sigma Y = A + (1-t)\sigma Y, \quad (38)$$

such that $A(1)$ coincides with A and $A(0)$ is a pure gauge. More generally, one obtains a family of t -dependent auxiliary fields given by

$$\beta(t) = \Gamma^{-1}(t)(d - \sigma)\Gamma(t) - A(t) = \delta \hat{\beta}(t), \quad (39)$$

Now, consider the auxiliary fields which have form degree $p > 1$. Expressing (39) by means of the Campbell-Hausdorff formula, one gets

$$\beta_p(t) = (-1)^p \left[\frac{t^{p-1}}{(p-1)!} Ad_Y^{p-2}(D_{g^{-1}dg} Y) + \frac{t^p}{p!} Ad_Y^{p-1}(\sigma Y) \right], \quad (40)$$

where $Ad_Y^p(X) = \underbrace{[Y, [Y, \dots [Y, X]]}_{p \text{ times}}$, and taking the derivative with respect to t ,

one obtains

$$\begin{aligned} \frac{d\beta_p}{dt} = \dot{\beta}_p(t) &= (-1)^p \left[\frac{t^{p-2}}{(p-2)!} Ad_Y^{p-2}(D_{g^{-1}dg} Y) + \frac{t^{p-1}}{(p-1)!} Ad_Y^{p-1}(\sigma Y) \right] = \\ &= \frac{1}{2} (-1)^p \delta \left[\frac{t^{p-2}}{(p-2)!} Ad_Y^{p-2}(D_{g^{-1}dg} X) + \frac{t^{p-1}}{(p-1)!} Ad_Y^{p-1}(\sigma X) \right] = \delta \Xi_p \end{aligned} \quad (41)$$

where we used the fact that $\delta Y = 0$ and $\delta g = 0$. The Ξ_p can be collected into a non homogeneous form

$$\Xi(t) = \frac{1}{2} \Gamma^{-1}(t) [(d - \sigma)X_g] \Gamma(t) + \frac{1}{2} \sigma X, \quad (42)$$

obeying $\delta \Xi(t) = \dot{\beta}(t)$, and where $X_g = gXg^{-1}$ and $Y_g = gYg^{-1}$ are related by $Y_g = -\frac{1}{2}\delta X_g$. Evidently $\hat{\beta}$ is defined by the previous relation only modulo a δ -exact term and some integration constant. The simplest choice compatible with the condition $\hat{\beta}_1 = 0$ is

$$\hat{\beta}(t) = \frac{1}{2} \int_0^t \Gamma^{-1}(\tau) [(d - \sigma)X_g] \Gamma(\tau) d\tau + \frac{1}{2} t \sigma X. \quad (43)$$

which finally expresses the $\hat{\beta}$ in terms of the particular parametrization of Γ . The transformation law with respect to σ can be obtained by applying the nilpotent operator $(d - \sigma)$ to the expression (43). One can then easily prove that:

$$\sigma \hat{\beta} = D_A \hat{\beta} + \frac{1}{2} [\beta, \hat{\beta}] + f + \delta R \quad (44)$$

where

$$f = -\frac{1}{2} D_A \sigma X - \frac{1}{4} [\sigma Y, \sigma X], \quad R = \frac{1}{2} \int_0^1 [\sigma X - \hat{\beta}(\tau), \hat{\beta}(\tau)] d\tau. \quad (45)$$

Note that $\delta f = -F(A)$. Notice that the transformation law of $\hat{\beta}$ is not algebraic; this is the price one has to pay in order to express everything in terms of non homogeneous forms. Evidently, from eqs. (43) and (44) we can find $\hat{\beta}_p$ for any p and calculate explicitly the transformation laws $\sigma \hat{\beta}_p$, which are indeed algebraic.

For the case $p = 2$, eq. (44) reduces to $\sigma \hat{\beta} = 0$ i.e.

$$s \begin{pmatrix} -1,-1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} 0,-1 \\ \beta_2 \end{pmatrix}, \quad \bar{s} \begin{pmatrix} -1,-1 \\ \hat{\beta}_2 \end{pmatrix} = - \begin{pmatrix} -1,0 \\ \beta_2 \end{pmatrix}. \quad (46)$$

For the case $p = 3$, R_3 is easily calculated and gives

$$R_3 = \begin{pmatrix} (-1,-2) \\ \beta_3 \end{pmatrix} - \begin{pmatrix} (-2,-1) \\ \beta_3 \end{pmatrix} \quad (47)$$

The transformation laws then read

$$\begin{aligned} s \begin{pmatrix} (-2,-1) \\ \hat{\beta}_3 \end{pmatrix} &= \frac{1}{4}(D_A \begin{pmatrix} (-1,-1) \\ \hat{\beta}_2 \end{pmatrix} + 2 \begin{pmatrix} (-1,-1) \\ \beta_3 \end{pmatrix}), \quad \bar{s} \begin{pmatrix} (-2,-1) \\ \hat{\beta}_3 \end{pmatrix} = - \begin{pmatrix} (-2,0) \\ \beta_3 \end{pmatrix} \\ \bar{s} \begin{pmatrix} (-1,-2) \\ \hat{\beta}_3 \end{pmatrix} &= \frac{1}{4}(D_A \begin{pmatrix} (-1,-1) \\ \hat{\beta}_2 \end{pmatrix} - 2 \begin{pmatrix} (-1,-1) \\ \beta_3 \end{pmatrix}), \quad s \begin{pmatrix} (-1,-2) \\ \hat{\beta}_3 \end{pmatrix} = \begin{pmatrix} (0,-2) \\ \beta_3 \end{pmatrix} \end{aligned} \quad (48)$$

