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Topological Yang-Mills Theory from Non-Commutative Geometry

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Abstract

We show that the topological Yang-Mills theory of Witten can be induced from the usual Yang-Mills theory of the second rank antisymmetric tensor field when we incorporate the matrix derivative of non-commutative geometry proposed by Connes in the superconnection framework. It is done by identifying consistently the antisymmetric tensor field B with the usual field strength F by $B = -F$.

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1. Introduction

The gauge symmetry of the second rank antisymmetric tensor field has been studied intensively [1], and Siegel discovered that the ghost for ghost is necessary to quantize this gauge theory completely [2]. Baulieu and Thierry-Mieg quantized this theory using the so-called horizontality condition [3]. Through these works the quantization of the gauge theory of antisymmetric tensor field is well understood. In this theory we have the vector ghosts β_μ and $\bar{\beta}_\mu$ due to the antisymmetric tensor field as well as the scalar ghosts α and $\bar{\alpha}$ of the ordinary one form gauge field.

Witten proposed the topological Yang-Mills gauge theory and showed that it generates the Donaldson invariants [4]. The classical action of this theory is given by

$$\mathcal{S}_C = \int_M \text{tr}[FF]. \quad (1)$$

Baulieu and Singer quantized this classical action using the BRST symmetry and reproduced the Witten's action [5]. Then Perry and Teo quantized it using both BRST and anti-BRST symmetries [6]. In their geometrical interpretation Perry and Teo got the BRST and anti-BRST transformation rules using the following two equations.

$$(d + s + \bar{s})(A + c + \bar{c}) + \frac{1}{2}[A + c + \bar{c}, A + c + \bar{c}] = F + \psi + \bar{\psi} + \phi + \bar{\phi} + \lambda \quad (2)$$

$$(d + s + \bar{s})(F + \psi + \bar{\psi} + \phi + \bar{\phi} + \lambda) + [A + c + \bar{c}, F + \psi + \bar{\psi} + \phi + \bar{\phi} + \lambda] = 0. \quad (3)$$

Baulieu and Singer also considered the same equations with only the BRST symmetry [5].

We will show in this paper that the role of the two equations (2) and (3) can be performed by one horizontality condition of the standard $\tilde{F} = F$ type when we consider this theory in the non-commutative geometry proposed by Connes [7, 8]. The basic reason why it is possible is that the matrix derivative of non-commutative geometry relates the odd and even forms. In this work there are three

keys. The first one is the horizontality condition, the second one is the concept of the superconnection, and the third one is the matrix derivative of non-commutative geometry.

2. Horizontality condition, superconnection and matrix derivative of non-commutative geometry

In this section we will describe briefly the essential aspects of the horizontality condition, the superconnection and the matrix derivative of non-commutative geometry which are used in the present work. The horizontality condition for the ordinary gauge theory with one form gauge field A is written as [9-13]

$$\tilde{F} = F, \quad (4)$$

where

$$\begin{aligned} F &= dA + AA \\ A &= A_\mu dx^\mu, \quad c = c_N dy^N, \quad \bar{c} = \bar{c}_{\bar{N}} d\bar{y}^{\bar{N}} \\ d &= dx^\mu \frac{\partial}{\partial x^\mu}, \quad s = dy^N \frac{\partial}{\partial y^N}, \quad \bar{s} = d\bar{y}^{\bar{N}} \frac{\partial}{\partial \bar{y}^{\bar{N}}} \\ \tilde{F} &= \tilde{d}\tilde{A} + \tilde{A}\tilde{A} \\ \tilde{d} &= d + s + \bar{s}, \quad \tilde{A} = A + c + \bar{c}. \end{aligned} \quad (5)$$

The horizontality condition (4) gives rise to the BRST and anti-BRST transformation rules which satisfy the nilpotency condition $s^2 = \bar{s}^2 = s\bar{s} + \bar{s}s = 0$.

