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WITH LONG-RANGE INTERACTION

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**$T = 0$ FINITE-SIZE SCALING FOR A QUANTUM SYSTEM
WITH LONG-RANGE INTERACTION**

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ABSTRACT

The general idea of finite-size scaling is tested by the example of the pure quantum version of an exactly solvable model of structural phase transition with long range interactions decaying at large distances r as $r^{-d-\sigma}$, where d is the space dimensionality and $\sigma > 0$. The finite-size shift of the quantum parameter (playing the same role as that of the temperature for classical systems) is calculated for the general geometry $L^{d-d'} \times \infty^{d'}$, with $0 \leq d' < d$. The root mean square order parameter is evaluated for $0 \leq d' < \sigma/2$.

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May 1994

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1- Introduction

To test the traditional idea of finite-size scaling [1] for systems with long-range interactions decaying at large distances r as $r^{-d-\sigma}$, where d is the space dimensionality and $0 < \sigma \leq 2$ is a parameter, several authors (see ref. [2] and references therein) have dealt with the spherical model. The particular case $\sigma = 2$ has been widely studied during the last decade [3-12]. It is, commonly, assumed that this one is the only model, which combines exact solvability and direct relevance to the physical reality. However the phase transition in such model occurs at temperature $T > 0$ i.e. the fluctuations considered are classical. Quantum models have also been considered [13-15].

In this paper we investigate the finite-size scaling for the pure quantum version (i.e. $T = 0$) of a model of structural phase transition with long-range interaction in the general geometry $L^{d-d'} \times \infty^{d'}$ subject to periodic boundary conditions along the $\tilde{d} = d - d'$ finite size directions (L is some scale length and $0 \leq d' < d$). Unlike the spherical model, the remarkable property of the model considered here comes from the presence of both quantum and classical critical fluctuations, depending on the temperature and the space dimensionality. The Hamiltonian of this model for any finite d -dimensional lattice $\mathbb{Z}^d = L_1 \times L_2 \times \dots \times L_d$ reads [16]

$$H = \sum_l \left(\frac{P_l^2}{2M} - \frac{A}{2} Q_l^2 \right) + \frac{1}{4} \sum_{ll'} \Phi_{ll'} (Q_l - Q_{l'})^2 + \frac{B}{4N} \left(\sum_l Q_l^2 \right)^2. \quad (1.1)$$

Where Q_l and $P_l = -i\hbar \nabla_{\mathbf{q}}$ are canonically conjugated momenta and coordinates, respectively, of the l -th atom of mass M in an hypercubical lattice of size $L \equiv N^{1/d}$. The parameter $A \equiv M\omega_0^2 > 0$ determines the frequency of a mode unstable in the harmonic approximation, and the parameter $B > 0$ "switches on" an anharmonic interaction, this being inversely proportional to the particle number N . The harmonic strength constant $\Phi_{ll'}$ is non zero only for finite numbers of neighbors, determining the range of the interaction.

In order to allow the reader to follow the results obtained throughout

this paper, we will review, briefly, the thermodynamics of system (1.1). For a review, which includes a more complete discussion and references to the model, see Ref. [17].

Using the Approximating Hamiltonian Method and some finite-size arguments, the free energy of model Hamiltonian (1.1) in the fully finite (block) geometry is [18]

$$f_{[T, \lambda, \bar{\Delta}]}^N = \frac{N}{T} \sum_q \ln 2 \operatorname{sh} \frac{\lambda q}{2T} \left[1 - \frac{8b}{(1 + \bar{\Delta})^2} \right] \quad (1.2)$$

where

$$\lambda = \hbar v_0, \quad b = \frac{B}{8A^2}$$

and in the long wave length approximation

$$\phi_z^q(\bar{\Delta}) \equiv \Delta + \frac{\lambda}{1} (\phi_0^q - \phi^q) = \Delta + |q|^a. \quad (1.3)$$

