Virasoro character identities from the Andrews–Bailey construction

Omar Foda and Yas-Hiro Quano*

Department of Mathematics, University of Melbourne
Parkville, Victoria 3052, Australia

Abstract

We prove \( q \)-series identities between bosonic and fermionic representations of certain Virasoro characters. These identities include some of the conjectures made by the Stony Brook group as special cases. Our method is a direct application of Andrews’ extensions of Bailey’s lemma to recently obtained polynomial identities.

1 Introduction

1.1 Aim

In an impressive series of papers that include [1], the Stony Brook group conjectured a large number of \( q \)-series identities. For reviews and complete references, see [2]. Let us restrict our attention to those conjectures related to the following Virasoro characters [1]:

\[ \chi^{(p, p+1)}_{r, s}(q) \]
\[ \chi^{(p+2)}_{(p+1)/2, (p+1)/2}(q) \]
\[ \chi^{(p, kp+1)}_{1, k}(q) \]

Here, \( \mathcal{M}(p, p') \) is a Virasoro minimal model specified by two coprime integers \( (p, p') \), and \( \chi^{(p, p')}_{r, s}(q) \) is a conformal character of \( \mathcal{M}(p, p') \), specified by two integers \( (r, s) \), where \( 1 \leq r \leq p - 1 \), \( 1 \leq s \leq p' - 1 \) [3].

These identities are of great interest for a number of physical and mathematical reasons which are beyond the scope of this work. The first step towards proving them was taken by Melzer [4], who conjectured a polynomial identity which implies the \( q \)-series identities for (i) in the above list, and proved it for \( p = 3, 4 \). In [5] Berkovich proved these polynomial identities for arbitrary \( p \) and for \( s = 1 \), and thus proved the \( q \)-series identities involving (i) for \( s = 1 \). In [6, 7] we presented a polynomial identity which implies Gordon’s generalization of the Rogers–Ramanujan identities [8], and the \( q \)-series identities for \( \chi^{(2, 2k+1)}_{1, i}(q) \) [9, 10].

In this paper, we prove \( q \)-series identities for

\[ (\text{II}) \quad \begin{cases} 
\chi_{i, k+1}^{(2k+1, k+1)}(q) \\
\chi_{i, k+1}^{(2k+1, k+1)+2}(q) \\
\chi_{i, k+1}^{(2k+1, k+1)+2k-1}(q) \\
\chi_{i, k+1}^{(2k+1, k+1)+2k-1}(q)
\end{cases} \]

for \( 1 \leq i \leq \kappa \).

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which include (ii) as special cases, respectively. In order to obtain the fermionic representations for (II), we apply Andrews’ extensions of Bailey’s lemma [11, 12, 13, 14] to the polynomial identities presented in [6, 7] and [4, 5], respectively. We wish to refer to these extensions as the Andrews-Bailey construction.

1.2 Plan

This paper is organized as follows. In the rest of this section, we formulate the problem and summarize our results. In section 2 we review a number of definitions and propositions concerning the Andrew-Bailey construction. In section 3 we obtain the desired q-series identities. Appendix A contains a summary of some of the q-series identities that can be obtained by a direct application of the Andrews-Bailey construction to Slater’s list of Bailey pairs [21].

1.3 Formulation of the problem

Rocha-Cardi [15] obtained the following expression for the Virasoro characters as an infinite series in q:

\[ \chi^{(p,p')}_{r,s}(q) = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left( q^{p' r^2 + (p' - s)p n} - q^{(p + r)(p' n + s)} \right), \]  

where \( p \) and \( p' \) are coprime positive integers, \( 1 \leq r \leq p - 1, \ 1 \leq s \leq p' \), and

\[ (a)_\infty \equiv (a; q)_\infty = \prod_{m=0}^{\infty} (1 - a q^m). \]

Starting from Bethe ansatz computations, the Stony Brook group [1] found different q-series expressions for large classes of these characters. For physical reasons their new expressions are referred to as the fermionic sum representations, whereas (1.1) is referred to as the bosonic sum representation.

For non-unitary minimal model \( M(2, 2k + 1) \), (1.1) reduces to

\[ \chi^{(2, 2k + 1)}_{1,i}(q) = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left( q^{(4k+2)n^2 + (2k - 2i + 1) n} - q^{(2k+1)(2k+1)n+i} \right) \]

\[ = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n[(2k+1)n+2k-2i+1]/2} \]

\[ = \prod_{n=\pm 1} \left( 1 - q^n \right)^{-1}, \]  

where the last equality is obtained using Jacobi’s triple product formula.

The fermionic expression corresponding to (1.2), which was obtained in [9, 10] from different approaches, is as follows:

\[ \chi^{(2, 2k + 1)}_{1,i}(q) = \chi^{(2, 2k + 1)}_{1,2k + 1 - i}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_k^2 \cdots + n_k^k}}{(q)^{n_1 - n_2 \cdots \cdots (q)^{n_{k-1} - n_k}}} (q)^{n_{k-1} - n_{k-1}}, \]

where \( 1 \leq i \leq k \), and

\[ (a)_n \equiv (a; q)_n = \frac{(a)_\infty}{(a^q)_\infty}, \]

2
Equating these two expressions, we reproduce Gordon's generalization of the Rogers–Ramanujan identity [8] for $1 \leq i \leq k$

$$\prod_{n=0}^{\infty} (1 - q^n)^{-1} = \sum_{n_1 \geq 2 \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + n_1 \cdots + n_k + n_{k-1}}}{(q)_{n_1 - n_2} \cdots (q)_{n_k - n_k - n_{k-1}} (q)_{n_k}}$$

This is the simplest q-series identity between bosonic and fermionic representations of Virasoro characters.