The above procedure using the horizontality condition works elegantly for the system of one form gauge fields. However, when we study a system which contains two form gauge fields, we need the concept of the superconnection which is the generalization of one form gauge field if we want the exactly same procedure to work as that of one form gauge field. The superconnection is given by [8, 14, 15]

$$\mathcal{A} = \begin{pmatrix} \omega_0 & L_{01} \\ L_{10} & \omega_1 \end{pmatrix} = \mathcal{A}_{ev} + \mathcal{A}_{od} = \begin{pmatrix} \omega_0 & 0 \\ 0 & \omega_1 \end{pmatrix} + \begin{pmatrix} 0 & L_{01} \\ L_{10} & 0 \end{pmatrix}, \quad (6)$$

where the even supermatrix elements ω_0 and ω_1 are odd degree forms, and the odd elements L_{01} and L_{10} are even degree forms. The superalgebra multiplication law is given as follows for general supermatrices $\mathcal{X}$ and $\mathcal{X}'$ which have arbitrary degree forms X and X' as their supermatrix elements [8, 14].

$$\mathcal{X}\mathcal{X}' = (a \otimes X)(a' \otimes X') = (-1)^{|X||a'|}(aa') \otimes (XX'), \quad (7)$$

where $|X|$ and $|a|$ are form grade and supermatrix grade respectively. $|X|$ is one for odd form, zero for even form. $|a|$ is zero for even supermatrix, and one for odd supermatrix. With the above superalgebra multiplication law we have the following Leibniz rule of the exterior derivative d for the superconnection $\mathcal{A}$ in (6) and a general supermatrix $\mathcal{X}$.

$$d(\mathcal{A}\mathcal{X}) = (d\mathcal{A})\mathcal{X} - \mathcal{A}(d\mathcal{X}), \quad (8)$$

where

$$\begin{aligned} d\mathcal{A} &= \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \omega_0 & L_{01} \\ L_{10} & \omega_1 \end{pmatrix} = \begin{pmatrix} d\omega_0 & -dL_{01} \\ -dL_{10} & d\omega_1 \end{pmatrix} \\ d\mathcal{X} &= \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} P & R \\ S & Q \end{pmatrix} = \begin{pmatrix} dP & -dR \\ -dS & dQ \end{pmatrix}. \end{aligned} \quad (9)$$

We note that (8) is the same as the Leibniz rule of the usual one form gauge field A which is given by $d(AX) = (dA)X - A(dX)$, where X is an arbitrary differential form. Here the minus sign in the second term of the right hand side obviously comes from the fact that A is one form. We can interpret the horizontality condition as being based on this Leibniz rule. Therefore, we can get the BRST and anti-BRST transformation rules of general form degree gauge fields following the exactly same procedure as that of one form gauge field if we work in the superconnection framework. That is, we can apply the horizontality condition $\tilde{\mathcal{F}} = \mathcal{F}$ of the type of (4) to get the BRST and anti-BRST transformation rules. The Leibniz rule for general supermatrices $\mathcal{X}$ and $\mathcal{X}'$, of which (8) is the special case, is given by

$$d(\mathcal{X}\mathcal{X}') = d((a \otimes X)(a' \otimes X')) = (d\mathcal{X})\mathcal{X}' + (-1)^{|a|+|X|}\mathcal{X}(d\mathcal{X}'). \quad (10)$$

The matrix derivative of non-commutative geometry of a general supermatrix $\mathcal{X}$ is defined as [7, 8]

$$\mathbf{d}_M \mathcal{X} = \mathbf{d}_M(a \otimes X) = \mathcal{H} \mathcal{X} - (-1)^{|a|+|X|} \mathcal{X} \mathcal{H}, \quad (11)$$

where

$$\mathcal{H} = \begin{pmatrix} 0 & \xi \\ \xi^\dagger & 0 \end{pmatrix} \quad (12)$$

with an arbitrary constant submatrix ξ . Using the superalgebra multiplication law (7), the matrix derivative in (11) can be expressed with the above ξ and $\xi^\dagger$ as

$$\mathbf{d}_M \mathcal{A} = \begin{pmatrix} \xi L_{10} + L_{01} \xi^\dagger & \xi \omega_1 - \omega_0 \xi \\ \xi^\dagger \omega_0 - \omega_1 \xi^\dagger & \xi^\dagger L_{01} + L_{10} \xi \end{pmatrix}. \quad (13)$$

The matrix derivative satisfies the following Leibniz rule for general supermatrices $\mathcal{X}$ and $\mathcal{X}'$.