Here

$$q = \left\{ \frac{2\pi p_1}{L_1}, \dots, \frac{2\pi p_d}{L_d} \right\}, \quad p_k = 0, \pm 1, \pm 2, \dots \pmod{L_k}, \quad k = 1, \dots, d.$$

In eq. (1.2) $\bar{\Delta}$ is the solution of the self consistency equation

$$\Delta + 1 = \frac{N}{b} \sum_q \frac{\phi^q(\bar{\Delta})}{\lambda} \coth \frac{b}{2T}. \quad (1.4)$$

In the thermodynamic limit the eq. (1.4) takes the following form

$$\Delta + 1 = \frac{b\lambda}{\pi} \int_{-\pi}^{\pi} d^d q \frac{1}{\sqrt{\Delta + |q|^a}} \coth \left(\frac{\lambda}{2T} \sqrt{\Delta + |q|^a} \right), \quad (1.5)$$

and the critical properties of the model system (1.1) can be determined [19] by the asymptotic behaviour of the integral in the right-hand side, when (for a different value of the ratio λ/T) $\Delta \rightarrow 0^+$.

In the case of short-range interaction ($a = 2$) finite-size scaling for the model considered here was investigated using an approach based on integral transformation by replacing the summand in the right-hand side of eq. (1.4) by its Laplace transform, this being the reduced Jacoby θ_3 function [20]. The aim

of this approach is to reduce the problem of the evaluation of d -dimensional sum to the corresponding one-dimensional problem. In the general case of long-range interaction ($0 < a < 2$) the application of the same method presents some difficulties because we do not know the Laplace transform of the right-hand side of the eq. (1.4). So, it is hard to explore the finite-size scaling for arbitrary values of T and λ i.e. in the intermediate region between the pure classical regime and the pure quantum one. However it is of a great interest to consider the two important limits:

a) the pure classical regime ($T > 0$ and $\lambda \rightarrow 0$). In this case the right-hand side of eq. (1.3) is expressible as a Watson type function and it may be treated using the Mittag-Leffler type kernels [21]. The results here will be similar to those of the spherical model [2].

b) The pure quantum regime ($\lambda > 0$ and $T \rightarrow 0$): in this case we can just predict the critical behaviour, taking into account the existence of the classical to quantum crossover. In this paper we present how this can be done using the same technique suggested in the case of the spherical model.

This paper is divided as follows. In section 2 we derive the finite-size corrections to the self consistency equation and to the free energy in the pure quantum regime of model (1.1). Section 3 is devoted to the evaluation of the shift of the critical quantum parameter λ . In section 4 we study the fluctuations in the root mean square order parameter. A discussion of the results obtained is given in section 5.

2- Large L-asymptotic form of the self-consistency equation

In the pure quantum regime ($\lambda/T \rightarrow \infty$) the free energy eq. (1.3) takes the form

$$f^N(\lambda, \bar{\Delta}) = \frac{\lambda}{2} U_{d,a}(\bar{\Delta}) - \frac{8b}{(1 + \bar{\Delta})^2}, \quad (2.1)$$

where

$$U_{d,\sigma}(\Delta) = \frac{1}{N} \sum_q \sqrt{\Delta + |q|^\sigma}. \quad (2.2)$$

In this limit the self consistency equation (1.4) is

$$1 + \Delta = \lambda b W_{d,\sigma}(\Delta), \quad (2.3)$$

where

$$W_{d,\sigma}(\Delta) = \frac{1}{N} \sum_q \frac{1}{\sqrt{\Delta + |q|^\sigma}}. \quad (2.4)$$

The method suggested to study the finite-size scaling for the spherical model [22] may be applied here, using the identity (see Appendix)

$$(1 + z^\sigma)^{-1/2} = \int_0^\infty dx e^{-xz} x^{\sigma/2-1} G_{\sigma/2}(-x^\sigma), \quad (2.5)$$

where

$$G_{\alpha,\beta}(t) = \pi^{-1/2} \sum_{k=0}^\infty \frac{\Gamma(k + 1/2)}{\Gamma(\alpha k + \beta)} \frac{t^k}{k!}, \quad \alpha > 0, \beta > 0. \quad (2.6)$$

