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Conditions**

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SMALL-AMPLITUDE SOLUTIONS OF THE SINE-GORDON EQUATION ON AN INTERVAL UNDER DIRICHLET OR NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. We give a complete classification of the small-amplitude finite-gap solutions of the Sine-Gordon (SG) equation on an interval under Dirichlet or Neumann boundary conditions. Our classification is based on an analysis of the finite-gap solutions of the boundary problems for SG equation by means of the Schottky uniformization approach.

Introduction.

The paper is devoted to analysis of the small-amplitude finite-gap solutions of the Sine-Gordon equation on an interval

$$u_{tt} = u_{xx} - \sin u, \quad 0 < x < \pi. \quad (\text{SG})$$

We consider this equation under Dirichlet or Neumann boundary conditions:

$$u(t, 0) \equiv u(t, \pi) \equiv 0 \quad (\text{D})$$

or

$$u_x(t, 0) = u_x(t, \pi) \equiv 0 \quad (\text{N})$$

The results and the proofs in (D)– and (N)–cases are parallel. So we mostly restrict ourselves to the Neumann problem and give brief reformulation of the main results for the Dirichlet problem.

The equation (SG)+(N) (as well as (SG)+(D)) defines a dynamical system in the phase-space Z of pairs $U(t, x) = (u(t, x), v = \dot{u}(t, x))$ (Z should be given some Sobolev norm $\|\cdot\|$, for example, $Z = \dot{H}^1(0, \pi) \times L_2(0, \pi)$ in the Dirichlet case). The equations (SG)+(N) and (SG)+(D) are well-known to be hamiltonian: one should supply the phase-space Z with the symplectic structure given by the 2-form ω_2 ,

$$\omega_2((u_1, v_1), (u_2, v_2)) = \int_0^\pi (u_1 v_2 - v_1 u_2) dx$$

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and consider the hamiltonian

$$\int_0^\pi \left(\frac{1}{2}(v^2 + u_x^2) + \cos u \right) dx.$$

To get an idea on qualitative behavior of solutions of (SG)+(N) let us consider the linearization of (SG):

$$u_{tt} = u_{xx} - u. \quad (1)$$

The equation (1)+(N) is a linear oscillating system with the frequencies $0^*, 1^*, 2^*, \dots$, where we denote

$$j^* = \sqrt{j^2 + 1}.$$

The solutions with frequency j^* have the form (u_j, v_j) , where

$$u_j(t, x) = I_j \sin j^*(t + \varphi_j) \cos jx, \quad I_j \geq 0.$$

Fix any $n \geq 1$ wave-numbers j ,

$$j \in \mathbf{V} = \{V_1^0, \dots, V_n^0\} \subset \mathbf{N} \cup \{0\}, \quad (2)$$

and consider superpositions (=sums) $U^n = (u^n, v^n)$ of solutions (u_j, v_j) with $j \in \mathbf{V}$, $u^n = u_1 + \dots + u_n$. They are time-quasiperiodic solutions of (1)+(N) with the frequency vector $\omega = (V_1^{0*}, \dots, V_n^{0*})$. Altogether the solutions $\tilde{U}^n$ fill the $2n$ -dimensional linear subspace E^{2n} of Z ,

$$E^{2n} := \text{span}\{(\cos V_j^0 x, 0), (0, \cos V_j^0 x) \mid j = 1, \dots, n\}. \quad (3)$$

Each solution $\tilde{U}^n$ lies in an invariant torus $T^n(I)$, where $\dim T^n(I) = n$ if all $I_j > 0$. So the space E^{2n} is foliated into invariant tori and

$$E^{2n} \simeq \mathbf{R}_+^n \times \mathbf{T}^n.$$

The equation (SG) has well-known finite-gap solutions, given by the theta-formula

$$u(t, x; \lambda, D) = 2i \log \frac{\theta(i(Vx + Wt + D + \Delta))}{\theta(i(Vx + Wt + D))}, \quad (4)$$

obtained first by Kozel and Kotlyarov [KK] and Its (see in [M]). The solution (4) defines (and is defined by) its spectral curve X which is a hyperelliptic Riemann curve with a real involution. In general any hyperelliptic curve X with a real involution determines a solution of the SG equation. Moreover, there are many connected components of the solutions corresponding to the same X , which makes a general picture rather complicated (for details see [BBEIM, EF]). The picture simplifies if we consider only small-amplitude solutions. In this case the genus g of the curve equals the number of nontrivial spectral branches of the corresponding L -operator (see [McK, EFM, BBEIM]); the branching points of X are $\{0, \infty\} \cup \{\lambda_1, \bar{\lambda}_1; \dots; \lambda_g, \bar{\lambda}_g\}$,

where $\lambda_j, \bar{\lambda}_j (j = 1, \dots, g)$ are the edges of the nontrivial spectral branches. The vector $(\lambda_1, \dots, \lambda_g) \in \mathbf{C}^g$ and $D \in \mathbf{T}^g$ are parameters of the solution.

In Section 1, following [Bo3] and [BiK], we single out among g -gap solutions (4) real-valued 2π -periodic solutions, satisfying the boundary conditions (N) or (D). Moreover, thus obtained solutions $U = (u, \dot{u})$ of (SG)+(N) form $2n$ -dimensional analytic varieties $\mathcal{T}^{2n}_\rho \subset Z$, $n = [g/2] + 1$, and similar with the solutions of the Dirichlet problem. The solutions in $\mathcal{T}^{2n}_\rho$ of the amplitude $< \rho$ form a smooth analytic manifold $\mathcal{T}^{2n}_\rho$, foliated to invariant tori of (SG)+(N):

$$\mathcal{T}^{2n}_\rho = \bigcup_X T^n(X). \quad (5)$$

The solutions with the same curve X lie in the same invariant torus. So the union in (5) is with respect to the curves X giving rise to solutions (4) which meet (N).

The tangent spaces to the manifolds $\mathcal{T}^{2n}_\rho$ at zero are exactly the spaces E^{2n} as in (3). So the spaces E^{2n} (or, equivalently, the vectors $\mathbf{V}$ as in (2)) parameterize the manifolds $\mathcal{T}^{2n}_\rho$.

Despite the fact that the solutions of the sine-Gordon equation are given by the explicit formula (4), their investigation is a serious problem due to a complicated parametrization. In the present paper to solve this problem we use an approach, suggested in [Bo1], [Bo2] and based on the Schottky uniformization of the spectral curve X . The Schottky uniformization provides a convenient coordinate system $\{\mu\}$ on the manifold formed by the curves X as in (5).

This approach allows us to investigate the manifolds $\mathcal{T}^{2n}_\rho$, including the nontrivial proof of their smoothness up to zero. Moreover, we obtain representations by convergent series for the frequencies $\lambda_j(\mu)$, the vectors $W(\mu)$ and the "Bloch-like" solutions of the linearized sine-Gordon equation.

We study also the linearization of the equation (LSG) about the solution (4) in the small-amplitude limit (Theorem 7).

In our paper [BoK2] we treat the nonlinear Klein-Gordon (NKG) equation

$$u_{tt} = u_{xx} - mu + f(u), \quad 0 < x < \pi, \quad (\text{NKG})$$

where

$$f(u) = \gamma u^3 + O(u^5), \quad \gamma > 0,$$

as the (rescaled) SG equation, perturbed by a higher order term. Using an infinite - dimensional KAM-theory [K1],[K2] it is proved that most of small- amplitude finite-gap solutions of the SG equation we study in this paper, which satisfy either Dirichlet or Neumann boundary conditions, persist in the NKG equation. The detailed description of the small-amplitude finite-gap solutions of the SG equation obtained in the present paper are crucial for this proof. In particular, critical for the perturbation technique is the nondegeneracy of the amplitude-frequency modulation for solutions forming the manifold $\mathcal{T}^{2n}_\rho$:

$$\det \partial W^n / \partial \mu |_{\mu=0} \neq 0$$

and the nonresonance properties (see [BoK2] for details), which both are proved based on the analysis of the present paper.

The similar results for the periodical finite-gap solutions of the KdV equation were obtained in [BoK1].

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1. Small-amplitude finite-gap solutions of boundary-valued problems for the Sine-Gordon equation.

We start with some basic facts from the finite-gap theory of the (SG) equation (see [McK, EF, DN, BBEIM] for the proofs and details). Let $X = \{P = (\lambda, \mu)\}$ be the hyperelliptic Riemann surface of the polynomial

$$\mu^2 = \lambda \prod_{i=1}^g (\lambda - \lambda_i)(\lambda - \bar{\lambda}_i), \quad (6)$$

where $\lambda_1, \dots, \lambda_g$ are pairwise different complex numbers from the upper half-plane $\mathbf{C}_+$ (we restrict ourself to the solutions with complex branching points because the small-amplitude finite-gap solutions we are interested in are of this type). We denote the hyperelliptic involution and the conjugation involution as follows:

$$\tau_1(\lambda, \mu) = (\lambda, -\mu), \quad \tau_2(\lambda, \mu) = (\bar{\lambda}, -\bar{\mu}).$$

Let us make on X the cut $\gamma_0 = [0, \infty)$ and the cuts γ_i , $i = 1, \dots, g$, where γ_i is a path from $\bar{\lambda}_i$ to λ_i ; let us choose the canonic basis of circles (a_i, b_i) , $i = 1, \dots, g$, on Γ in such a way that the circle a_j surrounds the cut γ_j (see Fig.1) and fix a basis of holomorphic differentials $d\omega_1, \dots, d\omega_g$ of X normalized by the conditions

$$\oint_{a_m} \omega_j = 2\pi i \delta_{mj} \quad j, m = 1, \dots, g.$$

The Riemann matrix $B = (B_{mj})$,

$$B_{mj} = \oint_{b_m} \omega_j, \quad j, m = 1, \dots, g,$$

defines the theta-function θ ,

$$\theta(z|B) = \sum \exp\left(\frac{1}{2}\langle Bm, m \rangle + \langle z, m \rangle\right).$$

This function has the matrix of periods $(2\pi i I, B)$.