$$\mathbf{d}_M(\mathcal{X} \mathcal{X}') = \mathbf{d}_M((a \otimes X)(a' \otimes X')) = (\mathbf{d}_M \mathcal{X}) \mathcal{X}' + (-1)^{|a|+|X|} \mathcal{X}(\mathbf{d}_M \mathcal{X}'). \quad (14)$$

Since the superconnection $\mathcal{A}$ in (6) has odd and even degree forms for even and odd supermatrix elements respectively, (14) gives

$$\mathbf{d}_M(\mathcal{A} \mathcal{X}) = (\mathbf{d}_M \mathcal{A}) \mathcal{X} - \mathcal{A}(\mathbf{d}_M \mathcal{X}), \quad (15)$$

where $\mathcal{X}$ is a general supermatrix. We note that (15) is the same Leibniz rule as (8). Then we extend the exterior derivative $\mathbf{d}$ to $\mathbf{d}_t$ as

$$\mathbf{d}_t = \mathbf{d} + \mathbf{d}_M, \quad (16)$$

and we define $\mathcal{F}_t$ and $\widetilde{\mathcal{F}}_t$ as

$$\begin{aligned} \mathcal{F}_t &= \mathbf{d}_t \mathcal{A} + \mathcal{A} \mathcal{A} \\ \widetilde{\mathcal{F}}_t &= \widetilde{\mathbf{d}}_t \widetilde{\mathcal{A}} + \widetilde{\mathcal{A}} \widetilde{\mathcal{A}}, \end{aligned} \quad (17)$$

where

$$\begin{aligned} \widetilde{\mathbf{d}}_t &= \mathbf{d}_t + \mathbf{s} + \bar{\mathbf{s}} = \mathbf{d} + \mathbf{d}_M + \mathbf{s} + \bar{\mathbf{s}} \\ \mathbf{s} \mathcal{A} &= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} \omega_0 & L_{01} \\ L_{10} & \omega_1 \end{pmatrix} = \begin{pmatrix} s \omega_0 & -s L_{01} \\ -s L_{10} & s \omega_1 \end{pmatrix} \\ \bar{\mathbf{s}} \mathcal{A} &= \begin{pmatrix} \bar{s} & 0 \\ 0 & \bar{s} \end{pmatrix} \begin{pmatrix} \omega_0 & L_{01} \\ L_{10} & \omega_1 \end{pmatrix} = \begin{pmatrix} \bar{s} \omega_0 & -\bar{s} L_{01} \\ -\bar{s} L_{10} & \bar{s} \omega_1 \end{pmatrix}, \end{aligned} \quad (18)$$

and $\widetilde{\mathcal{A}}$ is the superconnection which is defined on the tangent and cotangent manifolds of the fiber as well as on the space-time base manifold, that is, defined in the fiber-bundle space. $\widetilde{\mathcal{A}}$ will be given explicitly in the next section. Then we can apply the following horizontality condition which is of the type of (4) to get the BRST and anti-BRST transformation rules.

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_t. \quad (19)$$

In this paper we take $\xi = \frac{1}{2}I$ for the ξ in (12), where I is identity matrix. It will become clear in the next section that this choice of ξ is suitable in our present work. Then the matrix derivative in (13) is given by

$$\begin{aligned} \mathbf{d}_M \mathcal{A}_{ev} &= \mathbf{d}_M \begin{pmatrix} \omega_0 & 0 \\ 0 & \omega_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \omega_1 - \omega_0 \\ \omega_0 - \omega_1 & 0 \end{pmatrix} \\ \mathbf{d}_M \mathcal{A}_{od} &= \mathbf{d}_M \begin{pmatrix} 0 & L_{01} \\ L_{10} & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} L_{01} + L_{10} & 0 \\ 0 & L_{01} + L_{10} \end{pmatrix}. \end{aligned} \quad (20)$$

Now we are ready to investigate the BRST quantization of the topological Yang-Mills gauge theory geometrically whose classical action is given by (1).