With the aid of (2.5) we can express $W_{d,\sigma}(\Delta)$, from (2.4), in the form

$$W_{d,\sigma}(\Delta) = \Delta^{-1/2} \int_0^\infty dx \left[\prod_{i=1}^d Q_{L_i}(x) \right] x^{\sigma/4-1} G_{\sigma/2,\sigma/4}(-x^{\sigma/2}) \quad (2.7)$$

where

$$Q_L(x) = L^{-1} \sum_{-(L-1)/2}^{(L-1)/2} \exp[-(2\pi n/L)^2 x]. \quad (2.8)$$

Here we shall consider that system (1.1) is confined to the general geometry $L^{d-d'} \times \infty^{d'}$. Then by setting $L_1 = \dots = L_d = L$ and taking the limit $L_i \rightarrow \infty$ for i denoting the dimensions $d+1, \dots, d$, we present eq. (2.7) in the form

$$W_{d,\sigma}(\Delta) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty dx [Q_L(x)]^{d-d'} [\Phi(\pi x^{1/2})]^{d'} x^{\sigma/4-d'/2-1} G_{\sigma/2,\sigma/4}(-x^{\sigma/2} \Delta) \quad (2.9)$$

with

$$\lim_{L \rightarrow \infty} Q_L(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \exp[-ax^2] = (4\pi a)^{-1/2} \Phi(\pi a^{1/2}),$$

where $\Phi(\cdot)$ is the error function.

The use of the Poisson summation formula allows one to transform eq. (2.9) into

$$W_{d,d',\sigma}(\Delta) = W_{d,\sigma}(\Delta) + \delta W_{d,d',\sigma}(\Delta), \quad (2.10)$$

where

$$W_{d,\sigma}(\Delta) = \frac{\Delta^{d/2-1/2}}{(4\pi)^{d/2}} \int_0^\infty dx [\Phi(\pi x^{1/2} \Delta^{-1/\sigma})]^d x^{\sigma/4-d/2-1} G_{\sigma/2,\sigma/4}(-x^{\sigma/2}) \quad (2.11)$$

is the bulk term determining the critical behaviour in the thermodynamic limit, and

$$\delta W_{d,d',\sigma}(\Delta) = \frac{L^{\sigma/2-d}}{(4\pi)^{d/2}} \sum'_{l(d-d')} \int_0^\infty dx \exp\left[-\frac{|l|^2}{4x}\right] x^{\sigma/4-d/2-1} G_{\sigma/2,\sigma/4}(-x^{\sigma/2} L^\sigma \Delta) \quad (2.12)$$

is the finite-size correction. In eq. (2.12) the symbol $l(d-d')$ means that the summation is over $l \in \mathbb{Z}^{d-d'}$ and the prime denotes that the term with $l=0$ is omitted. At the end it is important to notice that equation (2.10) may give us the necessary information about the finite-size scaling for the model (1.1) in the pure quantum limit.

By use of a similar procedure and the identity (see Appendix)

$$(1 + z^\sigma)^{1/2} = 1 + \frac{\sigma}{2} \int_0^\infty dx (1 - e^{-xz}) x^{-\sigma/2-1} G_{\sigma/2,1-\sigma/4}(-x^{\sigma/2}), \quad (2.13)$$

it is easy to find the expression of the finite-size correction to the free energy as

$$\delta U_{d,d',\sigma}(\Delta) = (4\pi)^{-d/2} \sum'_{l(d-d')} \int_0^\infty dx \exp\left[-\frac{|l|^2 L^2}{4x}\right] x^{\sigma/4-d/2-1} G_{\sigma/2,1-\sigma/4}(-x^{\sigma/2} \Delta). \quad (2.14)$$