The function $\sqrt{\lambda}$ is not single-valued on X . To correlate the local parameters $\sqrt{\lambda}$ at the points $\lambda = 0$ and $\lambda = \infty$ we should fix a branch of $\sqrt{\lambda}$ on X . This branch is fixed if a contour $\mathcal{L}$ on X is specified, where $\sqrt{\lambda}$ has a jump alternating its sign ($\sqrt{\lambda}$ is analytic on $X - \mathcal{L}$ and boundary values of $\sqrt{\lambda}$ at two edges of $\mathcal{L}$ differ by

a sign $\sqrt{\lambda}|_{\mathcal{L}_+} = -\sqrt{\lambda}|_{\mathcal{L}_-}$). We choose $\mathcal{L}$ to be a union (see Fig.1) of the contours surrounding the cuts γ_i , which are mapped to γ_j 's by the projection $(\lambda, \mu) \rightarrow \lambda$. Let us consider the Abelian differentials $d\Omega_\infty, d\Omega_0$ with zero a -periods and such that $d\Omega_\infty$ has the only pole in ∞ and $d\Omega_0$ has the only pole in zero:

$$\begin{aligned} d\Omega_\infty(P) &= d(\sqrt{\lambda}) & P \rightarrow \infty, \\ d\Omega_0(P) &= d\left(\frac{1}{\sqrt{\lambda}}\right) & P \rightarrow 0. \end{aligned} \quad (7)$$

We denote the b -periods of $d\Omega_\infty, d\Omega_0$ as B^∞, B^0 :

$$B_n^{\infty,0} = \int_{b_n} d\Omega_{\infty,0},$$

and define the vectors

$$V = \frac{1}{4}(B^\infty - B^0), \quad W = \frac{1}{4}(B^\infty + B^0).$$

The antiholomorphic involution τ_2 acts on the basis of the cycles and on the local parameters as follows:

$$\tau_2 a_\kappa = a_\kappa, \quad \tau_2 b_\kappa = -b_\kappa + a_\kappa, \quad \tau_2^* \sqrt{\lambda} = -\sqrt{\lambda}.$$

These relations imply

$$\tau_2^* d\Omega^\infty = -\overline{d\Omega^\infty}, \quad \tau_2^* d\Omega^0 = -\overline{d\Omega^0}$$

and prove the realvaluedness of the g -vectors V, W .

The finite-gap (theta functional) solutions of (SG) are given by the formula (4), where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_g)$, $V = V(\boldsymbol{\lambda})$, $W = W(\boldsymbol{\lambda})$; $i\Delta = i(\pi, \dots, \pi)$ is the vector of the half-periods and $D \in \mathbf{T}^g = \mathbf{R}^g / 2\pi\mathbf{Z}^g$ is the phase of the solution.

The construction just described corresponds to each vector¹ $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_g) \in \mathbf{M}^g$, where

$$\mathbf{M}^g = \{(\lambda_1, \dots, \lambda_g) \mid \lambda_j \in \mathbf{C}_+, \lambda_j \neq \lambda_k \forall j \neq k\}, \quad (8)$$

the toroidal family of the finite-gap solutions (1.4), where the phase D varies in the g -torus.

We remind that a solution $U = (u, v)$ of (SG) satisfies the Neumann boundary conditions (N) if it satisfy "even periodic" boundary conditions with the doubled period:

$$U(t, x) \equiv U(t, x + 2\pi), \quad U(t, x) \equiv U(t, -x). \quad (\text{EP})$$

Similarly $U(t, x)$ satisfies the Dirichlet boundary conditions (D) if it satisfies the "odd periodic" boundary conditions:

$$U(t, x) \equiv U(t, x + 2\pi), \quad U(t, x) \equiv -U(t, -x). \quad (\text{OP})$$

¹in fact, to each set $\{\lambda_1, \dots, \lambda_g\}$. With some abuse of notations we do not distinguish a vector $(\lambda_1, \dots, \lambda_g)$ from the set $\{\lambda_1, \dots, \lambda_g\}$.

Lemma 1. *If the set $\lambda = \{\lambda_1, \dots, \lambda_g\}$ is symmetric with respect to the inversion*

$$\lambda \rightarrow \lambda^{-1}$$

and the divisor D satisfies

$$D_j = D_{g+1-j}, \quad j = 1, \dots, g, \quad (9)$$

then the solution (4) is even. If the set λ is as above and the divisor D satisfies

$$D_j = D_{g+1-j} + \pi,$$

then the solution (4) is odd.

Proof. Since the set λ is invariant with respect to the inversion, the spectral curve X (see (6)) is invariant with respect to the holomorphic involution τ_3 , where

$$\tau_3(\lambda, \mu) = \left(\frac{1}{\lambda}, \frac{\mu}{\lambda^{g+1}} \right). \quad (10)$$

The local parameters and the basis of cycles are transformed as follows:

$$\tau_3^* \sqrt{\lambda} = \frac{1}{\sqrt{\lambda}}, \quad \tau_3 a_n = a_{g+1-n}, \quad \tau_3 b_n = b_{g+1-n},$$

which implies

$$\tau_3^* d\Omega_\infty = d\Omega_0.$$

So for the periods of $d\Omega_0$, $d\Omega_\infty$ and $d\omega$ we have

$$T_3 B^\infty = B^0, \quad T_3 B T_3 = B,$$

where $g \times g$ matrix T_3 is equal to

$$T_3 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix}.$$

The vectors V and W are antisymmetric and symmetric respectively,

$$T_3 V = -V, \quad T_3 W = W,$$

and the theta-function is symmetric,

$$\theta(T_3 z) = \theta(z) = \theta(-z).$$

So

$$u(t, -x) = 2i \log \frac{\theta(T_3 i(Vx + Wt + D + \Delta) + i(D - T_3 D))}{\theta(T_3 i(Vx + Wt + D) + i(D - T_3 D))}.$$

This function equals to $u(t, x)$ if $T_3 D = D$ and to $-u(t, x)$ if $T_3 D = D + \Delta$ (because the transformation $D \mapsto D + \Delta$ interchanges the numerator and the denominator of the logarithm's argument).

Remark. One can show that the symmetries of X, D described above, are necessary for the solution $u(x, t)$ to be even or odd. This fact can be proved using the "direct approach" [BBEIM] of constructing the finite-gap solutions.

Due to the complete analogy between (OP)- and (EP)-case in what follows we prove the results for the (N) boundary conditions only and briefly reformulate them for the (D)-case.

For a vector $\lambda \in \mathbf{M}^g$ we denote by $\lambda^{-1} \in \mathbf{M}^g$ the g -vector with the inverse components

$$(\lambda^{-1})_j = (\lambda_j)^{-1}, \quad j = 1, \dots, g.$$

The vector $\lambda \in \mathbf{M}^g$ gives rise to an (EP)-solution, if (possibly, after some renumeration of its components)

$$|\lambda_1| \leq 1, \dots, |\lambda_n| \leq 1 \quad \text{and} \quad \lambda^{-1} = \lambda, \quad (11)$$

$$(V_1, \dots, V_g)(\lambda) \in \mathbf{Z}^g, \quad (12)$$

where

$$n = n(g) = 1 + \left\lfloor \frac{g-1}{2} \right\rfloor.$$

Let $\mathbf{M}_{\text{sym}}^g$ be the set of all $\lambda \in \mathbf{M}^g$, satisfying (11). Since all λ_j 's are different,

$$|\lambda_n| = 1 \quad \forall \lambda \in \mathbf{M}^g \quad \text{if } g \text{ is odd.}$$

As we have already proved, for $\lambda \in \mathbf{M}_{\text{sym}}^g$ the vectors V, W inherit the symmetries of the surface X . We formulate this fact as

Lemma 2. *If $\lambda \in \mathbf{M}_{\text{sym}}^g$, then the vector W is symmetric and V is antisymmetric with respect to the involution T_3 (i.e. $T_3 W = W$, $T_3 V = -V$). These vectors are given by the formulas*

$$V_k = \frac{1}{4}(B_k^\infty - B_{g+1-k}^\infty), \quad W_k = \frac{1}{4}(B_k^\infty + B_{g+1-k}^\infty). \quad (13)$$