3. Topological Yang-Mills theory from non-commutative geometry

In this section some parts are equivalent to those of Perry and Teo [6], however, we include such parts in order to show in a self-contained way that the BRST formalism for the usual Yang-Mills theory also applies to the topological Yang-Mills theory when we incorporate the matrix derivative of non-commutative geometry. We combine one form gauge field A and two form antisymmetric tensor gauge field B in the superconnection framework as

$$\mathcal{A} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad (21)$$

where

$$A = A_\mu dx^\mu, \quad B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu. \quad (22)$$

Here A and B are Lie algebra valued, that is, $A = A_a T_a$, $B = B_a T_a$ where T_a 's are gauge group generators, for example, $SU(N)$ group generators. In the below ghost and auxiliary fields are also Lie algebra valued in the same way. When we incorporate the matrix derivative of non-commutative geometry, the supercurvature is given by

$$\begin{aligned}\mathcal{F}_t &= \mathbf{d}_t \mathcal{A} + \mathcal{A} \mathcal{A} \\ &= (\mathbf{d} + \mathbf{d}_M) \mathcal{A} + \mathcal{A} \mathcal{A} \\ &= \begin{pmatrix} dA + AA + B + BB & -dB - AB + BA \\ -dB - AB + BA & dA + AA + B + BB \end{pmatrix},\end{aligned}\quad (23)$$

where we applied the superalgebra multiplication law in (7) and used the matrix derivative of $\mathcal{A}$ in (20).

The superconnection in the fiber-bundle space is given by

$$\tilde{\mathcal{A}} = \begin{pmatrix} A + \alpha + \bar{\alpha} & B + \beta + \bar{\beta} + \phi + \rho + \bar{\phi} \\ B + \beta + \bar{\beta} + \phi + \rho + \bar{\phi} & A + \alpha + \bar{\alpha} \end{pmatrix}, \quad (24)$$

where

$$\begin{aligned}A + \alpha + \bar{\alpha} &= A_\mu dx^\mu + \alpha_N dy^N + \bar{\alpha}_{\bar{N}} d\bar{y}^{\bar{N}} \\ B + \beta + \bar{\beta} + \phi + \rho + \bar{\phi} &= \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu + \beta_{\mu N} dx^\mu \wedge dy^N + \bar{\beta}_{\mu \bar{N}} dx^\mu \wedge d\bar{y}^{\bar{N}} \\ &\quad + \frac{1}{2} \phi_{MN} dy^M \wedge dy^N + \rho_{M \bar{N}} dy^M \wedge d\bar{y}^{\bar{N}} \\ &\quad + \frac{1}{2} \bar{\phi}_{\bar{M} \bar{N}} d\bar{y}^{\bar{M}} \wedge d\bar{y}^{\bar{N}}.\end{aligned}\quad (25)$$

In the above ϕ , ρ and $\bar{\phi}$ fields are ghost for ghost fields which are necessary to fix the residual gauge symmetries of β and $\bar{\beta}$. The forms in (25) are defined on the tangent and cotangent manifolds of the fiber as well as on the space-time base manifold, that is, they are defined in the fiber-bundle space. The component fields with an odd form in dy and $d\bar{y}$ directions have wrong spin-statistics and reduce the physical degrees of freedom. Here, α , $\bar{\alpha}$, β and $\bar{\beta}$ are such fields. Note that we changed the names of c , $\bar{c}$, ψ , $\bar{\psi}$ and λ in (2) and (3) to α , $\bar{\alpha}$, β , $\bar{\beta}$ and ρ , respectively.

The supercurvature in the fiber-bundle space with the matrix derivative incorporated is given by

$$\tilde{\mathcal{F}}_t = \widetilde{\mathbf{d}}_t \tilde{\mathcal{A}} + \tilde{\mathcal{A}} \tilde{\mathcal{A}}, \quad (26)$$

where

$$\widetilde{\mathbf{d}}_t = \mathbf{d} + \mathbf{d}_M + \mathbf{s} + \bar{\mathbf{s}}. \quad (27)$$

The $\mathbf{s}$ and $\bar{\mathbf{s}}$ above have been defined in (18). Now we can obtain the BRST and anti-BRST transformation rules from the horizontality condition which is given by

$$\tilde{\mathcal{F}}_t = \mathcal{F}_t. \quad (28)$$

By expanding (28) term by term for each form degree, we can get the BRST and anti-BRST transformation rules for each component field. The main point of this paper is that the matrix derivative of $\tilde{\mathcal{A}}$ which is a part of the left hand side of (28) is given by

$$\mathbf{d}_M \tilde{\mathcal{A}} = \begin{pmatrix} B + \beta + \bar{\beta} + \phi + \rho + \bar{\phi} & 0 \\ 0 & B + \beta + \bar{\beta} + \phi + \rho + \bar{\phi} \end{pmatrix}. \quad (29)$$