In [1] the Stony Brook group proved identities of the above type up to a finite power in $q$, using explicit computations, and conjectured their validity to all powers in $q$. Proving some of these conjectures is the problem we address in this work.

### 1.4 A summary of results

The q-series identities we shall prove are as follows:

**Theorem 1.1** The following identities hold:

$$\frac{1}{(q)} = \sum_{n=-\infty}^{\infty} \frac{q^{(2\alpha+1)(2\alpha+1)+2n^2 + \beta(2\alpha+1,k+2)-(k+1)(2\alpha+1)n}}{(2\alpha+1)(2\alpha+1+k+2)n_0} - \frac{(2\alpha+1)(2\alpha+1+k+2)n_0^{\alpha+1}}{(2\alpha+1)(2\alpha+1+k+2)n_0^{\alpha}}$$  

$$= \sum_{n_1 \geq 2 \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\alpha+1)(n_1 + \cdots + n_k)}}{(q)_{n_1 - n_2} \cdots (q)_{n_k - n_k - n_{k-1}} (q)_{n_k}}$$  

$$\times \prod_{\mu=1}^{\infty} \left[ 2n_k + \kappa - \mu + 1 - 2(n_1 + \cdots + n_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\mu} \right] q^\nu.$$  

$$\frac{1}{(q)} = \sum_{n=-\infty}^{\infty} \frac{q^{(2\alpha+1)(2\alpha+1)+2n^2 + \beta(2\alpha+1,k+2)-(k+1)(2\alpha+1)n}}{(2\alpha+1)(2\alpha+1+k+2)n_0} - \frac{(2\alpha+1)(2\alpha+1+k+2)n_0^{\alpha+1}}{(2\alpha+1)(2\alpha+1+k+2)n_0^{\alpha}}$$  

$$= \sum_{n_1 \geq 2 \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\alpha+1)(n_1 + \cdots + n_k)}}{(q)_{n_1 - n_2} \cdots (q)_{n_k - n_k - n_{k-1}} (q)_{n_k}}$$  

$$\times \prod_{\mu=1}^{\infty} \left[ 2n_k + \kappa - \mu + 1 - 2(n_1 + \cdots + n_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\mu} \right] q^\nu.$$  

As a corollary we have

**Corollary 1.2**

$$\lambda_{\alpha,\beta,\kappa+1}(q) = \sum_{n_1 \geq 2 \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\kappa+1)(n_1 + \cdots + n_k)}}{(q)_{n_1 - n_2} \cdots (q)_{n_k - n_k - n_{k-1}} (q)_{n_k}}$$  

$$\times \prod_{\mu=1}^{\infty} \left[ 2n_k + \kappa - \mu + 1 - 2(n_1 + \cdots + n_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\mu} \right] q^\nu.$$  

\footnote{There exists an extensive literature devoted to this topic but not discussed in this paper [16, 17, 18, 19].}
\[
\chi_{l,k,k+1+1}^{(2k+1),(2k+1)(k+1)+2}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\kappa-l)(n_1 + \cdots + n_k)}}{q^{n_1^2 + \cdots + \nu_{\lambda-1}^2 + \nu_{\lambda} + \cdots \nu_{\kappa-1}^2}} \\
\times \prod_{\mu=1}^{\kappa-1} \left[ 2n_k + \kappa - l + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_{\mu} - \nu_{\mu+1} - \alpha_\mu^e \right] \right]
\]

(1.8)

Setting \( k = 1, l = \kappa \) and \( 2n_1 = m_1, 2\nu_1 = m_1 - m_2, \cdots, 2\nu_{\kappa-1} = m_{\kappa-1} - m_\kappa \) in (1.7), we reproduce the corresponding expressions in [1].

**Theorem 1.3** The following identities hold:

\[
\frac{1}{\langle q \rangle} \sum_{n=-\infty}^{n=\infty} \left( q^{2(2k+1)}((2k+1)k+2k-1)n^2 + (c((2k+1)k+2k-1)(\kappa+l-1)(2k+1)n - q^{((2k+1)n+1)((2k+1)k+2k-1)n+(k+l-1)} \right) \\
= \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\kappa-l)(n_1 + \cdots + n_k)} \times \prod_{\nu=0}^{\kappa-1} 2^{2n_k + \kappa - l + 1 - 2(\nu_1 + \cdots + \nu_{\nu-1}) - \nu_{\nu} - \nu_{\nu+1} - \alpha_\nu^e}}{\langle q \rangle} \\
\times \prod_{\mu=1}^{\kappa-1} \left[ 2n_k + \kappa - l + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_{\mu} - \nu_{\mu+1} - \alpha_\mu^e \right] \right]
\]

(1.9)

As a corollary we have

**Corollary 1.4**

\[
\chi_{l,k,k+1}^{(2k+1),(2k+1)(k+1)-2}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\kappa-l)(n_1 + \cdots + n_k) + 2n_1 + 2n_k + \cdots + \nu_{\lambda-1}^2 + \nu_{\lambda} + \cdots \nu_{\kappa-1}^2 \times \prod_{\nu=0}^{\kappa-1} 2^{2n_k + \kappa - l + 1 - 2(\nu_1 + \cdots + \nu_{\nu-1}) - \nu_{\nu} - \nu_{\nu+1} - \alpha_\nu^e}}} {q^{n_1^2 + \cdots + \nu_{\lambda-1}^2 + \nu_{\lambda} + \cdots \nu_{\kappa-1}^2}} \\
\times \prod_{\mu=1}^{\kappa-1} \left[ 2n_k + \kappa - l + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_{\mu} - \nu_{\mu+1} - \alpha_\mu^e \right] \right]
\]

(1.11)
\[ \chi_{i, k,(k+1)\ldots}^{(2k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2 + 2n_k^2 + (e-1)(n_1 + \cdots + n_{k-1} + 2n_k)} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+\kappa+1} \times \sum_{\nu_1 \geq \cdots \geq \nu_{k-1} \geq 0} q^{e_1^2 + \cdots + e_{k-1}^2 + \nu_1 - (e-1)} \times \prod_{\mu=1}^{k-1} 2n_k + \kappa - \nu_1 + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - \alpha_{\kappa-\mu}^k \] (1.12)

2 The Andrews–Bailey construction revisited

This section is devoted to a review of a number of definitions and propositions from the pioneering work of Bailey [11] and its extensions by Andrews [12].