By Lemma 1, to extract from the set of even (odd) solutions (4) the solutions of (SG)+(OP) ((SG)+(EP)) we should solve the equation (12) with $\lambda \in \mathbf{M}_{\text{sym}}^g$. We start an analysis of this equation with simple small-gap limits for V and W vectors when $\lambda \in \mathbf{M}_{\text{sym}}^g$ tends to a real vector l with positive components:

$$V(\lambda) \rightarrow V^0(l), \quad W(\lambda) \rightarrow W^0(l) \quad \text{as} \quad \lambda \rightarrow l \in \mathbf{R}_+^g,$$

where

$$V_j^0(l) = V_j^0(l_j) = \frac{1}{2}(\sqrt{l_j} - \frac{1}{\sqrt{l_j}}), \quad W_j^0(l) = W_j^0(l_j) = \frac{1}{2}(\sqrt{l_j} + \frac{1}{\sqrt{l_j}}) \quad (14)$$

(see [McK], [EFM] and Theorem 3 below). From (11) we obtain the estimates for components of the limiting vector $\mathbf{l}$:

$$0 < l_1, \dots, l_n \leq 1 < l_{n+1}, \dots, l_g.$$

We suppose that all components of the vector $\mathbf{l}$ are different. Then, after unessential reordering of the first and the last n of them, we have:

$$0 < l_n < \dots < l_1 \leq 1 < l_g \dots < l_{n+1}, \quad l_j \cdot l_{g+1-j} = 1 \quad \forall j. \quad (15)$$

After this reordering the components of the vector V^0 are increasing:

$$V_n^0 < \dots < V_1^0 \leq 0 < V_g^0 < \dots < V_{n+1}^0, \quad V_j^0 = -V_{g+1-j}^0.$$

As $V^0 \in \mathbf{Z}^g$, then we obtain from (14) that

$$\begin{aligned} \{l_1, \dots, l_n\} &\in \{n\text{-vectors with components of the form} \\ &\quad (\sqrt{j^2 + 1} - j)^2 \mid j = 0, 1, 2, \dots\}, \\ \{l_{n+1}, \dots, l_g\} &\in \{(g-n)\text{-vectors with components of the form} \\ &\quad (\sqrt{j^2 + 1} - j)^2 \mid j = -1, -2, \dots\}. \end{aligned} \quad (16)$$

We shall prove below in Theorem 3 that each vector $l \in \mathbf{R}_+^g$ satisfying (15), (16) gives rise to a family of small-amplitude solutions of (SG)+(EP). As a suitable parameter for these families of solutions we choose the integer n -vector $\mathbf{V} = -(V_1^0, V_2^0, \dots, V_n^0)$, varying in the set $\mathcal{L}^g$, where

$$\mathcal{L}^g = \{\mathbf{V} = (V_1, \dots, V_n) \in \mathbf{Z}^{n(g)} \mid V_n > \dots > V_1 \geq 0, \quad V_1 = 0 \text{ iff } g \text{ is odd}\}.$$

For $\mathbf{V} \in \mathcal{L}^g$ fixed we denote

$$\mathbf{N}_n = \mathbf{N}_n(\mathbf{V}) = (\mathbf{N} \cup \{0\}) \setminus \{-V_1^0, \dots, -V_n^0\}.$$

We treat $\mathbf{V} = \{-V_1^0, \dots, -V_n^0\}$ and $\mathbf{N}_n$ as the lists of open and closed gaps of the solution (4).

By (14) and (16) components W_j^0 of the limiting vector W^0 have the form

$$W_j^0 = (V_j^0)^* = \sqrt{(V_j^0)^2 + 1}, \quad 1 \leq j \leq g.$$

Small-amplitudes solutions we are discussing now correspond to the situation when all the cuts in Fig.1 are small. Our analysis of these solutions is based on the uniformization of Riemann surfaces with small cuts $[\overline{\lambda_j}, \lambda_j]$, given by the Schottky parameters (see Section 2). Below in Theorem 3 we give the final results of this analysis. For the proofs we refer the reader to Section 2.

Theorem 3. For every $\mathbf{V} \in \mathcal{L}^g$ there exists $\rho > 0$ and real-analytic map

$$\lambda : M_\rho^C = \{\mu \in \mathbf{C}^n \mid |\mu_j| < \rho \ \forall j\} \rightarrow \mathbf{C}^g, \quad \mu \mapsto \lambda(\mu),$$

such that

a) for $\mu \in M_\rho^+ = M_\rho^C \cap \mathbf{R}_+^n$ the vector $\lambda(\mu)$ lies in $\mathbf{M}_{\text{sym}}^g \subset \mathbf{C}_+^g$ and the Riemann surface (6) with $\lambda = \lambda(\mu)$ satisfies (11), (12);

b) the maps

$$\mu \mapsto U(t, x; \lambda(\mu), D), \quad \mu \mapsto W(\lambda(\mu))$$

are analytic in M_ρ^C and $U(t, x; \lambda(0), D) \equiv 0$, $W_j(0) = W_j^0$;

c) the vector $V(\lambda(\mu))$ equals to V^0 for all μ ;

d) the matrix $\partial W / \partial \mu$ at the point $\mu = 0$ equals to

$$\partial W_j / \partial \mu_k|_{\mu=0} = \begin{cases} -16/W_j^0, & j \neq k, \\ -12/W_j^0, & j = k; \end{cases}$$

e) for $\mu = (0, \dots, \mu_j, \dots, 0)$, where $\mu_j \geq 0$,

$$U(0, x; \mu, D) = 16\sqrt{\mu_j} (\cos V_j^0 x \cos D_j, -W_j^0 \cos V_j^0 x \sin D_j) + O(\mu). \quad (17)$$

Corollary 4. The map

$$M_\rho^C \rightarrow \mathbf{C}^n, \quad \mu \mapsto (W_1, \dots, W_n)(\mu)$$

is an analytic diffeomorphism on its image, provided ρ is small enough.

Proof. We should check that $\det \partial W_j / \partial \mu_k \neq 0$ at $\mu = 0$. This determinant differs by a nonzero factor from the determinant of the matrix $m = (m_{jk})$, where $m_{jj} = 3$ and $m_{jk} = 4$ if $j \neq k$. The matrix m clearly defines an invertible linear map, so $\det m \neq 0$.

Thus, g -gap solutions $U(t, x; \mu, D)$ of (SG)+(N) analytically depend on μ, D and are parameterized by the discrete parameter $\mathbf{V} \in \mathcal{L}^g$.

Due to the symmetry relations, the vectors V, W and D are uniquely defined by their first n components (belonging to $\mathbf{R}^n$ and $\mathbf{T}^n$). With some abuse of notations we denote these n -vectors by the same symbols V, W and D .

The coordinate system (μ, D) is singular in the points, where some μ_j vanish, because for $\mu = 0$ the zone $[\lambda_j, \bar{\lambda}_j]$ shrinks to a point and the solution U does not depend on the phase D_j . This observation and asymptotics (11) hint that the functions $\{(\sqrt{2\mu_j}, D_j) \mid j = 1, \dots, n\}$ form a "good" polar coordinate system and the solution u analytically depends on the corresponding Cartesian coordinates (p, q) ,

$$p_j = \sqrt{2\mu_j} \cos D_j, \quad q_j = \sqrt{2\mu_j} \sin D_j. \quad (18)$$

Direct calculations, given in Section 3, Lemma 16, proves this conjecture:

Lemma 5. *The map*

$$\Phi_0 : D_\rho^{2n} := \{(p, q) \mid p_j^2 + q_j^2 < 2\rho \quad \forall j\} \rightarrow H_s, \quad \Phi_0(p, q)(x) = U(0, x; p, q),$$

is real-analytic for every $s \in \mathbf{N}$, and

$$\frac{\partial}{\partial p_j} \Phi_0(0) = 8\sqrt{2} (\cos V_j x, 0), \quad \frac{\partial}{\partial q_j} \Phi_0(0) = 8\sqrt{2} (0, -W_j^0 \cos V_j x). \quad (19)$$

Moreover, the map Φ_0 is odd: $\Phi_0(p, q)(x) \equiv -\Phi_0(-p, -q)(x)$.

In the lemma we denote by H_s the Sobolev space of vector-valued even periodic functions $U(x) = (u(x), v(x))$. That is,

$$H_s = \{U(x) \mid U(x) \equiv U(-x) \equiv U(x + 2\pi), \quad \int_0^{2\pi} |\partial_x^l U(x)|^2 dx < \infty \quad \forall l \leq s\}.$$

The formula (19) results from (17). The last statement of the lemma follows directly from the formula (4), since the transformation $D \mapsto D + \Delta$ interchanges the numerator and the denominator of the logarithm's argument in (4).

The following statement (with ρ small enough) is an immediate consequence of the lemma:

Corollary 6. *The set $\mathcal{T}_\rho = \Phi_0(D_\rho^{2n})$ is a $2n$ -dimensional analytic submanifold of H_s . This manifold passes through zero $0 \in H_s$ with the tangent space*

$$T_0 \mathcal{T}_\rho = E^{2n} := \text{span}\{(\cos V_j^0 x, 0), (0, \cos V_j^0 x) \mid j = 1, \dots, n\}.$$

The manifold is invariant under the flow of (SG)+(N) and is foliated by the invariant analytic tori of the form

$$\Phi_0(T^n(\mu)), \quad T^n(\mu) = \{p_j^2 + q_j^2 = 2\mu_j \geq 0 \mid j = 1, \dots, n\}.$$

The dimension of the torus $T^n(\mu)$ equals n in general case and drops by one if some μ_j vanishes.