We note that there exist β and $\bar{\beta}$ in the even supermatrix elements of the above $\mathbf{d}_M \tilde{\mathcal{A}}$, and that this fact gives rise to the topological Yang-Mills symmetry under which the classical action (1) is invariant when we expand (28) in terms of each form degree. (28) is written as follows for each form degree.

$$\begin{aligned}(dx)^1(dy)^1 &: sA = -D\alpha - \beta \\ (dx)^1(d\bar{y})^1 &: \bar{s}A = -D\bar{\alpha} - \bar{\beta} \\ (dy)^2 &: s\alpha = -\alpha\alpha - \phi \\ (d\bar{y})^2 &: \bar{s}\bar{\alpha} = -\bar{\alpha}\bar{\alpha} - \bar{\phi} \\ (dy)^1(d\bar{y})^1 &: s\bar{\alpha} + \bar{s}\alpha = -[\alpha, \bar{\alpha}] - \rho \\ (dx)^2(dy)^1 &: sB = -D\beta - [\alpha, B] \\ (dx)^2(d\bar{y})^1 &: \bar{s}B = -D\bar{\beta} - [\bar{\alpha}, B] \\ (dx)^1(dy)^2 &: s\beta = -D\phi - [\alpha, \beta]\end{aligned}\quad (30)$$

$$\begin{aligned}
(dx)^1(dy)^2 & : \quad \bar{s}\bar{\beta} = -D\bar{\phi} - [\bar{\alpha}, \bar{\beta}] \\
(dx)^1(dy)^1(d\bar{y})^1 & : \quad s\bar{\beta} + \bar{s}\beta = -[\alpha, \bar{\beta}] - [\bar{\alpha}, \beta] - D\rho \\
(dy)^3 & : \quad s\phi = -[\alpha, \phi] \\
(d\bar{y})^3 & : \quad \bar{s}\bar{\phi} = -[\bar{\alpha}, \bar{\phi}] \\
(dy)^2(d\bar{y})^1 & : \quad \bar{s}\phi + s\rho = -[\alpha, \rho] - [\bar{\alpha}, \phi] \\
(dy)^1(d\bar{y})^2 & : \quad s\bar{\phi} + \bar{s}\rho = -[\bar{\alpha}, \rho] - [\alpha, \bar{\phi}],
\end{aligned}$$

where $[\ , \]$ is the graded commutator and $Df \equiv df + [A, f]$.

The equations of $(dy)^1(d\bar{y})^1$, $(dx)^1(dy)^1(d\bar{y})^1$, $(dy)^2(d\bar{y})^1$ and $(dy)^1(d\bar{y})^2$ in the above (30) do not completely specify the transformation rules of the component fields. Therefore we need to introduce the auxiliary fields t , m , n and $\bar{n}$ given in the below to solve these equations.

$$\begin{aligned}
s\bar{\alpha} &\equiv t \quad , \quad \bar{s}\alpha = -t - [\alpha, \bar{\alpha}] - \rho \\
s\bar{\beta} &\equiv m \quad , \quad \bar{s}\beta = -m - [\alpha, \bar{\beta}] - [\bar{\alpha}, \beta] - D\rho \\
s\rho &\equiv n \quad , \quad \bar{s}\phi = -n - [\alpha, \rho] - [\bar{\alpha}, \phi] \\
s\bar{\phi} &\equiv \bar{n} \quad , \quad \bar{s}\rho = -\bar{n} - [\bar{\alpha}, \rho] - [\alpha, \bar{\phi}].
\end{aligned} \tag{31}$$

From (31) we get the following transformation rules of the auxiliary fields.

$$\begin{aligned}
st &= sm = sn = s\bar{n} = 0 \\
\bar{s}t &= -[\bar{\alpha}, t] + \bar{n} \\
\bar{s}m &= -[D\alpha, \bar{\phi}] - D\bar{n} - [\beta, \bar{\phi}] - [\bar{\beta}, t] - [\bar{\alpha}, m] \\
\bar{s}n &= -[\alpha\alpha, \bar{\phi}] - [\alpha, \bar{n}] - [\bar{\alpha}, n] - [\phi, \bar{\phi}] - [\rho, t] \\
\bar{s}\bar{n} &= -[\bar{\alpha}, \bar{n}] - [\bar{\phi}, t].
\end{aligned} \tag{32}$$