2.1 A Bailey pair

Definition 2.1 Let \( \alpha = \{\alpha_n(a,q)\}_{n \geq 0} \) and \( \beta = \{\beta_n(a,q)\}_{n \geq 0} \) be sequences of functions in \( a \) and \( q \). They form a Bailey pair relative to \( a \), if they satisfy the relation

\[ \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(a,q)_{n+r}}. \] (2.1)

Note that (2.1) has the form \( \beta = M\alpha \), where \( \alpha = \mathcal{f}(\alpha_0, \alpha_1, \alpha_2, \ldots) \), \( \beta = \mathcal{f}(\beta_0, \beta_1, \beta_2, \ldots) \); and that \( M \) is an invertible matrix of infinite size, because it is a lower triangular matrix with non-zero diagonal elements. Using an identity of \( q \)-hypergeometric series [20, eq.(1.4.3)], Andrews [12] proved the following proposition

Proposition 2.1 A pair \( (\alpha, \beta) \) is a Bailey pair relative to \( a \) if and only if

\[ \alpha_n = 1 - a q^{2n} \frac{(a)_{n+r} (-1)^{n-r} q^{-r}}{(q)_{n-r}} \beta_r, \] (2.2)

Note that the RHS of (2.2) has no singularity at \( a = 1 \).

2.2 A dual Bailey pair

Given a Bailey pair, Andrews [12] proposed a method to construct a new Bailey pair as follows:

Proposition 2.2 If \( \alpha = \{\alpha_n(a,q)\}_{n \geq 0} \) and \( \beta = \{\beta_n(a,q)\}_{n \geq 0} \) form a Bailey pair, Then the sequences \( A = \{A_n(a,q)\}_{n \geq 0} \) and \( B = \{B_n(a,q)\}_{n \geq 0} \) defined by

\[ A_n(a,q) = a^n q^n \alpha_n(a^{-1}, q^{-1}), \] \[ B_n(a,q) = a^{-n} q^{-n} \beta_n(a^{-1}, q^{-1}), \] (2.3)

form another Bailey pair relative to \( a \).
One can prove Proposition (2.2) by substituting \( \alpha_n(a^{-1}, q^{-1}) \), \( \beta_n(a^{-1}, q^{-1}) \) into (2.1). We refer to \((A, B)\) as the dual of \((\alpha, \beta)\) and vice versa.

2.3 A Bailey chain

Using Saalschütz theorem [20, eq.(1.7.2)], one can prove the celebrated Bailey’s lemma [11]:

**Lemma 2.3** Let \( \alpha \) and \( \beta \) be a Bailey pair relative to \( a \), and set

\[
\alpha'_n = a^n q^n \alpha_n, \quad \beta'_n = \sum_{r=0}^{n} \frac{a^r q^{r^2}}{(q)_{n-r}} \beta_r.
\]

Then \( \alpha' \) and \( \beta' \) is also a Bailey pair relative to \( a \).

Andrews [12] proposed a prescription to construct an infinite sequence of Bailey pairs starting from a given pair using Bailey’s lemma. Given a Bailey pair \((\alpha^{(0)}, \beta^{(0)})\), one can obtain another Bailey pair \((\alpha^{(1)}, \beta^{(1)})\) by applying (2.4). Repeating this procedure, one can obtain \((\alpha^{(k)}, \beta^{(k)})\) inductively from \((\alpha^{(k-1)}, \beta^{(k-1)})\). The resulting infinite sequence of pairs is called a Bailey chain.

Substituting \( \alpha^{(k)} \) and \( \beta^{(k)} \) into the defining relation (2.1) to get

\[
\sum_{r=0}^{n} \frac{a^r q^{r^2}}{(q)_{n-r}(aq)_{n+r}} = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_k=0}^{n_{k-1}} \frac{a^{n_1+\cdots+n_k} q^{n_1^2+\cdots+n_k^2}}{(q)_{n-n_1}(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}} \beta^{(0)}_{n_k},
\]

and taking the limit \( n \to \infty \), Andrews [12] obtained the following identity:

**Corollary 2.4** Let \((\alpha, \beta)\) be a Bailey pair relative to \( a \). The following holds:

\[
\frac{1}{(aq)_{\infty}} \sum_{n=0}^{\infty} a^k q^n \alpha_n = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{a^{n_1+\cdots+n_k} q^{n_1^2+\cdots+n_k^2}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}} \beta^{(0)}_{n_k}.
\]

The following remark is the central to this work: for certain \( \alpha_n \), we will be able to rewrite the LHS of (2.5) in the form of (1.1), thus obtaining a \( q \)-series identity of the type we wish to prove.