Thus, equations (11), (12) defines an n -dimensional analytic subvariety of the g -dimensional domain $\mathbf{M}_{\text{sym}}^g$. Due to Theorem 3, this subvariety has nonempty components $\mathbf{M}_{\mathbf{V}}^g$, parametrized by the vectors $\mathbf{V}$ from $\mathcal{L}^g$. The g -gap solutions of (SG)+(N), corresponding to vectors from $\mathbf{M}_{\mathbf{V}}^g$, forms in H_s a $2n$ -dimensional variety $\mathcal{T} = \mathcal{T}^{2n}(\mathbf{V})$, diffeomorphic to $\mathbf{M}_{\mathbf{V}}^g \times \mathbf{T}^n$. The intersection of $\mathcal{T}^{2n}(\mathbf{V})$ with small enough neighborhood of zero in the phase-space forms smooth analytic manifold; its closure is a $2n$ -dimensional smooth analytic manifold $\mathcal{T}_\rho = \mathcal{T}_\rho(\mathbf{V})$, diffeomorphic to the $2n$ -dimensional polydisk D_ρ^{2n} .

Due to Corollary 6, manifold $\mathcal{T}_\rho$ is stratified as follows:

$$\mathcal{T}_\rho = \mathcal{T}_\rho^0 \cup (\cup_{g' < g} \mathcal{T}_{\rho, g'}),$$

where $\mathcal{T}_\rho^0 = \mathcal{T}^{2n}(\mathbf{V}) \cap \mathcal{T}_\rho$ is an open part of $\mathcal{T}_\rho$, filled with g -gap solutions, and analytic submanifolds $\mathcal{T}_{\rho, g'}$, are filled with $(g' < g)$ -gap solutions of (SG)+(N).

We consider equation (SG) linearized about the g -gap solution $U = (u, v)$:

$$\delta \ddot{u} = \delta u_{xx} - (\cos u(t, x)) \delta u, \quad (\text{LSG})$$

supplemented by (N) (or (D)) boundary conditions.

There is a natural way to construct solutions $\delta u(t, x)$ of (LSG): to write $u(t, x; \mu, D) \equiv u(t, x; \lambda(\mu), D)$ as a degenerate $(g+2)$ -zone solution

$$u(t, x; \mu, D) = u^{n+1}(t, x; \mu, \mu_{n+1}; D, D_{n+1})|_{\mu_{n+1}=0},$$

where u^{n+1} is a $(g+2)$ -gap solution of (SG)+(N), corresponding to a vector $\mathbf{V}^{n+1} = (\mathbf{V}, V_{n+1}^0) \in \mathcal{L}^{g+2}$ ($\mathbf{V} \in \mathcal{L}^g$ corresponds to the solution u and $V_{n+1}^0 \in \mathbf{N}_n$). After this we obtain a solution of (LSG) as

$$\lim_{\mu_{n+1} \rightarrow 0} \frac{1}{\sqrt{\mu_{n+1}}} \frac{\partial u^{n+1}}{\partial D_{n+1}}, \quad (20)$$

where the factor $\mu_{n+1}^{-1/2}$ appears in the formula because not (D_{n+1}, μ_{n+1}) but (p_{n+1}, q_{n+1}) (see (18)) form a smooth coordinate system near $\mu_{n+1} = 0$. The solution (20) depends on the choice of the phase D_{n+1} . Different solutions are parameterized by elements of the set $\mathbf{N}_n$ which enumerates the closed gaps of the solution U .

Theorem 7. *For each $j \in \mathbf{N}_n$ there exists a linear combination δu_j of the solutions (2.2) with $V_{n+1}^0 = j$ of the form*

$$\delta u_j(D, t; \mu)(x) = e^{i w_j(\mu) t} \Psi^j(W(\mu) t + D, \mu)(x),$$

where w_j and Ψ^j are analytic functions. The frequency $w_j(\mu)$ equals to the $(n+1)$ '-th component of the W -vector of the solution u^{n+1} with $\mu_{n+1} = 0$. It can be analytically extended to some complex polydisc $M_\rho^c = \{|\mu_j| < \rho\}$, where

$$|w_j(\mu) - j^*| \leq C \min(|\mu|, (1+j)^{-1}).$$

The function Ψ^j is even in (p, q) . It can be analytically extended to some domain

$$\mathcal{O}_\rho = \{(p, q) \in D_\rho^c\} \times \{x \in \mathbb{C} \mid |\operatorname{Im} x| < \rho\},$$

where

$$D_\rho^c = \{(p, q) \in \mathbb{C}^{2n} \mid |p_j|^2 + |q_j|^2 < 2\rho \quad \forall j\}.$$

The function Ψ^j can be represented as

$$\Psi^j = \cos jx + \Psi^{j0}(Wt + D, \mu)(x)$$

where

$$\Psi^{j0}(D, \mu) = \frac{1}{2}(e^{ijx} \Psi^{j1}(D, \mu)(x) + e^{-ijx} \Psi^{j1}(D, \mu)(-x)).$$

The function Ψ^{j1} is analytic in x and (p, q, x) -variables and everywhere in $\mathcal{O}_\rho$

$$|\Psi^{j1}| \leq C |\mu| (1+j)^{-1}. \quad (21)$$

Proof. See Lemma 18 in Section 2. To obtain the estimate (21) one should use the direct and inverse estimates for the norm of an analytic function in a complex strip via its Fourier coefficients (see [A] and [K1], Appendix B to Part 3).

2. Proofs: Schottky uniformization of the spectral curve and parameters of the finite-gap solutions..

Here we describe a different parametrization of the spectral curve,² which allows us to investigate the behavior of the solutions in the small-amplitude regime. This parametrization is provided by the Schottky uniformization of the spectral curve. Details of the theory one can find in [Ba, Bu, Sch] (the Schottky uniformization) and in [Bo1, Bo2] (application to investigation of the finite gap solutions).

An arbitrary hyperelliptic curve shown in Fig.1 with the symmetry τ_3 can be uniformized in the following way in the z -plane. Let $C_1, \dots, C_n$ be disjoint circles (in this section we use the notation z for the uniformizing variable) orthogonal to the real axis and intersecting it only between the points $z = 0$ and $z = 1$. The symmetry

$$\tau_3 z = \frac{1}{z} \quad (22)$$

maps C_k onto a circle C_{g+1-k} , and the symmetry

$$\tau_1 z = -z \quad (23)$$

maps C_k and C_{g+1-k} onto circles $C'_k = \tau_1 C_{g+1-k}$. These circles comprise the boundary of a $2g$ -connected domain F (see Fig. 2).

The linear transformation σ_k

$$\frac{\sigma_k z + A_k}{\sigma_k z - A_k} = -\mu_k \frac{z + A_k}{z - A_k}, \quad 0 < \mu_k < 1, \quad 0 < A_k < 1, \quad k = 1, \dots, n$$

maps the outside of the boundary circle C_k into inside of the boundary circle C'_k , $\sigma_k C_k = C'_k$ and preserves the real axis. The points A_k and $-A_k$ are the fixed points of the transformation σ_k , A_k is inside C_k . The numbers³ A_k, μ_k are uniquely determined by the circle C_k : the radius of C_k is equal

$$\frac{2A_k}{1/\sqrt{\mu_k} + \sqrt{\mu_k}}$$

and the center of C_k is given by

$$A_k \frac{1 - \sqrt{\mu_k}}{1 + \sqrt{\mu_k}}.$$

The linear transformations σ_{g+1-k} are defined by the rule

$$\sigma_{g+1-k}(z) = \frac{1}{\sigma_k(\frac{1}{z})}.$$

They map the outside of C_{g+1-k} onto inside of C'_{g+1-k} and are given by the formula

$$\frac{\sigma_{g+1-k} z + A_{g+1-k}}{\sigma_{g+1-k} z - A_{g+1-k}} = -\mu_{g+1-k} \frac{z + A_{g+1-k}}{z - A_{g+1-k}}, \quad A_{g+1-k} = A_k^{-1}, \quad \mu_{g+1-k} = \mu_k.$$

²We restrict ourselves to the case of even genus $g = 2n$.

³Note that to identify formulas of this Section with analogous formulas of [Bo1, Bo2] one should replace $\mu \rightarrow -\mu$.

The elements $\sigma_1, \dots, \sigma_g$ generate a classical Schottky group [Sch, Bu], which in this case preserves the real axis. F is the fundamental domain of G . Identifying⁴ the boundary points of F by the holomorphic transformations $\sigma_1, \dots, \sigma_g$, we get a real hyperelliptic Riemann surface of genus g , possessing an additional holomorphic involution τ_3 . As a matter of fact, τ_1 is a hyperelliptic involution of X with $2g + 2$ fixed points: $z = 0$, $z = \infty$ and the pairs of mutually conjugate points given by $\sigma_m z = -\tau_1 z$, or, equivalently,

$$(z - A_m)^2 = -\mu_m(z + A_m)^2. \quad (24)$$

Also we see that

$$\tau_2 z = -\bar{z}$$

is an anti-holomorphic involution of X . Fixed points of τ_2 comprise a real oval $i\mathbf{R}$. Another antiholomorphic involution $(\tau_1 \tau_2)z = \bar{z}$ has a fixed oval $F \cap \mathbf{R}$ (all the intervals of $F \cap \mathbf{R}$ are glued in one oval of X by the transformations $\sigma_1, \dots, \sigma_g$). The fundamental domain F is invariant with respect to τ_3 , which means that X possesses a holomorphic involution (22) with two fixed points $z = \pm 1$.