3- Shift of the quantum parameter

It is known that when some sizes of the system are finite, the phase transition occurring in the system at the thermodynamic limit persists, if the

dimension d' of infinite sizes is greater than the lower critical dimension . In this case the value of the critical quantum parameter $\lambda^c(\omega)$ at which some thermodynamic functions exhibit a singularity is shifted to $\lambda^c(L)$ critical quantum parameter for a system confined to the geometry $L^{d-d'}\omega^d$. The shift is given by

$$\delta\lambda = 1/\lambda^c(L) - 1/\lambda^c(\omega) . \quad (3.1)$$

Setting $\Delta = 0$ in Eq. (2.10) we obtain the following result

$$\delta\lambda = b \, \delta W_{d',d',\omega}(0) . \quad (3.2)$$

Now let us evaluate the right-hand side of eq. (3.2). From eq. (2.12) we

obtain

$$\delta W_{d',d',\omega}(0) = \frac{1}{\int_0^\infty dx \exp\left(-\frac{1}{4} \frac{T_z}{x^{d/4-d/2-1}}\right)} \frac{(4\pi)^{d/2} \Gamma(d/4)}{\sum_{\mathbf{r}} I(\mathbf{r}-\mathbf{d}') I(\mathbf{r}-\mathbf{d})} .$$

Here, for $d > d' = d/2$ (d' is the lower critical dimensionality) the integral

converges and it is evaluated in

$$\delta W_{d',d',\omega}(0) = \frac{\Gamma(d/2-d/4)}{\Gamma(d/2) \Gamma(d/4)} \sum_{\mathbf{r}} I(\mathbf{r}-\mathbf{d}') I(\mathbf{r}-\mathbf{d}) \left(\frac{1}{2}\right)^{-d+d'/2} . \quad (3.4)$$

The sum in eq. (3.4) can be evaluated by the Epstein zeta function (see Ref. [23]), which can be regarded as the generalized ($d-d'$)-dimensional analogue

of the Riemann zeta function $\zeta\left(\frac{z}{d} - \frac{d'}{d}\right)$. This is convergent for $d' > d'$. The Epstein Zeta function has only a simple pole at $d' = d'$ and may be analytically continued for $0 \leq d' < d'$. Then, for example, in the case of the fully finite ($d' = 0$) two-dimensional ($d = 2$) purely quantum system one gets for $\sigma = 2$

$$\delta\lambda = \frac{n}{\zeta(1/2)} \frac{b/L}{\zeta(1/2, 1/4) - \zeta(1/2, 3/4)} \left(\approx -0.620 \dots b/L \right) , \quad (3.5)$$

where $\zeta(x, a)$ is the generalized Riemann zeta function.

It is interesting, however, to note that for $d' < d'$ in the self consistency equation it is not possible to set $\Delta = 0$ because in this case the r.h.s.

of Eq. (2.3) is divergent, therefore there is no phase transition in the system and in this case $\lambda^c(L)$ is the value of the quantum parameter (which we shall call the pseudocritical quantum parameter) corresponding to the center of the rounding of the singularities of the thermodynamic functions, holding at the thermodynamic limit. It is possible to calculate a shift for the pseudocritical quantum parameter $\delta\lambda$ by using the method employed in Ref. [24] to carry out the shift of the pseudocritical temperature for the spherical model. We mention that the two methods give the same value for the shift for $d' = 0$.

The postulates of finite-size scaling [3], developed for classical systems, may be exploited here. From eq. (3.1) and (3.4) we have $\delta\lambda \propto L^{-1/\nu}$. This, however, is true only for $d' < d < d'$ ($d' = 3\sigma/2$ is the upper critical dimension) where $\nu = 1/(d-d')$ is the critical exponent measuring the divergence of the bulk correlation length ξ . For dimensions above d' the shift is smaller than $L^{-1/\nu}$, because $\nu = 1/2$.