2.4 A Bailey lattice

In a Bailey chain, the parameter \( a \) remains constant throughout the chain. In [13, 14], yet another extension of Bailey’s lemma was presented, that allows one to vary \( a \). The resulting structure is called a Bailey lattice. Using \( q \)-hypergeometric series identities [20, eqs.(1.7.2),(2.2.4) and (2.3.4)] one can prove the following Proposition [13, 14]:

**Proposition 2.5** Let \( \alpha \) and \( \beta \) be a Bailey pair relative to \( a \) and set

\[
\alpha'_n = \begin{cases} \alpha_n, & n = 0, \\ (1 - a) a^n q^{n^2-n} \left\{ \frac{\alpha_n}{1 - a q^{2n}} - \frac{aq^{2n-2} \beta_{n-1}}{1 - a q^{2n-2}} \right\}, & n > 0, \end{cases}
\]

\[
\beta'_n = \sum_{r=0}^{n} \frac{a^r q^{r^2-r}}{(q)_{n-r}} \beta_r.
\]

Then \( \alpha' \) and \( \beta' \) is also a Bailey pair relative to \( a q^{-1} \).
Using the above extension, we can construct a new Bailey pair \((\alpha^{(k)}(aq^{-1}, q), \beta^{(k)}(aq^{-1}, q))\) from a given pair \((\alpha^{(0)}(a, q), \beta^{(0)}(a, q))\) as follows. In the first \(k - i - 1\) steps, we transform \((\alpha^{(0)}(a, q), \beta^{(0)}(a, q))\) by (2.4) as before. At the \((k - i)\)th step, we use (2.6). After that, we transform \((\alpha^{(k-i)}(aq^{-1}, q), \beta^{(k-i)}(aq^{-1}, q))\) by (2.4) \(i\) times. Substituting \((\alpha^{(k)}(aq^{-1}, q), \beta^{(k)}(aq^{-1}, q))\) into the defining relation (2.1), and taking the limit \(n \to \infty\), we obtain the following Corollary [13].

**Corollary 2.6** Let \((\alpha, \beta)\) be a Bailey pair. Then the following identity holds:

\[
\sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{a^{n_1 + \cdots + n_k} q^{n_1^2 + \cdots + n_k^2 - n_1 - \cdots - n_k} \beta_{n_k}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-1} - n_k}} = \frac{1}{(a)_{\infty}} \left\{ \alpha_0 + (1 - a) \sum_{n=1}^{\infty} \left[ \frac{\alpha^{(k-n)}(aq^{-1}, q)}{1 - a q^{2n}} - \frac{\alpha^{(k-n-1)}(aq^{-1}, q)}{1 - a q^{2n-2}} \right] \right\}. \tag{2.7}
\]

**2.5 Example**

As a starting point, let us choose the following pair of sequences [13]

\[
\alpha^{(0)}_n = \begin{cases} 
1, & n = 0, \\
(-1)^n q^{\frac{n^2}{2}} \frac{1 - a q^{2n}}{a (q)_{n}}, & n > 0, 
\end{cases} \quad \beta^{(0)}_n = \delta_{n0}. \tag{2.8}
\]

From Proposition 2.1 \((\alpha^{(0)}, \beta^{(0)})\) is a Bailey pair. In this example the dual Bailey pair is the original one itself.

Consider the Bailey chain \(\{(\alpha^{(k)}(\beta^{(k)}))\}_{k \geq 0}\), for \(a = q^j (j = 0, 1)^2\). Applying Corollary 2.4 to \((\alpha^{(0)}, \beta^{(0)})\), we obtain

\[
\frac{1}{(q)_{\infty}} \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (-1)^r q^{r(5r+1)/2} = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + j(n + \cdots + n_k + 1)}}{(q)_{n_1 - n_2} \cdots (q)_{n_k - 1}}. \tag{2.9}
\]

Setting \(k = 1\), (2.9) reduces to Euler’s pentagonal numbers theorem [22]

\[
(q)_{\infty} = 1 + \sum_{r=1}^{\infty} (-1)^r (q^{r(3r-1)/2} + q^{r(3r+1)/2}); \tag{2.10}
\]

and setting \(k = 2\), (2.9) reduces to the Rogers–Ramanujan identity

\[
\frac{1}{(q)_{\infty}} \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (-1)^r q^{r(5r+1)/2} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_{n}}. \tag{2.11}
\]

Applying Corollary 2.6 to \((\alpha^{(0)}, \beta^{(0)})\) for \(a = q\) and replace \(i\) by \(i - 1\), we obtain Gordon’s generalization of Rogers–Ramanujan identity [8]

\[
\frac{1}{(q)_{\infty}} \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} (-1)^r q^{r(2k+1)^2 + 2k - 2l + 1)/2} = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + j(n + \cdots + n_k)}}{(q)_{n_1 - n_2} \cdots (q)_{n_k - 1}}, \tag{2.12}
\]

where \(1 \leq i \leq k\). By comparing with (1.4) one can see that the LHS and RHS of (2.12) are the bosonic and fermionic representations of \(\chi_{1,i}^{(2k+1)}(q)\), respectively.