Let us choose a canonical basis of cycles of X such that a_m coincide with the C'_m positively oriented, b_m joins the points $z_m \in C_m$ and $\sigma_m z_m \in C'_m$ (see Fig. 2). This is the same basis as we used before (see Fig. 1) in Section 1.

Abelian differentials on X are well determined by the (-2) -dimensional Poincaré theta series. For the Schottky group with invariant circle (a real axis in our case) these series converge [Bu, Fo]. We denote by G_m the subgroup of G generated by σ_m . Cosets G/G_{m_1} and $G_{m_2} \setminus G/G_{m_1}$ are sets of all elements $\sigma = \sigma_{i_1}^{j_1} \dots \sigma_{i_k}^{j_k}$ such that $i_k \neq m_1$ and for $G_{m_2} \setminus G/G_{m_1}$ additionally $i_1 \neq m_2$. The following result is classical [Bu, Ba]:

Theorem 8. *The series*

$$d\omega_m = \sum_{\sigma \in G/G_m} \left(\frac{1}{z - \sigma(-A_m)} - \frac{1}{z - \sigma A_m} \right) dz$$

define holomorphic differentials normalized in the basis fixed above. The period matrix is given by

$$B_{m_1 m_2} = \sum_{\sigma \in G_{m_2} \setminus G/G_{m_1}} \log\{-A_{m_2}, A_{m_2}, \sigma(-A_{m_1}), \sigma(A_{m_1})\}, \quad m_1 \neq m_2, \quad (25)$$

$$B_{mm} = \log(-\mu_m) + \sum_{\sigma \in G_m \setminus G/G_m, \sigma \neq I} \log\{-A_m, A_m, \sigma(-A_m), \sigma A_m\},$$

where $\{\}$ denotes the cross-ratio

$$\{z_1, z_2, z_3, z_4\} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

⁴In a more invariant way X can be defined as $X = \Omega(G)/G$, where $\Omega(G)$ is the discontinuity set of G .

G is a discrete subgroup of $PLS(2, \mathbf{C})$. We denote the coefficients of $\sigma \in G$ as follows:

$$\sigma z = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2, \mathbf{C}).$$

For the generators σ_m they are equal to

$$\sigma_m \iff \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\sqrt{-\mu_m} + \frac{1}{\sqrt{-\mu_m}}) & -\frac{A_m}{2}(\frac{1}{\sqrt{-\mu_m}} - \sqrt{-\mu_m}) \\ \frac{1}{2A_m}(\sqrt{-\mu_m} - \frac{1}{\sqrt{-\mu_m}}) & \frac{1}{2}(\sqrt{-\mu_m} + \frac{1}{\sqrt{-\mu_m}}) \end{pmatrix}. \quad (26)$$

The hyperelliptic involution (23) implies a group involution

$$\sigma \rightarrow \sigma^* = \tau_1 \sigma \tau_1, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}. \quad (27)$$

Finally, to identify two descriptions of the spectral curve we construct the function $\lambda(z)$. It has double zero at $z = 0$ and double pole at $z = \infty$. Hence it is given by

$$\lambda(z) = \frac{1}{c} \sum_{\sigma \in G} [(\sigma z)^2 - (\sigma 0)^2] \quad (28)$$

with some constant c . Due to (27) the following asymptotics hold

$$\lambda \rightarrow \frac{z^2}{c}, \quad z \rightarrow \infty, \quad (29)$$

$$\lambda \rightarrow \frac{Q}{c} z^2, \quad Q = \sum_{\sigma \in G} \delta^{-4}(1 - 2\beta\gamma), \quad z \rightarrow 0.$$

To determine c let us identify the reductions (22) and (10)

$$\lambda\left(\frac{1}{z}\right)\lambda(z) = 1.$$

Using (29) we get

$$c = \sqrt{Q} > 0$$

since $\lambda(z) \rightarrow +\infty$, when $z \rightarrow \pm\infty$. The contour $\mathcal{L}$ (see Section 1) coincides with the boundary of F , therefore the branch of $\sqrt{\lambda}$ is single-valued on the fundamental domain F . At the points $z = \infty$ and $z = 0$ it behaves as follows

$$\begin{aligned} \sqrt{\lambda} &\rightarrow \frac{z}{q}, & z &\rightarrow \infty; \\ \sqrt{\lambda} &\rightarrow qz, & z &\rightarrow 0; & q^2 &= c. \end{aligned} \quad (30)$$

The asymptotics (7), (30) show that the abelian integral $\Omega^\infty = \int^z d\Omega^\infty$ is equal to

$$\Omega^\infty(z) = \frac{1}{q} \sum_{\sigma \in G} (\sigma z - \sigma 0).$$

Its periods can be calculated directly or using the reciprocity law, expressing them in terms of the normalized holomorphic differentials

$$B_m^\infty = \frac{1}{q} \sum_{\sigma \in G/G_m} [\sigma(-A_m) - \sigma A_m]. \quad (31)$$

Bellow we consider the limit of small μ 's

$$\mu_m = \varepsilon \nu_m, \quad \varepsilon \rightarrow 0, \quad m = 1, \dots, g, \quad (32)$$

with finite ν 's. and calculate all the parameters of the finite -gap solutions as the series with respect to ε . This limit corresponds to small gaps $[\lambda_i, \bar{\lambda}_i]$, since due to (24), (28)

$$|\lambda_m - \bar{\lambda}_m| \approx 8A_m^2 \sqrt{\mu_m}.$$

As it was already mentioned, since G preserves the real axis all the series in this section are absolutely convergent. For small C_m 's (which is the case (32)) these series can be rewritten as absolutely convergent series with respect to ε .

Lemma 9. *The series*

$$R_m = \sum_{\sigma \in G/G_m} (\sigma(A_m) - \sigma(-A_m)) \quad (33)$$

converges absolutely and in the limit (32) has the following leading terms

$$R_m = 2A_m + \varepsilon \sum_{l=1}^g \nu_l r_{ml} + O(\varepsilon^2), \quad (34)$$

$$r_{ml} = \frac{16A_m A_l^2}{A_m^2 - A_l^2}, \quad l \neq m; \quad r_{mm} = 0.$$

Proof. For the difference $\sigma z_1 - \sigma z_2$ one has

$$\sigma z_1 - \sigma z_2 = \frac{z_1 - z_2}{(\gamma z_1 + \delta)(\gamma z_2 + \delta)}.$$

If σ is one of the generators $\sigma_m^{\pm 1}$ and both z_1, z_2 are outside of C_m and C'_m then, using (26) we see that

$$\sigma_m^{\pm 1} z_1 - \sigma_m^{\pm 1} z_2 = (z_1 - z_2) O(\mu_m).$$

For the terms of (33) corresponding to $\sigma = \sigma_{i_1}^{j_1} \dots \sigma_{i_k}^{j_k}$ it implies

$$\sigma A_m - \sigma(-A_m) = O(\varepsilon^{|j_1| + \dots + |j_k|}).$$

The detailed estimates proving absolute convergence of the series are presented in [Ba, Bu]. To get the terms of the zero and of the first order we have to take into account only $\sigma = I$ and the generators $\sigma = \sigma_1, \sigma_1^{-1}, \dots, \sigma_g, \sigma_g^{-1}$:

$$R_m = 2A_m + \sum_{l=1}^g (\sigma_l A_m - \sigma_l(-A_m) + \sigma_l^{-1} A_m - \sigma_l^{-1}(-A_m)) + O(\varepsilon^2).$$

Direct calculation of this sum gives (34).

Lemma 10. *The following asymptotics for Q holds*

$$Q = \sum_{\sigma \in G} \delta^{-4}(1 - 2\beta\gamma) = 1 + 32\varepsilon \sum_{k=1}^n \nu_k + O(\varepsilon^2). \quad (35)$$

Proof. . It is easy to see that the leading terms are given by the generators. Because of (26), for $\sigma = \sigma_k, \sigma_k^{-1}, \sigma_{g+1-k} = \tau_3 \sigma_k \tau_3, \sigma_{g+1-k}^{-1} = \tau_3 \sigma_k^{-1} \tau_3$ we have

$$\frac{1 - 2\beta\gamma}{\delta^4} \approx 8\mu_k,$$

which proves (35).

Two possible choices of the square root (30) for q correspond to the simple transformation of the solution

$$u(t, x) \rightarrow u(-t, -x).$$

We choose q negative. Then (35) implies

$$q^{-1} = -1 + 8\varepsilon \sum_{k=1}^n \nu_k + O(\varepsilon^2). \quad (36)$$

Lemma 11. *The components of the vectors V, W are given by absolutely convergent series ($V_m = -V_{g+1-m}, W_m = W_{g+1-m}$):*

$$V_m = -\frac{1}{4q}(R_m - R_{g+1-m}), \quad W_m = -\frac{1}{4q}(R_m + R_{g+1-m}). \quad (37)$$

The asymptotics at $\varepsilon \rightarrow 0$ are differentiable and start with the following terms:

$$V_k = \sinh a_k + \varepsilon \sum_{l=1}^n \nu_l v_{kl} + O(\varepsilon^2), \quad (38)$$

$$W_k = \cosh a_k + \varepsilon \sum_{l=1}^n \nu_l w_{kl} + O(\varepsilon^2),$$

$$v_{kk} = -8 \sinh a_k + \frac{2}{\sinh a_k}, \quad w_{kk} = -8 \cosh a_k - \frac{2}{\cosh a_k},$$

$$v_{kl} = -8 \sinh a_k \left(1 + \frac{\cosh^2 a_l}{\sinh(a_k + a_l) \sinh(a_l - a_k)} \right),$$

$$w_{kl} = -8 \cosh a_k \left(1 + \frac{\sinh^2 a_l}{\sinh(a_k + a_l) \sinh(a_l - a_k)} \right), \quad k \neq l$$

where $e^{a_k} = A_k$.