The equations (30), (31) and (32) constitute a complete set of the BRST and anti-BRST transformation rules of the component fields. From (30) we can show easily that the transformation rules of B are the same as those of $-(dA + AA)$. Therefore, we can identify B with $-F$ consistently when we are concerned only

with their BRST and anti-BRST transformations, where $F \equiv dF + AA$. With this identification the superconnection in the fiber-bundle space given in (24) becomes

$$\tilde{\mathcal{A}} = \begin{pmatrix} A + \alpha + \bar{\alpha} & -F + \beta + \bar{\beta} + \phi + \rho + \bar{\phi} \\ -F + \beta + \bar{\beta} + \phi + \rho + \bar{\phi} & A + \alpha + \bar{\alpha} \end{pmatrix}. \tag{33}$$

Now let us consider the BRST and anti-BRST invariant quantum Lagrangian. We define the quantum action in four dimensional compact manifold M as

$$\mathcal{S}_Q = \int_M tr[F F] - \frac{1}{4} \int_M \mathcal{P}_G^0 Tr\{ \bar{s} \tilde{\mathcal{A}}^* \tilde{\mathcal{A}} \pm \tilde{\mathcal{A}} \tilde{\mathcal{A}}^* \}, \tag{34}$$

where $\star$ denotes taking the dual over the base manifold(x^μ) for each differential form entry of supermatrix and taking the hermitian conjugate for the supermatrix. In (34) $\mathcal{P}_G^0$ is a projection operator which projects ghost number zero parts, and Tr is the trace over the whole supermatrix whereas tr is only over the submatrix which is the supermatrix element made of Yang-Mills gauge group generators. We use the convention $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, $\epsilon_{0123} = 1$, and our definition for the dual of space-time form is given by [16]

$$* (dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}) = \frac{1}{(n-p)!} \epsilon^{i_1 i_2 \cdots i_p i_{p+1} \cdots i_n} dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_n} \tag{35}$$

in n dimensional space-time, which satisfies $**\omega_p = (-1)^{p(n-p)}\omega_p$ for a p -form ω_p . Then the quantum Lagrangian $\mathcal{L}_Q$ from (34) is given by

$$\begin{aligned}
\mathcal{L}_Q &= tr[FF] + \mathcal{L}'_Q \\
\mathcal{L}'_Q &= tr\{s\bar{s}(-\frac{1}{2}A_\mu A^\mu + \frac{1}{2}(\bar{\alpha}\alpha + \alpha\bar{\alpha}) \\
&\quad + \frac{1}{4}F_{\mu\nu}(F^{\mu\nu} \pm \hat{F}^{\mu\nu}) - \bar{\beta}_\mu \beta^\mu + \frac{1}{2}(\bar{\phi}\phi + \phi\bar{\phi} + \rho\rho)\},
\end{aligned} \tag{36}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $\hat{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. In (36)

$$tr[s\bar{s}(\bar{\alpha}\alpha + \alpha\bar{\alpha})] = 0, \tag{37}$$

and using the transformation rules in (30), (31) and (32) we can show that

$$tr[s\bar{s}(\bar{\phi}\phi + \phi\bar{\phi} + \rho\rho)] = 0. \tag{38}$$

Then we have

$$\mathcal{L}'_Q = \text{tr}[s\bar{s}(-\frac{1}{2}A_\mu A^\mu + \frac{1}{4}F_{\mu\nu}(F^{\mu\nu} \pm \hat{F}^{\mu\nu}) - \bar{\beta}_\mu \beta^\mu)]. \quad (39)$$

Each term in (39) is given as follows from (30), (31) and (32).