\footnote{When \(j = 0\) (resp. \(j = 1\)) the pair \((\alpha^{(1)}, \beta^{(1)})\) coincides B(1) (resp. B(3)) in Slater’s table [21].}
3 Proofs of $q$-series identities

3.1 Polynomial identity

Our starting point is the following polynomial identity which implies Gordon’s generalization of the Rogers–Ramanujan identities (2.12):

**Proposition 3.1** Let $\nu, \kappa, \iota$ be fixed non-negative integers such that $1 \leq \iota \leq \kappa$. Then the following polynomial identities hold.

\[
\sum_{\rho=0}^{\infty} (-1)^{\rho} q^{\rho (2k+1) \rho + 2k \iota - 2k\iota + 1}/2} / \left( \prod_{\rho=0}^{\infty} \right) q^3 \nu^2 \prod_{\mu=1}^{\kappa-1} \left[ \nu - 2(\nu_1 + \ldots + \nu_{\mu-1}) - \nu_\mu + 1 \right] \left( q^{\nu_\mu} - q^{\nu_\mu+1} \right) \right] \bigg|_{q}.
\]

Here, $[x]$ is the greatest integer part of $x$.

\[
\left[ \begin{array}{c} N \\ M \end{array} \right]_q = \begin{cases} \frac{(q)_N}{(q)_M(q)_N-M}, & \text{if } 0 \leq M \leq N, \\ 0, & \text{otherwise.} \end{cases}
\]

is a $q$-binomial coefficient, and

\[
\alpha_{\nu}^{(\kappa)} = \begin{cases} 0, & \text{if } 1 \leq \mu \leq \iota - 1, \\ \mu - \iota + 1, & \text{if } \iota \leq \mu \leq \kappa - 1. \end{cases}
\]

The proof is given in [6]. For $\kappa = 1, 2$, this identity appears in [8].

3.2 An identity for $\mathcal{M}(2\kappa + 1, (2\kappa + 1)\iota + 2)$

Set $\nu = 2n + \kappa - \iota$ and divide (3.1) by $(q^{\kappa-\iota+1})_{2n}$. Then we have

\[
\sum_{r=1}^{\infty} \frac{(q)_{n-2(2k+1)r+2k\iota-2k\iota+1}}{(q)_{2k-1})((2\kappa+1)r)}} q^{(4k+2)r+2k\iota-2k\iota+1} \nu^{(4k+2)r} + \sum_{r=0}^{\infty} \frac{(q)_{n-2k-\iota+1}}{(q)_n} q^{(4k+2)r-2k\iota+1} \nu^{(4k+2)r} \\
- \sum_{r=1}^{\infty} \frac{(q)_{n-2k-\iota+1}}{(q)_n} q^{(2k+1)r} \nu^{(2k+1)r} - \sum_{r=0}^{\infty} \frac{(q)_{n-2k-\iota+1}}{(q)_n} q^{(2k+1)r+1} \nu^{(2k+1)r+1} \\
= \frac{1}{(q^{\kappa-\iota+1})_{2n}} \sum_{\nu_1 + \ldots + \nu_{\kappa-1} = 0}^{\kappa-1} q^{\nu_1^2 + \ldots + \nu_{\kappa-1}^2 - \nu_\mu + 1} \prod_{\mu=1}^{\kappa-1} \left[ \nu - 2(\nu_1 + \ldots + \nu_{\mu-1}) - \nu_\mu + 1 \right] \left( q^{\nu_\mu} - q^{\nu_\mu+1} \right) \bigg|_{q}.
\]

\[
\times \prod_{\mu=1}^{\kappa-1} \left[ \nu - 2(\nu_1 + \ldots + \nu_{\mu-1}) - \nu_\mu + 1 \right] \left( q^{\nu_\mu} - q^{\nu_\mu+1} \right) \bigg|_{q}.
\]

\[
\times \prod_{\mu=1}^{\kappa-1} \left[ \nu - 2(\nu_1 + \ldots + \nu_{\mu-1}) - \nu_\mu + 1 \right] \left( q^{\nu_\mu} - q^{\nu_\mu+1} \right) \bigg|_{q}.
\]
From the above we can read the following Bailey pair relative to \( q^{l-1} \):

\[
\alpha_n = \begin{cases} 
1, & n = 0, \\
q^n((4\kappa+2)r+2\kappa+2l-2) & n = (2\kappa+1)r > 0, \\
q^n((4\kappa+2)r-2\kappa-2l+1) & n = (2\kappa+1)r - \kappa + \ell > 0, \\
-q^\ell((2\kappa+1)r-\kappa) & n = (2\kappa+1)r - \kappa > 0, \\
-q^\ell((2\kappa+1)r+\kappa+1) & n = (2\kappa+1)r + \kappa > 0, \\
0, & \text{otherwise},
\end{cases}
\]

(3.2)

\[
\beta_n = \frac{1}{(q^{\ell-1})_n q} \sum_{\nu_1 \geq \cdots \geq \nu_n = 0} q^{\nu_1^2 + \cdots + \nu_n^2 + \nu_1 + \cdots + \nu_n} \times 
\prod_{\ell = 1}^{\kappa-1} \left[ 2n + \kappa - \ell - 2(\nu_1 + \cdots + \nu_{\kappa-1}) - \nu_\ell - \nu_{\ell+1} - \alpha(\kappa) \right] q^n.
\]

(3.3)

Noting that

\[
\sum_{n=0}^{\infty} (q^{\ell-1})_n q^n \Delta_n = 1 + \sum_{r=1}^{\infty} (q^{\ell-1})_r (2(2\kappa+1)r+2\kappa+1) q^r 2(2\kappa+1)r+2\kappa+1
\]

\[
+ \sum_{r=1}^{\infty} (q^{\ell-1})_r ((2\kappa+1)r-\kappa+1) q^r ((2\kappa+1)r-\kappa+1) q^r \times
\]

\[
- \sum_{r=1}^{\infty} (q^{\ell-1})_r ((2\kappa+1)r-\kappa) q^r ((2\kappa+1)r-\kappa) q^r \times
\]

\[
- \sum_{r=1}^{\infty} (q^{\ell-1})_r ((2\kappa+1)r+\kappa+1) q^r ((2\kappa+1)r+\kappa+1) q^r
\]