Proof. The expressions (37) follow from (31), (33) and (13). Combining together in (34) the terms with the same ν 's, we have

$$\begin{aligned} R_k - R_{g+1-k} &= [2A_k + \varepsilon \nu_k r_{k,g+1-k} + \varepsilon \sum_{\substack{l=1 \\ l \neq k}}^n \nu_k (r_{kl} + r_{k,g+1-l})] - \\ &- [2A_k^{-1} + \varepsilon \nu_k r_{g+1-k,k} + \varepsilon \sum_{\substack{l=1 \\ l \neq k}}^n \nu_l (r_{g+1-k,l} + r_{g+1-k,g+1-l})] + O(\varepsilon^2). \end{aligned}$$

After some simple calculations we get

$$R_k - R_{g+1-k} = 4 \sinh a_k + \frac{8\varepsilon \nu_k}{\sinh a_k} - 32\varepsilon \sum_{\substack{l=1 \\ l \neq k}}^n \nu_k \frac{\sinh a_k \cosh^2 a_l}{\sinh(a_k + a_l) \sinh(a_l - a_k)}.$$

Multiplying this expression by (36), we get the first line in (38). The proof of the asymptotics for W is absolutely the same.

Lemma 12. *For any $V_0(l)$ and sufficiently small parameters $\mu = (\mu_1, \dots, \mu_n)$ an analytic map*

$$\mu \rightarrow a(\mu) = (a_1(\mu), \dots, a_n(\mu))$$

exists such that $V(\mu, a(\mu)) = V_0(l)$ is independent of μ . Furthermore, W is an analytic function of μ and

$$\begin{aligned} \frac{\partial}{\partial \mu_k} W_m(\mu, a(\mu))|_{\mu=0} &= -\frac{16}{\cosh a_m}, \quad k \neq m \\ \frac{\partial}{\partial \mu_m} W_m(\mu, a(\mu))|_{\mu=0} &= -\frac{12}{\cosh a_m}. \end{aligned} \tag{39}$$

Proof. The series (38) are invertible with small ε . The equality

$$V(\mu, a) = V^0(l)$$

due to the implicit function theorem determines $a = a(\mu)$ and

$$\partial a_m / \partial \mu_k |_{\mu=0} = -\frac{1}{\cosh a_m} v_{mk}.$$

It gives

$$\frac{\partial}{\partial \mu_k} W_m(\mu, a(\mu))|_{\mu=0} = w_{mk} + \sinh a_m \frac{\partial a_m}{\partial \mu_k} = w_{mk} - v_{mk} \tanh a_m$$

and finally (39). The analyticity of $W(\mu)$ follows from the analyticity of $W(\mu, a)$.

From the general g -gap solution described by the formula (4) we arrive by a limiting procedure to the case of small-amplitude waves. Now we describe asymptotics with respect to ε and also prove facts about the solutions of the linearized equation formulated in Section 1. First we consider general small-amplitude waves of the sine-Gordon equation. We impose the symmetries on the spectral data (reduction (22) and symmetry (9) of D) later on in final formulas.

So, from now on $C_1, \dots, C_g$ are disjoint circles orthogonal to the real axis, $C'_1, \dots, C'_g$ are the circles obtained from them by the map $z \rightarrow -z$, and $A_1, \dots, A_g, \mu_1, \dots, \mu_g$ are the uniformization parameters.

In the limit (32) the circles C_n and C'_n collapse to the points A_n and $-A_n$ respectively. In this degenerate case for all non-identity mappings σ and for arbitrary $a, b \in F$ the equality $\sigma a = \sigma b$ holds. Therefore in the series (25), (33) only the terms corresponding to $\sigma = I$ do not vanish. More detailed calculations as in the proof of Lemma 9 shows that the following asymptotics hold:

$$\exp\left(\frac{1}{2}B_{nn}\right) = \sqrt{-\mu_n} \prod_{\sigma \in G_n \setminus G/G_n, \sigma \neq I} \left(\frac{A_n - \sigma A_n}{A_n - \sigma(-A_n)} \right) = i\sqrt{\varepsilon} \sqrt{\nu_n} (1 + O(\varepsilon)), \quad (40)$$

$$\begin{aligned} \exp(B_{nm}) &= \left(\frac{A_m - A_n}{A_m + A_n} \right)^2 \prod_{\sigma \in G_m \setminus G/G_n, \sigma \neq I} \left(\frac{A_m - \sigma A_n}{A_m - \sigma(-A_n)} \right)^2 = \\ &= \left(\frac{A_m - A_n}{A_m + A_n} \right)^2 (1 + O(\varepsilon)), \quad n \neq m. \end{aligned} \quad (41)$$

Lemma 13. *For any $A^0 = (A_1^0, \dots, A_g^0)$ with all components different $A_i^0 \neq A_j^0$, $A_i^0 \neq -A_j^0$ and for*

$$z \in \mathcal{D}_c \equiv \{z \mid |\operatorname{Im} z_k| < c, \quad k = 1, \dots, g\}$$

there is $\varepsilon > 0$ such that, if all $|\mu_k| < \varepsilon, k = 1, \dots, g$, the theta function $\theta(z)$ is an analytic function of $A_1, \dots, A_g$, lying in a sufficiently small neighborhood of $A_1^0, \dots, A_g^0$ and of $\mu_1, \dots, \mu_g$. Its asymptotics for $\varepsilon \rightarrow 0$ starts as follows

$$\begin{aligned} \theta(z) &= 1 + \theta'(z) + O(\varepsilon), \\ \theta'(z) &= 2i \sum_{k=1}^g \sqrt{\mu_k} \cosh z_k, \end{aligned} \quad (42)$$

where $O(\varepsilon) < C\varepsilon$ uniformly for all $z \in \mathcal{D}_c$.

Proof. When $\varepsilon \rightarrow 0$ the theta function can be represented as a rapidly convergent series with respect to ε . The formulas (40), (41) show that

$$\exp\left(\frac{1}{2}B_{kk}m_k^2\right) \sim \varepsilon^{m_k^2/2},$$

whereas B_{kl} , $k \neq l$ remain finite. This shows that the theta function analytically depends on A 's and μ 's.

The zero order term ε^0 in the expansion of $\theta(z)$ is given by $m = (0, \dots, 0) \in \mathbf{Z}^g$. To calculate the term of order $\sqrt{\varepsilon}$ we should take all $m \in \mathbf{Z}^g$ of the form $m = (0, \dots, 0, \pm 1, 0, \dots, 0)$ with only one non-vanishing component equal to ± 1 , which gives $\theta'(z)$ as above. Since the theta function is $2\pi i \mathbf{Z}^g$ periodic, z can be taken on the compact set

$$\mathcal{D}^0 = \{z \mid 0 \leq \operatorname{Re} z \leq 2\pi, \quad |\operatorname{Im} z_k| \leq c, k = 1, \dots, g\},$$

which proves the uniformity of the estimates with respect to z .

Lemma 14. *The finite-gap solution (4) of the sine-Gordon equation in the limit (32) in the first approximation represents a linear superposition of g non-interacting Fourier modes of small amplitudes*

$$u(t, x) = 8\sqrt{\varepsilon} \sum_{k=1}^g \sqrt{\nu_k} \cos(V_k x + W_k t + D_k) + O(\varepsilon). \quad (43)$$

Proof. This lemma follows immediately from the first terms of the series (42)

$$u(t, x) = 2i \log \frac{\theta(z + \Delta)}{\theta(z)} \approx 2i \log \frac{1 + \sqrt{\varepsilon} \theta'(z + \Delta)}{1 + \sqrt{\varepsilon} \theta'(z)} \approx 2i \sqrt{\varepsilon} (\theta'(z + \Delta) - \theta'(z)),$$

where $z = i(Vx + Wt + D)$.