4- Fluctuations of the order parameter

By definition, in statistical mechanics, the order parameter is obtained by the use of a source term, proportional to an external ordering field h , which breaks the symmetry of the Hamiltonian and is made to vanish after the thermodynamic limit is taken. To apply this definition to finite (or partially finite) systems, it has been shown that the scale length L must be greater than the correlation length $\xi(h)$ of the order parameter fluctuations i.e. $L \gg \xi(h)$ [8,25], in this limit the expected critical behavior is that of the bulk system. For finite systems, however, the order parameter is identically zero when the field h vanishes. Instead of this definition another one is very useful, namely the root mean square order parameter

$$m_L = \left[\frac{1}{L} \sum_{\mathbf{r}} \left\langle \phi_{\mathbf{r}}^2 \right\rangle \right]^{1/2} , \quad (4.1)$$

where $\langle \dots \rangle^H$ denotes the thermal average taken with the Hamiltonian H . For the

model (1.1) both definitions coincide in the thermodynamic limit [26]

For calculating $m_\infty = \lim_{L \rightarrow \infty} m_L$, it is convenient to use the auxiliary

Hamiltonian

$$H = H + \rho N \left(\frac{1}{N} \sum_I Q_I \right)^2, \quad (4.2)$$

which has the same basic symmetry properties as H from (1.1). The free energy of the Hamiltonian (4.2) is

$$f_N(\rho) = \frac{\lambda}{2N} \sum_q \sqrt{\bar{\Delta} + |q|^{\sigma} + (2\rho/\nu_0^2) \delta_{q0}} - \frac{(1 + \bar{\Delta})^2}{8b}, \quad (4.3)$$

where δ_{q0} is the Kronecker symbol and $\bar{\Delta}$ is the solution of the equation

$$\Delta + 1 = \frac{\lambda}{N} b \sum_q \left(\Delta + |q|^{\sigma} + (2\rho/\nu_0^2) \delta_{q0} \right)^{-1/2} \quad (4.4)$$

The functions $f_N(\rho)$ are differentiable at $\rho = 0$. Then from the Griffiths-Fisher lemma [27] we obtain

$$m_\infty^2 = \lim_{\rho \rightarrow 0} \lim_{N \rightarrow \infty} \frac{\partial f_N(\rho)}{\partial \rho}. \quad (4.5)$$

The derivatives $\partial f_N(\rho)/\partial \rho$ are continuous at $\rho = 0$, so it is possible to inverse the order of the limits in eq. (4.5) to have the useful formulae

$$m_L^2 = \left. \frac{\partial f_N(\rho)}{\partial \rho} \right|_{\rho=0} \quad (4.6)$$

From eqs. (4.3) and (4.6) we obtain the following result for the order parameter for system (1.1) in a block geometry under periodic boundary conditions

$$m_L^2 = \frac{\lambda}{2\nu_0^2} \frac{1}{L^d} \chi_L^{1/2}, \quad (4.7)$$

where the usual zero field susceptibility $\chi_L = 1/\bar{\Delta}$ is obtained using an external ordering field h . The result (4.7) is different in comparison with $m_L^2 \propto \chi_L$, which holds in the classical limit. However the bulk critical exponent of the order parameter is $\beta = 1/2$ in both cases.

If the system is confined to the geometry $L^{d-d'} \times \omega^{d'}$ with $d' < d_c$, in Eq.

(4.7) instead of $v = L^d$ an effective volume $L^{d-d'} \omega^{d'}$ [7] appears and the order parameter must be given by

$$m_L^2 = \frac{\lambda}{2\nu_0^2} \frac{1}{L^{d-d'} \omega^{d'}} \chi_L^{1/2}. \quad (4.8)$$