\[
= \sum_{r=1}^{\infty} (q^{\ell-1})_r ((2\kappa+1)r+2\kappa+1) q^r ((2\kappa+1)r+2\kappa+1)
\]

\[
= (q)_{\infty} (2\kappa+1) (2\kappa+2) (q),
\]

and applying Corollary 2.4 to the Bailey pair (3.2-3.3), we obtain the following fermionic expression for the Viraoro character:

\[
\chi_{(2\kappa+1),(2\kappa+1)+2}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\kappa-1)(n_1+\cdots+n_k)}}{q^{n_1^2 + \cdots + n_k^2 + (\kappa-1)+n_k}} \times
\]

\[
\prod_{\ell = 1}^{\kappa-1} \left[ 2n_k + \kappa - \ell - 2(\nu_1 + \cdots + \nu_{\kappa-1}) - \nu_\ell - \nu_{\ell+1} - \alpha(\kappa) \right] q^n.
\]

(3.4)

Setting \( k = 1, \ell = \kappa \) and \( 2n_1 = m_1, 2n_2 = m_2, \cdots, 2n_{\kappa-1} = m_{\kappa-1} - m_\kappa \) in (3.4), we reproduce the corresponding expressions in [1].

Substituting \( \nu = 2n + \kappa - \ell + 1 \) into (3.1) and dividing by \((q^{\ell-1+2})_{2n}\) we obtain another Bailey pair relative
to \( q^{n-i+1} \)

\[
\alpha_n = \begin{cases} 
1, & n = 0, \\
q^n(4\kappa+2)r+2\kappa-2n+1, & n = (2\kappa+1)r > 0, \\
q^n((4\kappa+2)r-2\kappa+2n-1), & n = (2\kappa+1)r - \kappa + i - 1 > 0, \\
-q^{2n+1}(2\kappa+1)+i, & n = (2\kappa+1)r + i, \\
-q^{2n-1}(2\kappa+1)+i, & n = (2\kappa+1)r - \kappa - 1 > 0, \\
0, & \text{otherwise},
\end{cases}
\] (3.5)

\[
\beta_n = \frac{1}{(q^{-i+2})_{2m}} \sum_{\nu_1 \geq \cdots \geq \nu_{2m+1} \geq 0} q^{\nu_1^2 + \cdots + \nu_{2m+1}^2 + \nu_1 + \cdots + \nu_{2m+1}} \\
\times \prod_{\mu=1}^{\kappa-1} \left[ 2n_k + \kappa - i + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_{\mu} - \nu_{\mu+1} - \alpha_{\nu_{\mu}}^i(q) \right].
\] (3.6)

Applying Corollary 2.4 to this Bailey pair we obtain the following fermionic expression for the Virasoro character:

\[
\chi_{i,k}(2\kappa, (2\kappa+1)k+1) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + \kappa - i + 1}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k + \kappa - i + 1}} \\
\times \prod_{\mu=1}^{\kappa-1} \left[ 2n_k + \kappa - i + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_{\mu} - \nu_{\mu+1} - \alpha_{\nu_{\mu}}^i(q) \right].
\] (3.7)

Furthermore, applying Corollary 2.6 to this pair (3.5–3.6) we get

\[
\sum_{j=0}^{i} q^{j(\kappa-i+1)} \chi_{i,(\kappa-i)k+1-i+2j} = \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2 + (\kappa-i)(n_1+\cdots+n_k)+(\kappa-i+1)(n_{k+1}+\cdots+n_k)}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k + \kappa - i + 1}} \\
\times \prod_{\mu=1}^{\kappa-1} \left[ 2n_k + \kappa - i + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_{\mu} - \nu_{\mu+1} - \alpha_{\nu_{\mu}}^i(q) \right].
\] (3.8)

where \( 0 \leq i \leq k \). Notice that the LHS of the above equation is a weighted sum of Virasoro characters. Extracting single character from such an equation is beyond the scope of this work.

### 3.3 An identity for \( \mathcal{M}(2\kappa+1, (2\kappa+1)k+2\kappa-1) \)

The dual Bailey pair to (3.2–3.3) is

\[
A_n = \begin{cases} 
1, & n = 0, \\
q^{r((2\kappa+1)(2\kappa+1)r+2\kappa+1)(\kappa+i+1)+2 \kappa+1)}, & n = (2\kappa+1)r > 0, \\
q^{r((2\kappa+1)(2\kappa+1)r-2\kappa+1)(\kappa+i+1)+2 \kappa+1)}, & n = (2\kappa+1)r - \kappa + i > 0, \\
-q^{(2\kappa+1)(2\kappa+1)r-2\kappa+1),} & n = (2\kappa+1)r - \kappa > 0, \\
-q^{(2\kappa+1)(2\kappa+1)r+2\kappa+1)}, & n = (2\kappa+1)r + i > 0, \\
0, & \text{otherwise},
\end{cases}
\] (3.9)
\[ B_n = \frac{q^{2\nu_1 + \cdots + \nu_{\mu-1}}}{(q^{\nu_1+\cdots+\nu_{\mu-1}})^{2n}} \sum_{v_0 = \nu_1 + \cdots + \nu_{\mu-1} + \nu_{\mu+1}}^{\nu_0 \leq \nu_1 + \cdots + \nu_{\mu-1} + \nu_{\mu+1}} q^{v_0^2 + \cdots + v_{\mu+1}^2 - v_0 (2m + \kappa - \iota)} \times \]
\[
\prod_{\mu=1}^{\kappa-1} \left[ 2n + \kappa - \iota - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - a_{i_\mu}^j(\kappa) \right]_{q} \].