Corollary 15. *The solution of the Neumann boundary problem in the small-amplitude case is of the form*

$$u(t, x) = 16\sqrt{\varepsilon} \sum_{k=1}^n \sqrt{\nu_k} \cos(V_k x) \cos(W_k t + D_k) + O(\varepsilon).$$

This corollary follows directly from (43) if we take $g = 2n$, impose the symmetries

$$D_k = D_{g+1-k}, \quad \nu_k = \nu_{g+1-k}, \quad k = 1, \dots, n, \quad (44)$$

explained in this Section and in Section 1, and take into account

$$V_{g+1-k} = -V_k, \quad W_{g+1-k} = W_k. \quad (45)$$

Lemma 16. *For small μ the theta function $\theta(z)$ is an analytic function of A_j , $\sqrt{\mu_j} \exp(\pm i z_j)$ and μ_j , $j = 1, \dots, g$*

Proof. The formula (40) shows that

$$\exp \frac{1}{2} B_{kk} = i \sqrt{\mu_k} \exp \frac{1}{2} \tilde{B}_{kk},$$

where $\exp \frac{1}{2} \tilde{B}_{kk}$ is an analytic function of μ and A . For the theta function this implies

$$\begin{aligned} \theta(z) &= \sum_{m \in \mathbb{Z}^g} \exp \left\{ \frac{1}{2} \sum_{k=1}^g B_{kk} m_k^2 + \sum_{k < l} B_{kl} m_k m_l + \sum_{k=1}^g m_k z_k \right\} = \\ &= \sum_{m \in \mathbb{Z}^g} \prod_{k=1}^g \left\{ (i\sqrt{\mu_k})^{m_k^2} e^{iz_k m_k} \right\} \exp \left\{ \sum_{k < l} B_{kl} m_k m_l + \frac{1}{2} \sum_k \tilde{B}_{kk} m_k^2 \right\}. \end{aligned} \quad (46)$$

Rewriting the first term in the product above as

$$(i\sqrt{\mu_k})^{m_k^2} e^{iz_k m_k} = (i\sqrt{\mu_k} e^{iz_k})^{m_k} (-\mu_k)^{\frac{m_k(m_k-1)}{2}},$$

we see that all the terms in (46) are analytic in A , μ and $\sqrt{\mu} \exp \pm iz$.

In the symmetrical case (44), (45), taking into account Lemma 12 we get the following

Lemma 17. *For small μ 's and for all t and x the solution $u(t, x, \mu_1, \dots, \mu_n, D_1, \dots, D_n)$ of the Neumann boundary problem for the sine-Gordon equation is an analytic function of*

$$\sqrt{\mu_k} e^{\pm i D_k}, \quad k = 1, \dots, n,$$

or, equivalently, of

$$p_k = \sqrt{2\mu_k} \cos D_k, \quad q_k = \sqrt{2\mu_k} \sin D_k.$$

Let us pass now to solutions of the linearized SG-equation (LSG). As it was explained in Section 1, to obtain them we should take a solution $u_{g+2}(t, x)$ of genus $g+2$ ($g=2n$), parameterized by

$$\mu_1, \dots, \mu_{n+1}, D_1, \dots, D_{n+1},$$

differentiate it by D_{n+1} and put $\mu_{n+1} = 0$. More exactly,

$$\delta u = \frac{1}{\sqrt{\mu_{n+1}}} \frac{\partial u_{g+2}(t, x)}{\partial D_{n+1}} \Big|_{\mu_{n+1}=0}$$

is a solution of the LSG-equation.

Let us denote by X_0 the limiting non-singular Riemann surface of genus g , uniformized by the Schottky group G_0 , generated by $\sigma_1, \dots, \sigma_n$. To simplify the notations we slightly change the numeration of cycles of X . The numeration of cycles and generators for X_0 is the same as above: the cycles $a_k, b_k, a_{g+1-k}, b_{g+1-k}$ correspond to σ_k (or, equivalently, to μ_k); the cycles a_{g+1}, b_{g+1} and a_{g+2}, b_{g+2} correspond to μ_{n+1} (with A_{n+1} and A_{n+1}^{-1} respectively).

Formulas (40) and the symmetry of the period matrix show that in the limit $\mu_{n+1} \rightarrow 0$ the period matrix behaves as follows:

$$B \longrightarrow \begin{pmatrix} B_0 & s & Ts \\ s^T & \log(i\sqrt{\mu_{n+1}}) + 2r & l \\ (Ts)^T & l & \log(i\sqrt{\mu_{n+1}}) + 2r \end{pmatrix}, \quad (47)$$

where B_0, s, r, l are finite, and $n \times n$ matrix T is equal to

$$T = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix}.$$

Here B_0 is the period of X_0 , s is a g -dimensional vector with the components

$$s_k = B_{k,g+1} = B_{g+1-k,g+2} = \sum_{\sigma \in G_0} \log \left(\frac{A_{n+1}^{-1} - \sigma(A_k^{-1})}{A_{n+1}^{-1} - \sigma(-A_k^{-1})} \right)^2 \quad (48)$$

and

$$r = \sum_{\sigma \in G_0, \sigma \neq I} \log \left(\frac{A_{n+1} - \sigma A_{n+1}}{A_{n+1} - \sigma(-A_{n+1})} \right).$$

Since iD_{n+1} always enters as an additive term in z_{g+1} and z_{g+2} , we have

$$\begin{aligned} \theta_{(n+1)}(z|B) &\equiv \frac{1}{\sqrt{\mu_{n+1}}} \frac{\partial}{\partial D_{n+1}} \theta(z|B) \big|_{\mu_{n+1}=0} = \\ &= \frac{i}{\sqrt{\mu_{n+1}}} \left(\frac{\partial}{\partial z_{g+1}} + \frac{\partial}{\partial z_{g+2}} \right) \theta(z|B) \big|_{\mu_{n+1}=0} = \\ &= \frac{i}{\sqrt{\mu_{n+1}}} \sum_{m \in \mathbf{Z}^{g+2}} (m_{g+1} + m_{g+2}) \exp\left(\frac{1}{2} \langle Bm, m \rangle + \langle z, m \rangle\right) \big|_{\mu_{n+1}=0}. \end{aligned}$$

In the limit (47) only the terms with $m_{g+1} = \pm 1$ or $m_{g+2} = \pm 1$ in the series above persist. So $\theta_{(n+1)}(z|B)$ converges to

$$\begin{aligned} &\frac{\exp \frac{1}{2} B_{g+1,g+1}}{\sqrt{\mu_{n+1}}} \times \\ &\left(\sum_{n \in \mathbf{Z}^g} \exp\left(\frac{1}{2} \langle B_0 n, n \rangle + \langle s, n \rangle + \langle z_0, n \rangle\right) e^{z_{g+1}} + \right. \\ &+ \sum_{n \in \mathbf{Z}^g} \exp\left(\frac{1}{2} \langle B_0 n, n \rangle - \langle s, n \rangle + \langle z_0, n \rangle\right) e^{-z_{g+1}} + \\ &+ \sum_{n \in \mathbf{Z}^g} \exp\left(\frac{1}{2} \langle B_0 n, n \rangle + \langle Ts, n \rangle + \langle z_0, n \rangle\right) e^{z_{g+2}} + \\ &+ \sum_{n \in \mathbf{Z}^g} \exp\left(\frac{1}{2} \langle B_0 n, n \rangle - \langle Ts, n \rangle + \langle z_0, n \rangle\right) e^{-z_{g+2}} \Big) = \\ &= ie^r \left((e^{z_{g+1}} \theta(z_0 + s|B_0) + e^{-z_{g+1}} \theta(z_0 - s|B_0) + \right. \\ &\left. + e^{z_{g+2}} \theta(z_0 + Ts|B_0) + e^{-z_{g+2}} \theta(z_0 - Ts|B_0) \right), \end{aligned}$$

where z_0 is a vector consisting of the first g components of

$$z = (z_0, z_{g+1}, z_{g+2})$$

Applying this formula to δu and substituting

$$\begin{aligned} z_{g+1} &= i(V_{n+1}x + W_{n+1}t + D_{n+1}), \\ z_{g+2} &= i(-V_{n+1}x + W_{n+1}t + D_{n+1}), \end{aligned}$$

we get

$$\begin{aligned} \delta u &= 4e^r (ve^{iD_{n+1}} + \bar{v}e^{-iD_{n+1}}), \\ v &= \frac{1}{2}e^{iW_{n+1}t} \left\{ e^{iV_{n+1}x} \left(\frac{\theta(z_0 + s + \Delta|B_0)}{\theta(z_0 + \Delta|B_0)} + \frac{\theta(z_0 + s|B_0)}{\theta(z_0|B_0)} \right) + \right. \\ &\quad \left. + e^{-iV_{n+1}x} \left(\frac{\theta(z_0 + Ts + \Delta|B_0)}{\theta(z_0 + \Delta|B_0)} + \frac{\theta(z_0 + Ts|B_0)}{\theta(z_0|B_0)} \right) \right\}, \end{aligned} \quad (49)$$

where $z_0 = i(Vx + Wt + D)$ and $V, W \in \mathbf{R}^g$ are determined by x_0 . Since D_{n+1} is arbitrary, v is also a solution of the LSG equation.