Now it remains to evaluate the susceptibility χ_L from the self consistency equation (eq. (2.3) in the general geometry $L^{d-d'} \times \omega^{d'}$). With the aid of the Jacobi identity, eq. (2.12) transforms into (for $0 < d' < d_c$)

$$\delta W_{d,d',\sigma}(\Delta) = L^{\sigma/2-d} \left\{ D_{d',\sigma}(L^{\sigma}\Delta)^{d'/\sigma-1/2} + F_{d,d',\sigma}(L^{\sigma}\Delta) \right\} \quad (4.9)$$

where

$$D_{d',\sigma} = \frac{4}{\sigma} \frac{\Gamma(1/2-d'/\sigma) \Gamma(d'/\sigma)}{(4\pi)^{(d'+1)/2} \Gamma(d'/2)} \quad (4.9')$$

and

$$F_{d,d',\sigma}(t) = \frac{\pi^{d'/2}}{(2\pi)^{\sigma/2}} \int_0^\infty dx x^{\sigma/4-d'/2-1} G_{\sigma/2,\sigma/4}(-t x^{\sigma/2}/(2\pi)^\sigma) \times \left[A^{\tilde{d}}(x) - \left(\frac{\pi}{x} \right)^{\tilde{d}/2} - 1 \right].$$

Here

$$A(u) = \sum_{l=-\infty}^\infty \exp(-u l^2).$$

Now from eqs. (2.3), (2.10) and (4.9), the self consistency equation becomes

$$1 + \Delta = b\lambda W_{d,\sigma}(\Delta) + b\lambda L^{\sigma/2-d} \left\{ D_{d',\sigma}(L^{\sigma}\Delta)^{d'/\sigma-1/2} + F_{d,d',\sigma}(L^{\sigma}\Delta) \right\}, \quad (4.10)$$

where it is possible to express the bulk term in terms of the hypergeometric function [28]

$$W_{d,\sigma}(\Delta) = \frac{2[(d(4\pi)^{d/2}\Gamma(d/2))]^{-1}}{(x_D^\sigma + \Delta)^{1/2}} x_D^d {}_2F_1(1, 1/2; 1+d/\sigma; \frac{x_D^\sigma}{x_D^\sigma + \Delta}).$$

In the last expression x_D is the radius of the sphericalized brillouin zone. Taking into account that the expansion of the hypergeometric function depends

upon the ratio d/σ , hereafter we will discuss eq. (4.10) considering different dimensional regimes.

a) $d_c < d < d_c$. In this case the bulk-critical exponents of the model

considered here depend on the dimensionality d and the range of the interactions. When the quantum parameter approaches its pseudocritical value $\lambda_c(L)$ (i.e. $\Delta(\lambda_c(L))$ is very small but $\neq 0$) the bulk term has the following asymptotic form

$$w_{d,\sigma}(\Delta) \approx w_{d,\sigma}(0) - |D_{d,\sigma}| \Delta^{d/\sigma-1/2}, \quad (4.11)$$

where $D_{d,\sigma}$ is given by (4.9'). Then from eqs. (4.10) and (4.11) we have

$$bD_{d,\sigma} L^{d'-d} \Delta^{d'/\sigma-1/2} = \tilde{\lambda} + b |D_{d,\sigma}| \Delta^{d/\sigma-1/2}, \quad (4.12)$$

where (see also eqs. (3.1))

$$\tilde{\lambda} \equiv 1/\lambda - 1/\lambda_c(L) = 1/\lambda - 1/\lambda_c(\infty) - L^{\sigma/2-d} F_{d,\sigma}(0),$$

Eq. (4.12) has a simple solution

$$\chi_L = \left[|D_{d,\sigma}| / D_{d,\sigma} \right]^{1/\sigma} L^{\sigma}, \quad (4.13)$$

at $\lambda = \lambda_c(L)$ i.e. in the critical region of the second order phase transition.