In order to obtain \( B_m \), we use

\[
\frac{N + M}{M} \Bigg|_{q^{-1}} = q^{-NM} \frac{N + M}{M} \Bigg|_{q},
\]

and

\[
\sum_{\mu=1}^{\kappa-1} (\nu_\mu - \nu_{\mu+1})(2(\nu_1 + \cdots + \nu_\mu) + a_{i_\mu}^j(\kappa)) = 2 \sum_{\mu=1}^{\kappa-1} \nu_\mu^2 + \sum_{\mu=1}^{\kappa-1} \nu_\mu.
\]

From

\[
\frac{1}{(q^{\kappa-2})^{kn}} = \chi_{k,k+\kappa-1}(2x+1,(2x+1)k+2x-1)(q),
\]

and Corollary 2.4, we obtain

\[
\chi_{k,k+\kappa-1(2x+1,(2x+1)k+2x-1)}(2x+1,(2x+1)k+2x-1)(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} (q^{\nu_1 + \cdots + \nu_k + \nu_{\mu+1} - \nu_0})^{2n_1 + \cdots + \nu_{\mu+1} - \nu_0} \times \]
\[
\prod_{\mu=1}^{\kappa-1} \left[ 2n_k + \kappa - \iota - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - a_{i_\mu}^j(\kappa) \right]_{q}.
\]

Substituting \( \nu = 2m + \kappa - \iota + 1 \) into (3.1) and dividing by \( (q^{\kappa-2})^{2n} \), we obtain another Bailey pair relative to \( q^{x+1} \)

\[
A_n = \begin{cases} 
1, & n = 0, \\
q^{(2k+1)(2k+2)(k+1)}(q^{\kappa-2})^{2n} & n = (2k + 1)r > 0, \\
q^{(2k-1)(2k+1)(k+1)}(q^{\kappa-1})^{2n} & n = (2k + 1)r + \kappa - \iota + 1 > 0, \\
-q^{(2k-1)(2k+1)(k+1)}(q^{\kappa-2})^{2n} & n = (2k + 1)r + \iota, \\
-q^{(2k-1)(2k+1)(k+1)}(q^{\kappa-2})^{2n} & n = (2k + 1)r - \kappa - 1 > 0, \\
0, & \text{otherwise}, 
\end{cases}
\]

\[
B_n = \frac{q^{n_2 + (\kappa - \iota + 1)n}}{(q^{\kappa-2})^{2n}} \sum_{v_0 \leq v_{\mu+1}}^{v_0 = \nu_1 + \cdots + \nu_{\mu+1}} q^{v_0^2 + \cdots + v_{\mu+1}^2 - v_0 (2n + \kappa - \iota)} \times \]
\[
\prod_{\mu=1}^{\kappa-1} \left[ 2n + \kappa - \iota + 1 - 2(\nu_1 + \cdots + \nu_{\mu-1}) - \nu_\mu - \nu_{\mu+1} - a_{i_\mu}^j(\kappa) \right]_{q}.
\]

Applying Corollary 2.4 to this Bailey pair we obtain the following fermionic expression for the Virasoro
\[
\chi_{\lambda',\lambda+1}^{(2x+1),(2x+1)k+2x-1}(q) = \sum_{n_1 \geq 2} \cdots \sum_{n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+2n_1^2+(\kappa-1)(n_1+\cdots+n_k-1)+2n_k}}{(q)n_1 \cdots (q)n_k} \times
\prod_{\nu_1^2+\cdots+\nu_\mu^2=2} \left[ \frac{\nu_1^2+\cdots+\nu_\mu^2+\nu_\mu}{\nu_1^2+\cdots+\nu_\mu^2+\nu_\mu} \right] \]

Furthermore, applying Corollary 2.6 to this pair (3.13–3.14) we get
\[
\sum_{j=0}^t q^{j(\kappa-i+1)} \chi_{\lambda',\lambda+1}^{(2x+1),(2x+1)k+2x-1}(q) = \sum_{n_1 \geq 2} \cdots \sum_{n_k \geq 0} \frac{q^{n_1^2+\cdots+n_k^2+2n_1^2+(\kappa-1)(n_1+\cdots+n_k-1)+2n_k}}{(q)n_1 \cdots (q)n_k} \times \prod_{\nu_1^2+\cdots+\nu_\mu^2=2} \left[ \frac{\nu_1^2+\cdots+\nu_\mu^2+\nu_\mu}{\nu_1^2+\cdots+\nu_\mu^2+\nu_\mu} \right] \]

where \(0 \leq i \leq k\).

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We would like to thank Professors G. E. Andrews and B. M. McCoy for their encouragement and interest in this work. This work is supported by the Australian Research Council.

**A Virasoro characters and Bailey pairs**

In this appendix, we list several fermionic representations obtained from Slater's table [21]. Let us begin by considering \(A\) series in her table [21]. In each case \(\alpha_0 = 1\).