Lemma 18. *The solution v_j of the LSG equation with the Neumann boundary conditions corresponding to the j -th opening gap ($V_{n+1} = j$) is given by the formula*

$$v_j = e^{iw_j(\mu)t} \left(e^{ijx} F(Vx + Wt + D, \mu) + e^{-ijx} F(-Vx + Wt + D, \mu) \right), \quad (50)$$

$$F(Vx + Wt + D, \mu) = \frac{1}{2} \left(\frac{\theta(Vx + Wt + D + s + \Delta|B_0)}{\theta(Vx + Wt + D + \Delta|B_0)} + \frac{\theta(Vx + Wt + D + s|B_0)}{\theta(Vx + Wt + D|B_0)} \right).$$

For

$$x \in \mathcal{D} = \{x \mid |\operatorname{Im} x| < c\}$$

there is $\varepsilon > 0$ such that for all $|\mu_k| < \varepsilon$, $k = 1, \dots, n$,

(i) $F(Vx + Wt + D, \mu) = \tilde{F}(Vx + Wt, p_1, q_1, \dots, p_n, q_n)$, where $\tilde{F}$ is an analytic function of all $p_k = \sqrt{2\mu_k} \cos D_k$, $q_k = \sqrt{2\mu_k} \sin D_k$, invariant with respect to the transformation $p_k, q_k \rightarrow -p_k, -q_k$ for all $k = 1, \dots, n$ simultaneously, and having the following asymptotics:

$$F = 1 + O\left(\frac{\varepsilon}{j}\right), \quad \varepsilon \rightarrow 0, \quad j \rightarrow \infty, \quad (51)$$

where $O(\varepsilon/j) < C\varepsilon/j$ uniformly for all $x \in \mathcal{D}$.

(ii) $w_j(\mu)$ is an analytic function of $\mu_1, \dots, \mu_n$ with the asymptotics

$$\begin{aligned} w &= j^* + O(\varepsilon), \quad \varepsilon \rightarrow 0, \\ w_j &= j^* + O(j^{-1}), \quad j \rightarrow \infty, \end{aligned}$$

where $j^* = \sqrt{j^2 + 1}$.

Proof. . Since $T(Vx + Wt + D) = -Vx + Wt + D$ (see Section 1) the formula (49) is equivalent to (50). First we investigate the asymptotics of w_j . We have seen already (38), that when $\varepsilon \rightarrow 0$

$$V_{n+1} = \sinh a_{n+1} + O(\varepsilon), \quad W_{n+1} = \cosh a_{n+1} + O(\varepsilon),$$

where $A_{n+1} = e^{a_{n+1}}$. Combined with the equality

$$V_{n+1} = -j,$$

it implies

$$w_j(\mu) = j^* + O(\varepsilon), \quad j^* = \sqrt{j^2 + 1}.$$

The calculation of the asymptotics with $j \rightarrow \infty$ is also straightforward. Formulas (37) (we remind the special numeration for the $n+1$ -th gap) imply

$$\frac{w_{n+1}}{v_{n+1}} \rightarrow \frac{w_j}{-j} = \frac{R_{g+1} + R_{g+2}}{R_{g+1} - R_{g+2}},$$

where

$$R_{g+1} = \sum_{\sigma \in G_0} (\sigma A_{n+1} - \sigma(-A_{n+1})),$$

$$R_{g+2} = \sum_{\sigma \in G_0} (\sigma(A_{n+1}^{-1}) - \sigma(-A_{n+1}^{-1})).$$

In the limit $A_{n+1} \rightarrow 0$ we have

$$R_{g+1} = \sum_{\sigma \in G_0} \frac{d}{dz} (\sigma z - \sigma(-z))|_{z=0} A_{n+1} + O(A_{n+1}^2),$$

$$R_{g+2} = 2A_{n+1}^{-1} + O(1),$$

which yields

$$w_j = j \left(\frac{1 + R_{g+1}/R_{g+2}}{1 - R_{g+1}/R_{g+2}} \right) = j + O(j^{-1}), \quad j \rightarrow \infty.$$

The asymptotics for F follows from the simple estimate

$$\left| \frac{\theta(z + \Delta + s)}{\theta(z + \Delta)} + \frac{\theta(z + s)}{\theta(z)} - 2 \right| \leq \sum_{k=1}^g \left| \frac{\frac{\partial}{\partial z_k} \theta(z + \Delta + \xi_k)}{\theta(z + \Delta)} + \frac{\frac{\partial}{\partial z_k} \theta(z + \xi_k)}{\theta(z)} \right| |s_k|, \quad (52)$$

with some ξ_k 's in $[z + \Delta, z + \Delta + s]$. Indeed, formula (48) shows that

$$s_k = \sum_{G_0} \log \left(\frac{1 - A_{n+1} \sigma(A_k^{-1})}{1 - A_{n+1} \sigma(-A_k^{-1})} \right)^2 = O(A_{n+1})$$

for all i . Since $A_{n+1} \approx 1/2j$ we get for $|s_i|$ in (52)

$$|s_i| = O(j^{-1}). \quad (53)$$

On the other hand, formula (42) implies

$$\frac{\frac{\partial}{\partial z_k} \theta(z + \Delta + \xi_k)}{\theta(z + \Delta)} + \frac{\frac{\partial}{\partial z_k} \theta(z + \xi_k)}{\theta(z)} = \frac{2i\sqrt{\mu_k} \sinh(z_k + \xi_k + \pi) + O(\varepsilon)}{1 + 2i \sum_{l=1}^g \sqrt{\mu_l} \cosh(z_l + \pi) + O(\varepsilon)} +$$

$$+ \frac{2i\sqrt{\mu_k} \sinh(z_k + \xi_k) + O(\varepsilon)}{1 + 2i \sum_{l=1}^g \sqrt{\mu_l} \cosh(z_l) + O(\varepsilon)} = O(\varepsilon). \quad (54)$$

These estimates are uniform with respect to $x \in \mathcal{D}$ since as in the proof of Lemma 13 we can consider x varying in a compact set. Combining (52-54) we get (51).

Finally, the change $(p, q) \rightarrow (-p, -q)$ in $\tilde{F}$ is equivalent to the common shift $z \rightarrow z + \Delta$ of the arguments of all theta functions. This shift preserves F .

References.

- [A] Arnold V.I. *Proof of a theorem of A.N.Kolmogorov on the conservation of quasiperiodic motions under a small change of the Hamiltonian function*, Uspekhi Mat. Nauk, **18:5** (1963), 13-40; English transl. in Russ. Math. Surv. **18:5** (1963), 9-36.
- [Ba] Baker H.F. *Abel's theorem and the allied theory, including the theory of theta-functions*, Cambridge, Cambridge University Press (1897)
- [BBEIM] Belokolos E.D., Bobenko A.I., Enolskii V.Z., Its A.R., Matveev V.B. *Algebro-geometric approach to nonlinear integrable equations*, Springer-Verlag, Berlin (to appear)
- [BiK] Bikbaev R.F., Kuksin S.B. *A periodic boundary value problem for the Sine - Gordon equation, small Hamiltonian perturbations of it, and KAM-deformations of finite-gap tori*, Algebra i Analiz **4:3** (1992), 42-78; English transl. in St. Petersburg Math. J. **4** (1993), 439-468
- [Bo1] Bobenko A.I. *Schottky uniformization and the finite-gap integration*, Dokl. Akad. Nauk SSSR **295** (1987), 268-272; English transl. in Soviet Math. Dokl. **36** (1988), 38-42
- [Bo2] Bobenko A.I. *Uniformization of Riemann surfaces and effectivization of theta-functional formulas*, Preprint TU Berlin 257 (1990), to appear as a chapter in [BBEIM]
- [Bo3] Bobenko A.I. *Eigenfunctions of the Dirichlet and Neumann boundary value problems for the elliptic sine-Gordon equation*, Zap. Nauchn. Sem. LOMI **179** (1989), 32-36; English transl. in J. Soviet Math. **57:3** (1991)
- [BoK1] Bobenko A.I., Kuksin S.B. *Finite-gap periodic solutions of the KdV equation are nondegenerate*, Phys. Lett. A **161:3** (1991), 274-276
- [BoK2] Bobenko A.I., Kuksin S.B. *The nonlinear Klein-Gordon equation on an interval as a perturbed sine-Gordon equation*, Comm. Math. Helv. (to appear)
- [Bu] Burnside W. *On a class of automorphic functions*, Proc London Math. Soc **23** (1892), 49-88
- [EF] Ercolani N., Forest M.G. *The geometry of real sine-Gordon wavetrains*, Comm. Math. Phys. **99** (1985), 1-49
- [EFM] Ercolani N., Forest M.G., McLaughlin D. *Geometry of the modulational instability*, Part 3, Physica **43 D** (1990), 349-384
- [Fo] Ford L.R. *Automorphic functions*, McGraw-Hill, New York (1929)
- [K1] Kuksin S.B. *Nearly integrable infinite-dimensional Hamiltonian systems*, Lect. Notes Math. **1556** (1993), Springer
- [K2] Kuksin S.B. *KAM-theory for partial differential equations*, In: Proceedings of the First European Congress of Mathematics (Paris, 6-10 July 1992) (to appear)
- [KK] Kozel V.A., Kotlyarov V.P. *Quasiperiodic solutions of the equation $U_{tt} - U_{xx} + \sin U = 0$* , Dokl. Akad. Nauk Ukr. SSR **10A** (1976), 878-881 [Russian]
- [M] Matveev V. *Abelian functions and solitons*, Preprint 373 Univ. of Wroclaw (1976)

[McK] McKean H.P. *The sine-Gordon and sinh-Gordon equation on the circle*,
Comm. Pure Appl. Math. **34** (1981), 197-257

[Sch] Schottky F. *Ueber eine specielle Funktion, welche bei einer bestimmten linearen Transformation ihres Arguments unverändert bleibt*, J. Reine Angew. Math. **101** (1887), 227-272

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Fig. 1. The spectral curve with the canonical basis

Fig. 2. Schottky uniformization of the spectral curve