Let us note that the susceptibility has the required finite-size scaling form $\chi_L \propto L^{\gamma/\nu}$, where $\gamma = \sigma/(d - d_c)$ measures the divergence of the bulk susceptibility, when $\lambda < \lambda_c(L)$ (i.e. in the region of first order phase transition),

the solution of eq. (4.12) is

$$\chi_L = \left[\tilde{\lambda} / b D_{d,\sigma} \right]^{1/\sigma} L^{\sigma} \quad (4.14)$$

b) $d = d_c$. The expression (4.13), solution of eq. (4.12), exhibits a characteristic singularity in the limit $d \rightarrow 3\sigma/2$. In this case, however, one should take into account the following asymptotic form of bulk term valid in

the vicinity of the pseudocritical quantum parameter $\lambda_c(L)$

$$w_{3\sigma/2,\sigma}(\Delta) \approx w_{d,\sigma}(0) + \left[(4\pi)^{3\sigma/4} \sigma \Gamma(3\sigma/4) \right]^{-1} \Delta \ln(\Delta/x_0^d). \quad (4.15)$$

By substitution of (4.15) in (4.10) it is easy to get the equation

$$\tilde{\lambda} = b \left[(4\pi)^{3\sigma/4} \sigma \Gamma(3\sigma/4) \right]^{-1} \Delta \ln \Delta + b D_{d,\sigma} L^{d'-d} \Delta^{d'/\sigma-1/2}, \quad (4.16)$$

which has the solutions

$$\chi_L = \left[(4\pi)^{3\sigma/4} \sigma \Gamma(3\sigma/4) D_{d,\sigma} \right]^{1/\sigma} (\ln L)^{\sigma/(d-d_c)} L^{\sigma} \quad (4.17)$$

at $\lambda = \lambda_c(L)$ and the expression (4.14) for $\lambda < \lambda_c(L)$. The presence of $\ln L$ in

(4.17) breaks the finite-size scaling in its simple form.

c) $d > d_c$. Above its upper critical dimension d_c the critical exponents are those of the mean-field theory. In this case we have

$$w_{d,\sigma}(\Delta) \approx w_{d,\sigma}(0) - w_{d,\sigma}(0) \Delta \quad (4.18)$$

and eq. (4.10) becomes

$$b D_{d,\sigma} L^{d'-d} \Delta^{d'/\sigma-1/2} = \tilde{\lambda} + \left(\frac{\lambda}{1} + w_{d,\sigma}(0) \right) \Delta. \quad (4.19)$$

This equation has

$$\chi_L = \left[\left(\frac{\lambda}{1} + w_{d,\sigma}(0) \right) / b D_{d,\sigma} \right]^{1/\sigma} L^{\sigma} \quad (4.20)$$

as a solution at $\lambda = \lambda_c(L)$ and the expression (4.14) for $\lambda < \lambda_c(L)$. From eq. (4.20) we see the failure of finite-size scaling for $d > d_c$, as is claimed for

the "upper" model, this is due to the presence of a dangerous irrelevant

variable. For model (1.1) this variable is B (respectively b). The susceptibility $\chi_L(b)$ is singular at $b = 0$ and at $\lambda \approx \lambda_c(L)$.

Now it is possible to determine the root mean square order parameter in the quantum limit. Let us just note that the behaviour of χ_L in L obtained

here is like that one obtained from a general consideration of fluctuations for classical systems [7] in the region of second order phase transition. In

the first order regime, however, the expression for the susceptibility (Eq.

(25) of Ref. [7]) has to be modified in the following one

$$\chi_L \propto L^{\sigma(d-d')/(d_c-d')}$$

in order to represent in a unified way both the classical and the quantum cases.

5. Discussion

We consider the finite-size scaling for the pure quantum version of an exactly solvable model of phase transition, with long-range interactions decaying at large distances as $r^{-d-\sigma}$. The model has the general geometry $L^{d-d'} \times \omega^{d'}$ with periodic boundary conditions imposed along the $\tilde{d} = d - d'$ finite directions. This model reflects the majority of the basic features inherent in the theory of phase transitions and critical phenomena. In its pure quantum version the phase transition occurring in the system is controlled by the variation of the quantum parameter $\lambda = \hbar\nu_0$ (this plays the same role as the temperature for classical system).