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(\alpha_3)</th>
<th>(\alpha_3^{-1})</th>
<th>(\alpha_3+1)</th>
<th>(\beta_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A(1))</td>
<td>(q^{6r-2r}+q^{6r+2r})</td>
<td>(-q^{6r-2r-4r+1})</td>
<td>(-q^{6r-2r+4r+1})</td>
<td>(1/(q)_{2n})</td>
</tr>
<tr>
<td>(A(2))</td>
<td>(q^{6r+2r})</td>
<td>(q^{6r-2r})</td>
<td>(-q^{6r+2r+4r+1})</td>
<td>(1/(q^2)_{2n})</td>
</tr>
<tr>
<td>(A(3))</td>
<td>(q^{6r-2r}+q^{6r+2r})</td>
<td>(-q^{6r-2r})</td>
<td>(-q^{6r+2r+4r+1})</td>
<td>(q^n/(q)_{2n})</td>
</tr>
<tr>
<td>(A(4))</td>
<td>(q^{6r+4r})</td>
<td>(q^{6r+2r})</td>
<td>(-q^{6r+4r+4r+1})</td>
<td>(q^n/(q^2)_{2n})</td>
</tr>
<tr>
<td>(A(5))</td>
<td>(q^{3r-2r})</td>
<td>(-q^{3r^2-2r})</td>
<td>(-q^{3r^2+2r})</td>
<td>(q^{3r^2}/(q)_{2n})</td>
</tr>
<tr>
<td>(A(6))</td>
<td>(q^{3r^2+2r})</td>
<td>(-q^{3r^2+2r})</td>
<td>(-q^{3r^2+4r+4r+1})</td>
<td>(q^{3r^2+2r}/(q^2)_{2n})</td>
</tr>
<tr>
<td>(A(7))</td>
<td>(q^{3r^2-2r}+q^{3r^2+2r})</td>
<td>(-q^{3r^2-4r+4r+1})</td>
<td>(-q^{3r^2+4r+4r+1})</td>
<td>(q^{3r^2-n}/(q)_{2n})</td>
</tr>
<tr>
<td>(A(8))</td>
<td>(q^{3r^2+2r})</td>
<td>(-q^{3r^2-2r})</td>
<td>(-q^{3r^2+4r+4r+1})</td>
<td>(q^{3r^2-n}/(q^2)_{2n})</td>
</tr>
</tbody>
</table>

Andrews [12] pointed out that Slater's \(A\) series consist of four Bailey pairs and their dual pairs. In fact, \(A(1)\) and \(A(5)\) are dual, the same holds between \(A(2)\) and \(A(8)\); \(A(3)\) and \(A(7)\); \(A(4)\) and \(A(6)\).
By applying Corollary 2.4 to A(1)–A(8) we obtain

\[
\begin{align*}
A(1) & \quad \lambda_{1,k}^{(3, k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k} \\
A(2) & \quad \lambda_{1,k}^{(3, k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2 + n_{k+1}} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+1} \\
A(3) & \quad \lambda_{1,k}^{(3, k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k} \\
A(4) & \quad \lambda_{1,k}^{(3, k+2)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2 + n_{k+1}} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+1} \\
A(5) & \quad \lambda_{1,k}^{(3, k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+1} \\
A(6) & \quad \lambda_{1,k}^{(3, k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2 + n_{k+1}} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+1} \\
A(7) & \quad \lambda_{1,k}^{(3, k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2 + n_{k+1}} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+1} \\
A(8) & \quad \lambda_{1,k}^{(3, k+1)}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2 + n_{k+1}} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+1} \\
\end{align*}
\]

(A.1)

Furthermore, by applying Corollary 2.6 to A(2), we obtain the following expression

\[
\sum_{j=0}^{i} q^j \lambda_{1, 2k-i+2j+1}(q) = \sum_{n_1 \geq \cdots \geq n_k \geq 0} q^{n_1^2 + \cdots + n_k^2 + n_{k+1}} (q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k} (q)_{2n_k+1} 
\]

(A.2)

where \(0 \leq i \leq k\). We can reproduce these equation (A.2) by setting \(\kappa = 1\) in (3.8).

Let us go on the B and E series in Slater’s table [21]. In each case \(\alpha_0 = 1\).
By applying Corollary 2.4 to these Bailey pairs, we have

\[ B(1) \quad \chi^{(2,2k+3)}_{1,k+1}(q) = \sum_{n_2 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}}. \]

\[ B(2) \quad \chi^{(2,2k+3)}_{1,k}(q) = \sum_{n_2 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2+n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}}. \]

\[ B(3) \quad \chi^{(2,2k+3)}_{1,1}(q) = \sum_{n_2 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2+n_k+n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}(q)_{n_k}}. \]

\[ E(1) \quad \chi^{(2,2k+2)}_{1,k+1}(q) = \sum_{n_2 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2+n_k+n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}(q)_{n_k}(-q)_{n_k}}. \]

\[ E(3) \quad \chi^{(2,2k+2)}_{1,1}(q) = \sum_{n_2 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2+n_k+n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}(q)_{n_k}(-q)_{n_k}}. \]

\[ E(4) \quad \chi^{(2,2k+2)}_{1,k}(q) = \sum_{n_2 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_k^2+n_k+n_k+n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-1}-n_k}(q)_{n_k}(q)_{n_k}(-q)_{n_k}(-q)_{n_k}}. \]

Furthermore, by applying Corollary 2.6 to B(3), we obtain the interpolating expression between B(1) and B(3):

\[ \chi^{(2,2k+3)}_{1,i}(q) = \chi^{(2,2k+3)}_{1,2k+3-i}(q) = \sum_{n_2 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_i^2+n_i+n_i+\cdots+n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{i-1}-n_i}(q)_{n_i}(q)_{n_k}}, \]

where \(1 \leq i \leq k+1\).

From each form of \((\alpha, \beta)\), we can easily guess that E(1), E(4) and E(3) are the counterpart of B(1), B(2) and B(3). Actually, using Corollary 2.6 we get a interpolating expression between E(1) and E(3)

\[ \chi^{(2,2k+2)}_{1,i+1}(q) = \sum_{n_2 \geq \cdots \geq n_k \geq 0} \frac{q^{n_1^2 + \cdots + n_i^2+n_i+\cdots+n_k}}{(q)_{n_1-n_2} \cdots (q)_{n_{i-1}-n_i}(q)_{n_i}(-q)_{n_k}}, \]

where \(0 \leq i \leq k\).

**References**