In the case of $p = 4$, the relation (45) gives for R_4 a non trivial expression, since the term containing $\hat{\beta}(t)$ contributes for the first time. Nevertheless, some algebraic identities conspire to give even to R_4 the simple form

$$R_4 = \begin{pmatrix} (-1,-3) \\ \hat{\beta}_4 \end{pmatrix} - \begin{pmatrix} (-3,-1) \\ \hat{\beta}_4 \end{pmatrix}, \quad (49)$$

One finally gets

$$\begin{aligned} s \begin{pmatrix} (-1,-3) \\ \hat{\beta}_4 \end{pmatrix} &= \begin{pmatrix} (0,-3) \\ \beta_4 \end{pmatrix}, \quad \bar{s} \begin{pmatrix} (-1,-3) \\ \hat{\beta}_4 \end{pmatrix} = \frac{1}{3}(D_A \begin{pmatrix} (-1,-2) \\ \hat{\beta}_3 \end{pmatrix} + \frac{1}{2}[\begin{pmatrix} (0,-1) \\ \beta_2 \end{pmatrix}, \begin{pmatrix} (-1,-1) \\ \hat{\beta}_2 \end{pmatrix}] - \begin{pmatrix} (-1,-2) \\ \beta_4 \end{pmatrix}) \\ s \begin{pmatrix} (-2,-2) \\ \hat{\beta}_4 \end{pmatrix} &= \frac{1}{3}(D_A \begin{pmatrix} (-1,-2) \\ \hat{\beta}_3 \end{pmatrix} + \frac{1}{2}[\begin{pmatrix} (0,-1) \\ \beta_2 \end{pmatrix}, \begin{pmatrix} (-1,-1) \\ \hat{\beta}_2 \end{pmatrix}] + 2 \begin{pmatrix} (-1,-2) \\ \beta_4 \end{pmatrix}) \\ \bar{s} \begin{pmatrix} (-2,-2) \\ \hat{\beta}_4 \end{pmatrix} &= \frac{1}{3}(D_A \begin{pmatrix} (-2,-1) \\ \hat{\beta}_3 \end{pmatrix} + \frac{1}{2}[\begin{pmatrix} (-1,0) \\ \beta_2 \end{pmatrix}, \begin{pmatrix} (-1,-1) \\ \hat{\beta}_2 \end{pmatrix}] - 2 \begin{pmatrix} (-2,-1) \\ \beta_4 \end{pmatrix}) \\ s \begin{pmatrix} (-3,-1) \\ \hat{\beta}_4 \end{pmatrix} &= \frac{1}{3}(D_A \begin{pmatrix} (-2,-1) \\ \hat{\beta}_3 \end{pmatrix} + \frac{1}{2}[\begin{pmatrix} (-1,0) \\ \beta_2 \end{pmatrix}, \begin{pmatrix} (-1,-1) \\ \hat{\beta}_2 \end{pmatrix}] + \begin{pmatrix} (-2,-1) \\ \beta_4 \end{pmatrix}), \quad \bar{s} \begin{pmatrix} (-3,-1) \\ \hat{\beta}_4 \end{pmatrix} = - \begin{pmatrix} (-3,0) \\ \beta_4 \end{pmatrix} \end{aligned} \quad (50)$$

These relations complete the set of BRST/anti-BRST transformations of the $\hat{\beta}$ fields, which appear in four dimension.

5 The transformations of the non Abelian B and π fields.

From the formula (28), it is clear that, even if B and π contain as non vanishing components only the physical fields shown in Fig. 1, the *rotation* to the new *tilde basis* produces non vanishing components of the non homogeneous forms $\tilde{B}$ and $\tilde{\pi}$ with a wider range of values for g and/or $\bar{g}$. In four dimensions,

these new *non physical* fields are: the 1-forms $\begin{pmatrix} (-1,2) \\ \tilde{B}_1 \end{pmatrix}$, $\begin{pmatrix} (-1,1) \\ \tilde{\pi}_1 \end{pmatrix}$, $\begin{pmatrix} (1,-1) \\ \tilde{\pi}_1 \end{pmatrix}$, $\begin{pmatrix} (2,-1) \\ \tilde{B}_1 \end{pmatrix}$, and the 2-forms $\begin{pmatrix} (-2,2) \\ \tilde{B}_2 \end{pmatrix}$, $\begin{pmatrix} (-2,1) \\ \tilde{\pi}_2 \end{pmatrix}$, $\begin{pmatrix} (-1,1) \\ \tilde{B}_2 \end{pmatrix}$, $\begin{pmatrix} (-1,0) \\ \tilde{\pi}_2 \end{pmatrix}$, $\begin{pmatrix} (0,-1) \\ \tilde{\pi}_2 \end{pmatrix}$, $\begin{pmatrix} (1,-1) \\ \tilde{B}_2 \end{pmatrix}$, $\begin{pmatrix} (1,-2) \\ \tilde{\pi}_2 \end{pmatrix}$, and $\begin{pmatrix} (2,-2) \\ \tilde{B}_2 \end{pmatrix}$. When eqs. (34) are projected on definite form degree, one sees that both *physical* and *non physical* fields appear in the transformations. Nevertheless, as in the Abelian case, *non physical* fields can be decoupled from the transformations, but here a non trivial redefinition of the physical fields is required. The key to this mechanism, whose details are given in the Appendix, is the possibility to write the non physical fields in terms of $s, \bar{s}$ variations and external derivatives of some fields $\mathcal{Z}$, which can then be absorbed in a redefinition of the basis of the physical field. In this new basis of fields, which, to simplify the notation we denote with B and π , one obtains a complete nilpotent algebra of BRST/anti-BRST transformation, which can be used to construct a quantum action for the Freedman-Townsend model. We write them in terms of the usual fields $b_p = 2^{p-1}\beta_p$, $\hat{b}_p = 2^{p-1}\hat{\beta}_p$ and make the previous redefinitions $\frac{1}{2} \begin{pmatrix} (1,1) \\ B_0 \end{pmatrix} \rightarrow \begin{pmatrix} (1,1) \\ B_0 \end{pmatrix}$ and $-\frac{1}{2}\pi \rightarrow \pi$. Thus one has:

$$\begin{aligned} s \begin{pmatrix} (0,0) \\ B_2 \end{pmatrix} &= D_A \begin{pmatrix} (1,0) \\ B_1 \end{pmatrix} + [\begin{pmatrix} (-1,0) \\ b_2 \end{pmatrix}, \begin{pmatrix} (2,0) \\ B_0 \end{pmatrix}] + [\begin{pmatrix} (0,-1) \\ b_2 \end{pmatrix}, \begin{pmatrix} (1,1) \\ B_0 \end{pmatrix}] - \frac{1}{2}[\begin{pmatrix} (-1,-1) \\ \hat{b}_2 \end{pmatrix}, \begin{pmatrix} (2,1) \\ \pi_0 \end{pmatrix}], \\ s \begin{pmatrix} (1,0) \\ B_1 \end{pmatrix} &= D_A \begin{pmatrix} (2,0) \\ B_0 \end{pmatrix}, \quad s \begin{pmatrix} (0,1) \\ B_1 \end{pmatrix} = D_A \begin{pmatrix} (1,1) \\ B_0 \end{pmatrix} - \pi_1, \\ s \begin{pmatrix} (2,0) \\ B_0 \end{pmatrix} &= 0, \quad s \begin{pmatrix} (1,1) \\ B_0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} (2,1) \\ \pi_0 \end{pmatrix}, \quad s \begin{pmatrix} (0,2) \\ B_0 \end{pmatrix} = - \begin{pmatrix} (1,2) \\ \pi_0 \end{pmatrix}, \\ s \begin{pmatrix} (1,1) \\ \pi_1 \end{pmatrix} &= \frac{1}{2} D_A \begin{pmatrix} (2,1) \\ \pi_0 \end{pmatrix}, \quad s \begin{pmatrix} (2,1) \\ \pi_0 \end{pmatrix} = 0, \quad s \begin{pmatrix} (1,2) \\ \pi_0 \end{pmatrix} = 0, \end{aligned} \quad (51)$$

$$\begin{aligned} \bar{s} \begin{pmatrix} (0,0) \\ B_2 \end{pmatrix} &= D_A \begin{pmatrix} (0,1) \\ B_1 \end{pmatrix} + [\begin{pmatrix} (0,-1) \\ b_2 \end{pmatrix}, \begin{pmatrix} (0,2) \\ B_0 \end{pmatrix}] + [\begin{pmatrix} (-1,0) \\ b_2 \end{pmatrix}, \begin{pmatrix} (1,1) \\ B_0 \end{pmatrix}] - \frac{1}{2}[\begin{pmatrix} (-1,-1) \\ \hat{b}_2 \end{pmatrix}, \begin{pmatrix} (1,2) \\ \pi_0 \end{pmatrix}], \\ \bar{s} \begin{pmatrix} (1,0) \\ B_1 \end{pmatrix} &= D_A \begin{pmatrix} (1,1) \\ B_0 \end{pmatrix} + \pi_1, \quad \bar{s} \begin{pmatrix} (0,1) \\ B_1 \end{pmatrix} = D_A \begin{pmatrix} (0,2) \\ B_0 \end{pmatrix}, \\ \bar{s} \begin{pmatrix} (2,0) \\ B_0 \end{pmatrix} &= \pi_0, \quad \bar{s} \begin{pmatrix} (1,1) \\ B_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1,2) \\ \pi_0 \end{pmatrix}, \quad \bar{s} \begin{pmatrix} (0,2) \\ B_0 \end{pmatrix} = 0, \\ \bar{s} \begin{pmatrix} (1,1) \\ \pi_1 \end{pmatrix} &= \frac{1}{2} D_A \begin{pmatrix} (1,2) \\ \pi_0 \end{pmatrix}, \quad \bar{s} \begin{pmatrix} (2,1) \\ \pi_0 \end{pmatrix} = 0, \quad \bar{s} \begin{pmatrix} (1,2) \\ \pi_0 \end{pmatrix} = 0. \end{aligned}$$

The algebra of the auxiliary fields is given by:

$$\begin{aligned}
s \hat{b}_2^{(0,-1)} &= 0, \quad \bar{s} \hat{b}_2^{(0,-1)} = s \hat{b}_2^{(-1,0)} = F(A), \quad \bar{s} \hat{b}_2^{(-1,0)} = 0, \\
s \hat{b}_3^{(0,-2)} &= 0, \quad \bar{s} \hat{b}_3^{(0,-2)} = s \hat{b}_3^{(-1,-1)} = D_A \hat{b}_2^{(0,-1)}, \\
\bar{s} \hat{b}_3^{(-1,-1)} &= s \hat{b}_3^{(-2,0)} = D_A \hat{b}_2^{(-1,0)}, \quad \bar{s} \hat{b}_3^{(-2,0)} = 0, \\
s \hat{b}_4^{(0,-3)} &= 0, \quad \bar{s} \hat{b}_4^{(0,-3)} = s \hat{b}_4^{(-1,-2)} = D_A \hat{b}_3^{(0,-2)} + \frac{1}{2} [\hat{b}_2^{(0,-1)}, \hat{b}_2^{(0,-1)}], \\
\bar{s} \hat{b}_4^{(-1,-2)} &= s \hat{b}_4^{(-2,-1)} = D_A \hat{b}_3^{(-1,-1)} + [\hat{b}_2^{(0,-1)}, \hat{b}_2^{(0,-1)}], \\
s \hat{b}_4^{(-2,-1)} &= 0, \quad \bar{s} \hat{b}_4^{(-2,-1)} = s \hat{b}_4^{(-3,0)} = D_A \hat{b}_3^{(-2,0)} + [\hat{b}_2^{(0,-1)}, \hat{b}_2^{(0,-1)}],
\end{aligned} \tag{52}$$