$$\begin{aligned} \text{tr}[s\bar{s}(\frac{1}{2}A_\mu A^\mu)] &= \text{tr}[s(A_\mu(\partial^\mu \bar{\alpha} + \bar{\beta}^\mu))] \\ &= \text{tr}[A_\mu(\partial^\mu t + m^\mu) - (\partial \bar{\alpha} + \beta)_\mu(D\alpha + \beta)^\mu], \end{aligned} \quad (40)$$

$$\begin{aligned} \text{tr}[s\bar{s}(\frac{1}{2}F_{\mu\nu}(F^{\mu\nu} \pm \hat{F}^{\mu\nu}))] &= \text{tr}[s((F^{\mu\nu} \pm \hat{F}^{\mu\nu})D^{[\mu}\bar{\beta}^{\nu]})] \\ &= \text{tr}[(F_{\mu\nu} \pm \hat{F}_{\mu\nu})(D^{[\mu}m^{\nu]} + [(D\alpha + \beta)^{[\mu}, \bar{\beta}^{\nu]})] \\ &\quad - (D_{[\mu}\bar{\beta}_{\nu]})(D^{[\mu}\beta^{\nu]} \pm \widehat{D^{[\mu}\beta^{\nu]}}) - [\alpha, (F^{\mu\nu} \pm \hat{F}^{\mu\nu})]], \end{aligned} \quad (41)$$

where $D^{[\mu}\beta^{\nu]} \equiv D^\mu\beta^\nu - D^\nu\beta^\mu$ and $\widehat{D^{[\mu}\beta^{\nu]}} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}D_{[\rho}\beta_{\sigma]}$, and

$$\begin{aligned} \text{tr}[s\bar{s}(\bar{\beta}_\mu \beta^\mu)] &= \text{tr}[s(\bar{\beta}_\mu m^\mu + (D_\mu \bar{\phi})\beta^\mu - \bar{\beta}_\mu(D^\mu \rho) + [\bar{\beta}_\mu, \bar{\beta}^\mu]\alpha)] \\ &= \text{tr}[m_\mu m^\mu - m_\mu(D^\mu \rho - 2[\alpha, \bar{\beta}^\mu]) + (D_\mu \bar{n})\beta^\mu + \bar{\beta}_\mu(D^\mu n) \\ &\quad + [(D\alpha + \beta)_\mu, \bar{\phi}]\beta^\mu + \bar{\beta}_\mu[(D\alpha + \beta)^\mu, \rho] - [\bar{\beta}_\mu, \bar{\beta}^\mu](\alpha\alpha + \phi) \\ &\quad + (D_\mu \bar{\phi})(D^\mu \phi - [\alpha, \beta^\mu])]. \end{aligned} \quad (42)$$

The BRST and anti-BRST invariant quantum Lagrangian given by (36) and (39) is the same as that considered by Perry and Teo except the sign of the last term $\bar{\beta}_\mu \beta^\mu$ [6]. Since the sign difference of this term is not significant, our quantum Lagrangian can be considered to be equivalent to theirs. We note that (39) has been obtained naturally from (34) in the superconnection framework.

4. Conclusion

We studied the Yang-Mills gauge symmetry under which the classical action $\int_M \text{tr}[FF]$ is invariant. This action has an unusual symmery of $\delta A_\mu = \beta_\mu$ as well as the usual gauge symmetry of $\delta A_\mu = D_\mu \alpha$. This action is invariant under $\delta A_\mu = \beta_\mu$

because of the Bianchi identity $D \wedge F = 0$. On the other hand, it is invariant under $\delta A_\mu = D_\mu \alpha$ because the field strength transforms as $\delta F_{\mu\nu} = [F_{\mu\nu}, \alpha]$ as well-known in the usual gauge theory. Therefore these two kinds of symmetries of $\int_M \text{tr}[FF]$ seem different in their origins.

In this paper we showed that these two kinds of symmetries can be interpreted in one structure when we incorporate the matrix derivative of non-commutative geometry in the superconnection framework. Then in this structure they can be treated like a single object. For example, as we did in this paper, we can get their BRST and anti-BRST transformation rules from one horizontality condition $\widetilde{\mathcal{F}}_t = \mathcal{F}_t$ which is of the $\tilde{F} = F$ type. That is, in this structure we do not need to rely on the Bianchi identity in getting the BRST and anti-BRST transformation rules, whereas in the references [5] and [6] it was necessary to use the Bianchi identity in (3) in order to get the transformation rules. We also studied the BRST and anti-BRST invariant quantum Lagrangian in this structure.

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