Using the technique proposed in Ref. [22] we have found the large L -asymptotic form of the self consistency equation and the free energy, which allowed us to test the postulates of finite-size scaling for a quantum model with long-range interactions. In calculating the shift of the quantum parameter $\delta\lambda$, we have found $\delta\lambda \propto L^{d_c-d}$ for $d_c < d$. When d' takes the value corresponding to the lower critical dimension $d_c = \sigma/2$, the shift is infinite.

For the fully finite classical systems, belonging to the class of universality of the spherical model, the susceptibility χ_L is proportional to m_L^2 (m_L is the root-mean-square order parameter for system with size L). From (4.7) we see that the situation is quite different for the quantum version of (1.1) where we have $m_L^2 \propto \chi_L^{1/2}$. The determination of χ_L in the regions of both first order (i.e. λ is smaller than the pseudocritical quantum order parameter $\lambda_c(L)$) and second order ($\lambda \approx \lambda_c(L)$) reveals that in the former case the results

obtained are similar for all dimensions $d > d_c$; in the latter they are different depending on whether d is less than, equal to or greater than d_c .

From the foregoing results and those obtained for the classical limit of the model (1.1) by deduction from the investigations on the spherical model [2-10,22] we can express in unified way the behaviour of the susceptibility for classical and quantum systems. For this end we consider two cases depending on whether the transition is of the first or of the second order :

(i) First order phase transition

$$\chi_L \propto L^{\sigma(d-d')/(d_c-d')}$$

(ii) Second order phase transition

$$\chi_L \propto \begin{cases} L^\sigma & d_c < d < d_+ , \\ L^\sigma (\ln L)^{\sigma/(d_+-d')} & d = d_+ , \\ L^{\sigma(d-d')/(d_+-d')} & d_+ < d , \end{cases}$$

where $d_c = \sigma$ and $d_+ = 2\sigma$ for the classical limit of the model under consideration.

Renormalization group investigations has shown that the failure of hyperscaling for $\dim d > d_+$ is a consequence of the presence of an irrelevant variable, which has a dangerous character [4,29]. To introduce such a variable for the spherical model use has been made of the so called "soft" spherical constraint [10]. For the model considered here the dangerous irrelevant variable is the "constant" B corresponding to the variable u in the "up⁴" theory.

Appendix

Let us start with the identity

$$(1+z)^{-1/2} = \int_0^\infty dx e^{-x} x^{\beta-1} G_{\alpha,\beta}(-x^\alpha z), \quad (A.1)$$

which is obtained by means of term-by-term integration with the use of the

series

$$G_{\alpha,\beta}(t) = \pi^{-1/2} \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)}{\Gamma(k + \beta)} \frac{k!}{t^k}, \quad \alpha > 0, \beta > 0. \quad (A.2)$$

These functions are entire functions of finite order of growth. For example we

have

$$G_{1,1/2}(t) = \pi^{-1/2} \exp(-t)$$

and

$$G_{2,1}(t) = J_0(t^{1/2}),$$

where $J_0(\cdot)$ is the Bessel function of the first kind.

If we set in (A.1) $z = y^{-\alpha}$, $y > 0$, and $x = ty$, we will obtain the Laplace

transform

$$y^{\alpha/2-\beta} \frac{(1+y^{\alpha})^{1/2}}{\int_0^{\infty} dx e^{-xt} t^{\beta-1} G_{\alpha,\beta}(-t^{\alpha})} = \quad (A.3)$$

from which we derive the identity (2.5) by setting $\beta = \alpha/2$.

The identity (2.13) is obtained by setting $\beta = 1 - \alpha/2$ in (A.3) and

integrating over y from 0 to z .

Acknowledgment

The author expresses his gratitude to Dr. N.S. Tonchev under whose help-

ful guidance this research has been conducted and in addition to Dr. J.G.

Brankov for useful discussions.

This work is supported by the Bulgarian Science Foundation (project F-

23/91).

The author would like to thank Professor Abdus Salam, the International

Atomic Energy Agency and UNESCO for hospitality at the International Centre for

Theoretical Physics, Trieste.

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