and moreover one has:

$$\begin{aligned}
s \hat{b}_2^{(-1,-1)} &= \hat{b}_2^{(0,-1)}, \quad \bar{s} \hat{b}_2^{(-1,-1)} = -\hat{b}_2^{(-1,0)}, \\
s \hat{b}_3^{(-1,-2)} &= \hat{b}_3^{(0,-2)}, \quad \bar{s} \hat{b}_3^{(-1,-2)} = \frac{1}{2} (D_A \hat{b}_2^{(-1,-1)} - \hat{b}_3^{(-1,-1)}), \\
s \hat{b}_3^{(-2,-1)} &= \frac{1}{2} (D_A \hat{b}_2^{(-1,-1)} + \hat{b}_3^{(-1,-1)}), \quad \bar{s} \hat{b}_3^{(-2,-1)} = -\hat{b}_3^{(-2,0)}, \\
s \hat{b}_4^{(-1,-3)} &= \hat{b}_4^{(0,-3)}, \\
\bar{s} \hat{b}_4^{(-1,-3)} &= \frac{2}{3} (D_A \hat{b}_3^{(-1,-2)} + \frac{1}{2} [\hat{b}_2^{(0,-1)}, \hat{b}_2^{(-1,-1)}] - \hat{b}_4^{(-1,-2)}), \\
\bar{s} \hat{b}_4^{(-2,-2)} &= \frac{2}{3} (D_A \hat{b}_3^{(-2,-1)} + \frac{1}{2} [\hat{b}_2^{(0,-1)}, \hat{b}_2^{(-1,-1)}] - \hat{b}_4^{(-2,-1)}), \\
s \hat{b}_4^{(-2,-2)} &= \frac{2}{3} (D_A \hat{b}_3^{(-1,-2)} + \frac{1}{2} [\hat{b}_2^{(0,-1)}, \hat{b}_2^{(-1,-1)}] + \hat{b}_4^{(-1,-2)}), \\
\bar{s} \hat{b}_4^{(-3,-1)} &= \frac{2}{3} (D_A \hat{b}_3^{(-2,-1)} + \frac{1}{2} [\hat{b}_2^{(0,-1)}, \hat{b}_2^{(-1,-1)}] + \hat{b}_4^{(-2,-1)}), \\
\bar{s} \hat{b}_4^{(-3,-1)} &= -\hat{b}_4^{(-3,0)}.
\end{aligned} \tag{53}$$

6 Construction of the quantum action.

The quantum action is written as the sum of an invariant action plus a gauge-fixing term¹, i.e. $S_{inv} + S_{g.f.} = S_{inv} + s\bar{s} \int \Omega$. Thus, the first step is the construction of an $s, \bar{s}$ invariant action, since the classical action is no more invariant under the BRST/anti-BRST transformations (51), (52) and (53). Assuming the triviality of the $s/\bar{s}$ cohomology on forms, a rapid inspection of the degree shows that

$$S_{inv} = \int \{ s\bar{s} \Lambda + \frac{1}{2} Tr (\Lambda * \Lambda) \}, \tag{54}$$

where

$$\Lambda = Tr (\hat{\beta} B) = Tr \left(\sum_{\substack{p=0, k+k=p \\ k \geq 0, k \geq 0}}^2 \hat{b}_{p+2}^{(-k-1, -k-1)} B_{p-2}^{(k, k)} \right) \tag{55}$$

In fact, since $s^2 = \bar{s}^2 = \{s, \bar{s}\} = 0$, S_{inv} satisfies the invariance requirements, and, moreover, $S_{inv}|_{\beta=\hat{\beta}=0}$ coincides with the classical action:

On the other hand, the gauge fixing term cannot be expressed in terms of non homogeneous forms. A simple choice, which, in fact, leads to a popular gauge fixing, is

$$\Omega = Tr \left\{ \frac{1}{2} B_2^{(0,0)} * B_2^{(0,0)} + B_1^{(0,1)} * B_1^{(1,0)} + \frac{1}{2} B_0^{(1,1)} * B_0^{(1,1)} + B_0^{(0,2)} * B_0^{(2,0)} \right\}. \tag{56}$$

Before giving explicitly the resulting S_{eff} , let us slightly simplify the notation by writing a generic field $\phi_p^{g,\bar{g}}$ as ϕ_p^{gh} , where $gh = g - \bar{g}$ is the ghost number; obviously the correspondence is in general not unique, however no ambiguity arises in four dimension, for the fields which appear in S_{eff} . In fact, using the

¹This is the action, which, supplemented with terms containing external sources, has to be employed in the construction of generating functionals. The notation S_{eff} is introduced to facilitate the comparison with results of ref.[4].

explicit form of the s and $\bar{s}$ transformations

$$\begin{aligned}
S_{eff} = & \int Tr \{ B_2^0 F(A) + \frac{1}{2} A * A \\
& - \frac{1}{2} [b_2^{-1}, b_2^{-1}] B_0^2 - [b_2^{-1}, b_2^1] B_0^0 - \frac{1}{2} [b_2^1, b_2^1] B_0^{-2} - \frac{1}{2} [\hat{\beta}_2^0, b_2^{-1}] \pi_0^1 - \frac{1}{2} [\hat{\beta}_2^0, b_2^1] \pi_0^{-1} \\
& + (D_A B_1^1 + [b_2^{-1}, B_0^2] + [b_2^1, B_0^0] - \frac{1}{2} [\hat{\beta}_2^0, \pi_0^1]) * (D_A B_1^{-1} + [b_2^1, B_0^{-2}] + [b_2^{-1}, B_0^0] - \frac{1}{2} [\hat{\beta}_2^0, \pi_0^{-1}]) \\
& + B_2^0 * (D_A \pi_1^0 + \frac{1}{2} [b_2^{-1}, \pi_0^1] + [b_2^1, \pi_0^{-1}]) + D_A B_0^{-2} * D_A B_0^2 - D_A B_0^0 * D_A B_0^0 \\
& + D_A \pi_0^{-1} * B_1^1 + B_1^{-1} * D_A \pi_0^1 + \pi_1^0 * \pi_1^0 - \frac{1}{4} \pi_0^{-1} * \pi_0^1 \}
\end{aligned} \tag{57}$$

The equations of motion for b_2^{-1} , b_2^1 and $\hat{\beta}_2^0$, imply

$$b_2^{-1} = - * \bar{s} B_2^0, \quad b_2^1 = * s B_2^0, \quad [\pi_0^1, * B_2^0 - \hat{\beta}_2^0] = 0. \tag{58}$$

(reproducing the gauge-fixing prescription of ref.[4], which we will discuss in the next section).

Elimination of b_2^{-1} and b_2^1 , by using eqs.(58), presents however some awkwardness. The equations above imply

$$\begin{aligned}
b_2^{-1} = & - * (D_A B_1^{-1} + [b_2^1, B_0^{-2}] + [b_2^{-1}, B_0^0] - \frac{1}{2} [\hat{\beta}_2^0, \pi_0^{-1}]) \\
b_2^1 = & * (D_A B_1^1 + [b_2^{-1}, B_0^2] + [b_2^1, B_0^0] - \frac{1}{2} [\hat{\beta}_2^0, \pi_0^1])
\end{aligned} \tag{59}$$

The solution of these equations has an infinity of terms, which replaced into the action would rend it *non-local*. This can be seen without much effort, by looking, for instance, to terms containing B_0^0 and setting B_0^2 , B_0^{-2} , π_0^1 and π_0^{-1} equal to zero. In this limit, the eqs. (59) decouple

$$\begin{aligned}
b_2^{-1} = & - * (D_A B_1^{-1} + [b_2^{-1}, B_0^0]) \\
b_2^1 = & * (D_A B_1^1 + [b_2^1, B_0^0])
\end{aligned} \tag{60}$$

A direct iteration gives

$$b_2^{\pm 1} = \pm \sum_{k=0}^{\infty} \{ * A d_{B_0^0}^{2k} (D_A B_1^{\pm 1}) - A d_{B_0^0}^{2k+1} (D_A B_1^{\pm 1}) \}, \tag{61}$$

Replacing these expressions into the action, one would get the terms

$$S_{non.lin} = - \int \sum_{k=0}^{\infty} \{ A d_{B_0^0}^{2k+1} (D_A B_1^{-1}) - * A d_{B_0^0}^{2k+2} (D_A B_1^{-1}) \} D_A B_1^1. \tag{62}$$

As these terms cannot be cancelled by others coming from the complete solution (59), we conclude that S_{eff} is non local. Nevertheless, we can establish the equivalence between this action and the standard local action arising from s -formalism alone [6, 3]. The equivalence is obtained after reduction to the pure s invariant action, setting $b_2^{-1} = \hat{\beta}_2^0 = 0$. The result is equivalent to the usual BRST gauge fixed action

$$S = S'_{inv} + s \int Tr \Psi, \tag{63}$$

where $S'_{inv} = S_{inv}|_{b_2^{\pm 1} = \hat{\beta}_2^0 = 0}$, and

$$\Psi = \Psi_0 + \frac{1}{2} B_2^0 \int_0^{\infty} \{ (D_{A(t)} * D_{A(t)}) B_1^{-1}(t) - (D_{A(-t)} * D_{A(-t)}) B_1^{-1}(-t) \} e^{-t} dt, \tag{64}$$

$$B_1^{-1}(t) = e^{-t B_0^0} B_1^{-1} e^{t B_0^0}, \quad A(t) = e^{-t B_0^0} A e^{t B_0^0} + t B_0^0. \tag{65}$$

In eq.(64), Ψ_0 is the usual gauge-fixing fermion

$$\Psi_0 = -(B_1^{-1} D_A * B_2^0 + B_0^{-2} D_A * B_1^1 + B_0^0 D_A * B_1^{-1}). \tag{66}$$

As physical quantities do not depend on the choice of Ψ , in what concerns on-shell matrix elements, the reduced theory obtained from S_{eff} in eq.(57) and the standard BRST theory with gauge-fixing fermion Ψ_0 , are equivalent.

7 The relation with the extended Batalin–Vilkovisky formalism.

In Ref.[6] the auxiliary fields necessary to construct a nilpotent BRST algebra, were identified with the *antifields* of the Batalin–Vilkovisky scheme. The extension of this identification to the $s/\bar{s}$ algebra can be easily done. However the algebra obtained at the end of Sec. 5 is not entirely convenient to build the extended BV action; in fact, the number of the antifields of the extended BV formalism, is greater than the number of the auxiliary fields present in our algebra. As it is well known, in the extended BV formalism developed in [4], to each field Φ_p are associated *three* antifields ²: $\Phi_{n-p}^{(g,\bar{g})}$, $\Phi_{n-p}^{(-g,-\bar{g}-1)}$ and $\Phi_{n-p}^{(-g-1,-\bar{g})}$

²It is convenient to assign the form degree $n-p$ to the antifields, where p is the form degree of the corresponding field and n is the dimension of the space.

$(-g-1, -\bar{g}-1)$

$\hat{\Phi}_{n-p}^*$. In theories with closed gauge algebra, the antifields appear in the quantum action only in linear terms, playing the role of the sources of the corresponding s , $\bar{s}$ and $\frac{1}{2}[s, \bar{s}]$ transformations of the fields. In our algebra (51), (52), (53), we have no antifield corresponding to the π fields and, moreover, the number of b fields is less than the number of sources of the s and $\bar{s}$ transformations of the B . For instance, only $\beta_3^{(-1,-1)}$ has the right bigrading assignments to play the role of the source coupled to $s B_1^{(0,-1)}$ or to $\bar{s} B_1^{(-1,0)}$. Another difference comes from the transformation of the fields $\hat{\beta}$, as given in eq. (53), with respect to the prescription of [4].

A hint toward the solution of this discrepancy comes from the observation that the BV antifields corresponding to the fields π have the same bi-grading of some of the *hat-fields*: for instance, the two antifields $\pi_3^{(-2,-1)}$ and $\pi_3^{(-1,-2)}$ than the field $\pi_1^{(1,1)}$ have the same bigrading of the fields $\beta_3^{(-2,-1)}$ and $\beta_3^{(-1,-2)}$. In fact, there is an obvious freedom to define a new field $\hat{\beta}'$ as

$$\hat{\beta}' = \hat{\beta} + \pi^*, \quad (67)$$

where π^* is a new non homogeneous form satisfying $\delta\pi^* = 0$, so that $\delta\hat{\beta}' = \delta\hat{\beta} = \beta$. This possibility can be exploited to recover the extended BV algebra from the one given in eq.(51), (52), (53).

Let us first write the algebra of the transformations of the auxiliary fields in the form

$$s\beta = \frac{1}{2}(D_A\beta + \beta^2 + F(A)), \quad \bar{s}\beta = \frac{1}{2}(D_A\beta + \beta^2 + F(A)), \quad (68)$$

$$s\hat{\beta} = \frac{1}{2}(\mathcal{F} + \beta), \quad \bar{s}\hat{\beta} = \frac{1}{2}(\mathcal{F} - \beta),$$

where the explicit expression of $\mathcal{F} = \sigma\hat{\beta}$ is given by eq.(44).

It then follows that:

$$s\mathcal{F} = -\frac{1}{2}(D_A\beta + \beta^2 + F(A)), \quad \bar{s}\mathcal{F} = \frac{1}{2}(D_A\beta + \beta^2 + F(A)). \quad (69)$$

Let us now introduce a new non homogeneous form π^* (which will contain the BRST/anti-BRST antifields of the π) and define the fields

$$\beta^* = -\frac{1}{2}(\mathcal{F} - \beta) - s\pi^*, \quad \bar{\beta}^* = \frac{1}{2}(\mathcal{F} + \beta) + \bar{s}\pi^*, \quad (70)$$

whose transformations, using eqs.(68) and (69), can be written

$$s\beta^* = \bar{s}\bar{\beta}^* = \frac{1}{2}(D_A\beta + \beta^2 + F(A)). \quad (71)$$

Imposing

$$\delta\pi^* = 0, \quad (72)$$

i.e. $s\pi^* = \bar{s}\pi^*$, one obtains

$$\beta^* + \bar{\beta}^* = \beta, \quad (73)$$

and

$$\bar{\beta}^* = s(\hat{\beta} + \pi^*), \quad \beta^* = -\bar{s}(\hat{\beta} + \pi^*). \quad (74)$$

In terms of the field $\hat{\beta}' = \hat{\beta} + \pi^*$, the two fields β^* and $\bar{\beta}^*$, into which β decomposes, (see eq.(73)) are given by:

$$\beta^* = -s\hat{\beta}', \quad \bar{\beta}^* = s\hat{\beta}', \quad (75)$$

in agreement with the prescription of Ref.[4]. An analogous decomposition $\pi^* = \pi^* + \bar{\pi}^*$ can be done, solving eq.(72) by imposing $\tilde{\pi} = \delta\pi$ or

$$\pi^* = -\bar{s}\tilde{\pi}, \quad \bar{\pi}^* = s\tilde{\pi}, \quad (76)$$

which automatically introduces the *hat-fields* for the π . Using eq.(76), one has $s\pi^* = \bar{s}\bar{\pi}^*$, and from (70) it follows

$$s\pi^* = \bar{s}\bar{\pi}^* = \frac{1}{2}(\bar{\beta}^* - \beta^* - \mathcal{F}), \quad (77)$$

where now $\mathcal{F}$ must be expressed in terms of $\hat{\beta}$, β^* , $\bar{\beta}^*$, π^* and $\bar{\pi}^*$. In this way we can accomodate for the full set of BV fields. The usual form of the extended BV algebra is obtained in terms of rescaled fields $b_p = 2^{p-1}\beta_p$, $\hat{b}_p = 2^{p-1}\hat{\beta}_p$.

We recall that in the extended BV method, the BRST/anti-BRST algebra is obtained only after the construction of a solution $S(\Phi, \Phi^*, \bar{\Phi}^*, \hat{\Phi})$ of the Master Equations (ME's)

$$\frac{1}{2}(S, S)_a + V_a S = 0, \quad a = 1, 2, \quad (78)$$

where one defines the *antibrackets*

$$(X, Y)_1 \equiv \frac{\delta_r X}{\delta \Phi} \frac{\delta_l Y}{\delta \Phi^*} - \frac{\delta_r X}{\delta \Phi^*} \frac{\delta_l Y}{\delta \Phi}, \quad (79)$$

$$(X, Y)_2 \equiv \frac{\delta_r X}{\delta \Phi} \frac{\delta_l Y}{\delta \bar{\Phi}^*} - \frac{\delta_r X}{\delta \bar{\Phi}^*} \frac{\delta_l Y}{\delta \Phi},$$

and the operators

$$V_1 = \bar{\Phi}^* \frac{\delta}{\delta \Phi}, \quad V_2 = -\Phi^* \frac{\delta}{\delta \bar{\Phi}}. \quad (80)$$

Any solution of the ME's allows the construction of a nilpotent algebra generated by the operators s and $\bar{s}$ given by

$$s = (\cdot, S)_1 + V_1, \quad \bar{s} = (\cdot, S)_2 + V_2 \quad (81)$$

The ME's insure the nilpotency of these transformations. It should also be stressed that the solution S of the ME's is not invariant under the action of the operators in (81). In fact one can show that an invariant action can be extracted from S , since it can be written in the form

$$S = S_{\text{inv}} + S_{\text{sources}}^{\text{BRST}}, \quad (82)$$

where $S_{\text{sources}}^{\text{BRST}}$ contains the terms linear in the antifields.

$$S_{\text{sources}}^{\text{BRST}} = (-)^{\text{deg}(\Phi)} (\Phi^* s \Phi + \bar{\Phi}^* \bar{s} \Phi^A) + \frac{1}{2} \hat{\Phi}_A [s, \bar{s}] \Phi^A, \quad (83)$$

and $s S_{\text{inv}} = \bar{s} S_{\text{inv}} = 0$. Evidently, S_{inv} , which is defined only modulo $s, \bar{s}$ closed terms, coincides with the classical action uniquely in the case of closed gauge algebras, otherwise new terms, quadratic in the antifields, appear³.

Since we have directly constructed a nilpotent algebra for the complete set of fields and antifields, for the Freedman-Townsend model, we can immediately produce a solution of the ME's through eqs. (82) and (83), by taking, for instance,

$$S_{\text{inv}} = \int \text{Tr} \left\{ \frac{1}{2} A * A - s \bar{s} (B \hat{b}^* + \pi \hat{\pi}^*) \right\}, \quad (84)$$

using the transformations of the rescaled fields b and $\hat{b}$. One can check that expression (84) coincides with the expression (54), which was constructed with the algebra (51), (52), (53). The resulting expression of S agrees with the solution of the ME's, found in [10]. The gauge-fixing can be done along the lines discussed in the previous section.

³It is worth mentioning that [9] it is possible to reduce the ME's equations to generalized antibrackets (introducing a variable conjugated to the $\hat{\Phi}$), so that S_{eff} is a solution of the corresponding ME's.

8 Conclusions

We give a constructive recipe to get the BRST/anti-BRST algebra of the Freedman-Townsend model. It consists in a suitable deformation of the nilpotent algebra of the Abelian 2-form, and generalizes a procedure introduced, for the pure BRST sector, in a previous work. Extension to non Abelian forms in higher dimensions and to topological BF theories should be straightforward, although calculations become lengthy. A clumsy feature is the occurrence of non locality in the gauge-fixed action. This sheds some doubts on the usefulness of the extended BRST/anti-BRST quantization. Notwithstanding, it has been proved that gauge invariant quantities, i.e. on-shell S-matrix elements, are the same as those of the standard BRST quantization with an appropriate choice of the gauge fermion. Similar issues have been recently discussed in the framework of the extended antifield formalism, of Batalin, Lavrov and Tyutin. The comparison is spelled out in Sect. 7.

9 Appendix. The elimination of the non physical fields in the BRST/anti-BRST algebra of the non-Abelian two form.

Consider the set of fields contained in the non homogeneous forms $\tilde{B}$ and $\tilde{\pi}$, which is obtained through eqs. (28), and which is relevant in four dimensions.

Projecting the relations (34), on definite form degrees, one obtains

$$\begin{aligned}
\bar{s} \tilde{B}_0 &= 0, \quad s \tilde{B}_0 + \bar{s} \tilde{B}_0 = 0, \quad s \tilde{B}_0 + \bar{s} \tilde{B}_0 = 0, \quad s \tilde{B}_0 = 0, \\
\bar{s} \tilde{B}_1 &= 0, \quad s \tilde{B}_1 + \bar{s} \tilde{B}_1 = D_A \tilde{B}_0, \\
s \tilde{B}_1 + \bar{s} \tilde{B}_1 &= D_A \tilde{B}_0, \\
s \tilde{B}_1 + \bar{s} \tilde{B}_1 &= D_A \tilde{B}_0, \quad s \tilde{B}_1 = 0, \\
\bar{s} \tilde{B}_2 &= 0, \quad s \tilde{B}_2 + \bar{s} \tilde{B}_2 = D_A \tilde{B}_1 + [\tilde{\beta}_2, \tilde{B}_0], \\
s \tilde{B}_2 + \bar{s} \tilde{B}_2 &= D_A \tilde{B}_1 + [\tilde{\beta}_2, \tilde{B}_0] + [\tilde{\beta}_2, \tilde{B}_0], \\
s \tilde{B}_2 + \bar{s} \tilde{B}_2 &= D_A \tilde{B}_1 + [\tilde{\beta}_2, \tilde{B}_0] + [\tilde{\beta}_2, \tilde{B}_0], \\
s \tilde{B}_2 + \bar{s} \tilde{B}_2 &= D_A \tilde{B}_1 + [\tilde{\beta}_2, \tilde{B}_0], \\
s \tilde{B}_2 &= 0,
\end{aligned} \tag{85}$$

The equations

$$s \tilde{B}_1 = 0, \quad \bar{s} \tilde{B}_1 = 0, \quad s \tilde{B}_2 = 0, \quad \bar{s} \tilde{B}_2 = 0. \tag{86}$$

are solved by

$$\tilde{B}_1 = s \mathcal{Z}_1, \quad \tilde{B}_1 = -\bar{s} \mathcal{Z}_1, \quad \tilde{B}_2 = s \mathcal{Z}_2, \quad \tilde{B}_2 = -\bar{s} \mathcal{Z}_2. \tag{87}$$

Making the redefinitions

$$\tilde{B}_1 = \tilde{B}_1 - \bar{s} \mathcal{Z}_1, \quad \tilde{B}_1 = \tilde{B}_1 + s \mathcal{Z}_1, \tag{88}$$

one obtains

$$\bar{s} \tilde{B}_1 = D_A \tilde{B}_0, \quad s \tilde{B}_1 + \bar{s} \tilde{B}_1 = D_A \tilde{B}_0, \quad s \tilde{B}_1 = D_A \tilde{B}_0. \tag{89}$$

An explicit computation gives also

$$\begin{aligned}
\tilde{B}_2 &= s \mathcal{Z}_2 + \bar{s} \mathcal{Z}_2 - D_A \mathcal{Z}_1 + [\tilde{\beta}_2, \tilde{B}_0], \\
\tilde{B}_2 &= -\bar{s} \mathcal{Z}_2 - s \mathcal{Z}_2 + D_A \mathcal{Z}_1 - [\tilde{\beta}_2, \tilde{B}_0].
\end{aligned} \tag{90}$$

A redefinition of $\tilde{B}_2$ through

$$\tilde{B}_2 = \tilde{B}_2 + s \mathcal{Z}_2 - \bar{s} \mathcal{Z}_2, \tag{91}$$

gives the transformation laws

$$\begin{aligned}
\bar{s} \tilde{B}_2 &= D_A \tilde{B}_1 + 2[\tilde{\beta}_2, \tilde{B}_0] + [\tilde{\beta}_2, \tilde{B}_0] + [\tilde{\beta}_2, \tilde{B}_0], \\
s \tilde{B}_2 &= D_A \tilde{B}_1 + 2[\tilde{\beta}_2, \tilde{B}_0] + [\tilde{\beta}_2, \tilde{B}_0] + [\tilde{\beta}_2, -\bar{s} \tilde{B}_0].
\end{aligned} \tag{92}$$

Similar manipulation can be done in the equations which are deduced from $\delta \tilde{B} = \tilde{\pi}$. For the physical fields one obtain the relations

$$\begin{aligned}
s \tilde{B}_0 - \bar{s} \tilde{B}_0 &= \pi_0, \quad s \tilde{B}_0 - \bar{s} \tilde{B}_0 = \pi_0, \\
s \tilde{B}_1 - \bar{s} \tilde{B}_1 &= \pi_1,
\end{aligned} \tag{93}$$

where one finds that no redefinition is needed for the $\tilde{\pi}$ fields, i.e.

$$\begin{aligned}
\pi_0 &= \tilde{\pi}_0, \quad \pi_0 = \tilde{\pi}_0, \\
\pi_1 &= \tilde{\pi}_1.
\end{aligned} \tag{94}$$

These relations complete the set of transformations of the *physical* fields B and π , from which the algebra reported in eqs.(51) in the text can be deduced.

Using the parametrization of Γ given in (16), one can write explicit expressions of the $\mathcal{Z}$. One finds

$$\begin{aligned}
\mathcal{Z}_1^{(-1,1)} &= -\frac{1}{2}[g^{-1} B_0^{(0,2)} g, X], & \mathcal{Z}_1^{(1,-1)} &= -\frac{1}{2}[g^{-1} B_0^{(2,0)} g, X], \\
\mathcal{Z}_2^{(-2,1)} &= -\frac{1}{8}[[g^{-1} B_0^{(0,2)} g, X], c], & \mathcal{Z}_2^{(1,-2)} &= -\frac{1}{8}[[g^{-1} B_0^{(2,0)} g, X], \bar{c}], \\
\mathcal{Z}_2^{(-1,0)} &= -\frac{1}{2}[g^{-1} B_1^{(0,1)} g, X] - \frac{1}{8}[[g^{-1} B_0^{(1,1)} g, X], c] - \frac{3}{8}[[g^{-1} B_0^{(0,2)} g, X], \bar{c}], \\
\mathcal{Z}_2^{(0,-1)} &= -\frac{1}{2}[g^{-1} B_1^{(1,0)} g, X] - \frac{1}{8}[[g^{-1} B_0^{(1,1)} g, X], \bar{c}] - \frac{3}{8}[[g^{-1} B_0^{(2,0)} g, X], c].
\end{aligned} \tag{95}$$

where $X = X_1^{(-1,-1)}$, $c = c_1^{(-1,0)} = \bar{s}X$ and $\bar{c} = c_1^{(0,-1)} = -sX$.

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