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Finite Temperature Quantum Field Theory in The Functional Schrödinger Picture

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We calculate the finite temperature Gaussian effective potential of the scalar ϕ^4 theory in the functional Schrödinger picture. Our method is the direct generalization of the variational method proposed by Eboli, Jackiw and Pi for the quantum mechanical systems, and gives the same result as that of Amelino-Camelia and Pi who used the self-consistent composite operator method.

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The quantum field theoretic techniques in computing the finite temperature effects of many-body systems have attracted considerable interests in connection with cosmological phase transitions, and various perturbative techniques have been developed[1]. For the precise determination of the critical temperature and the nature of the phase transitions, however, one needs to include the contributions from higher loop diagrams. Recently Amelino-Camelia and Pi[2] have used the self-consistent composite operator method to compute the finite temperature effective potential of $\lambda\phi^4$ theory, in which contributions of daisy and superdaisy diagrams are summed up. It is the purpose of this paper to establish the non-perturbative approximation method based on the functional Schrödinger picture formulation of the finite temperature quantum field theory proposed by Eboli, Jackiw and Pi[3]. Especially, Eboli, Jackiw and Pi have obtained the Gaussian approximation of the Helmholtz free energy of the quantum mechanical system of anharmonic oscillator. We have extended this Gaussian approximation method to the case of quantum field theory, and have found that the Gaussian approximation of the free energy of the scalar $\lambda\phi^4$ theory is the same as the result of the Amelino-Camelia and Pi.

The Gaussian effective potentials in the functional Schrödinger picture have been known to be useful for the scalar and fermionic field theories, and also for the scalar quantum electrodynamics[4]. Several authors have attempted to extend the Gaussian approximation method for the finite temperature case by using the particle picture[5], and the authors of ref.[3] have established the Gaussian approximation method for the systems not in thermal equilibrium.

In the quantum mechanical many-body systems we are interested in the density matrix

$$\rho \equiv e^{-\beta H} \quad (1)$$

where β is the inverse temperature and H is the Hamiltonian of the system. The density matrix can be considered as the solution of the differential equation[6],

$$-\frac{\partial \rho}{\partial \beta} = H \rho, \quad (2)$$

It is easy to show that, for the simple harmonic oscillator, the solution of Eq.(2) is of the Gaussian form as in well-known. The Gaussian approximation of the anharmonic oscillator, for example, can be obtained by choosing the solution of the simple harmonic oscillator as the trial density matrix of the anharmonic oscillator to minimize the Helmholtz free energy as have been done by Eboli, Jackiw and Pi[3].

Here, we intend to extend this procedure into the quantum field theory. We are going to calculate the effective potential of the 4-dimensional scalar $\lambda\phi^4$ theory by making a Gaussian Ansatz for the density matrix in the field theoretical Schrödinger picture. For field theories, the dynamics is described by function $\phi(x)$ of space-time and the density matrix is described by functional $\rho(\phi)$ of the field $\phi(x)$. Then equation (2), which is the differential equation with respect to x in the quantum mechanical case, becomes the functional differential equation with respect to the field variable $\phi(x)$.

We start with the Hamiltonian of $\lambda\phi^4$ theory:

$$H = \int d^3x \left[\frac{1}{2} \Pi^2(x) - \frac{1}{2} \phi(x) \nabla^2 \phi(x) + \frac{1}{2} \mu^2 \phi^2(x) + \frac{\lambda}{4!} \phi^4(x) \right], \quad (3)$$

where $\Pi(x)$ denotes the conjugate momentum of $\phi(x)$ and is represented as $\Pi = \frac{1}{i} \frac{\delta}{\delta \phi(x)}$ in the Schrödinger picture. In the case of the free theory, $\lambda = 0$, the density matrix which is solution of Eq.(2) has a Gaussian form;

$$\rho(\phi_1, \phi_2) = \det^{\frac{1}{2}} \left[\omega \tanh\left(\frac{\beta\omega}{2}\right) \right] \exp \left\{ -\frac{1}{2} \int d^4x \int d^4y \left[\phi_1(x) (\omega \coth \beta\omega)(x, y) \phi_1(y) + \phi_2(x) (\omega \coth \beta\omega)(x, y) \phi_2(y) - 2\phi_1(x) (\omega \operatorname{csch} \beta\omega)(x, y) \phi_2(y) \right] \right\}, \quad (4)$$

where

$$\omega^2 = -\nabla^2 + \mu^2. \quad (5)$$

The Gaussian approximation approximates the interacting Hamiltonian as the quadratic Hamiltonian by making proper parameterization. Therefore, by using the Gaussian ansatz of the density matrix,

$$\rho(\phi_1, \phi_2) = N \exp \left\{ -\frac{1}{2} \int d^4x \int d^4y \left[\phi_1(x) \Lambda(x, y) \phi_1(y) + \phi_2(x) \Lambda(x, y) \phi_2(y) - \phi_1(x) F(x, y) \phi_2(y) \right] \right\}, \quad (6)$$

where

$$\begin{aligned} N &= \det^{1/2} \left\{ \Omega \tanh\left(\frac{b\Omega}{2}\right) \right\} \\ \Lambda(x, y) &= \Omega(x, z) \coth b\Omega(z, y) \\ F(x, y) &= 2\Omega(x, z) \operatorname{csch} b\Omega(z, y), \end{aligned} \quad (7)$$

we can obtain the Gaussian approximation of the interacting theory. Here a density matrix ρ can be viewed as the average value of a density operator,

$$\hat{\rho} = \det(2 \sinh \frac{b\Omega}{2}) \exp \left\{ - \int d^3x \left[\frac{b}{2} \Pi^2(x) + \frac{1}{2} \int d^3y \phi(x) b\Omega^2(x, y) \phi(y) \right] \right\}, \quad (8)$$

that is,

$$\rho(\phi_1, \phi_2) = \langle \phi_1 | \hat{\rho} | \phi_2 \rangle. \quad (9)$$

Then by using the average values of observable $\hat{O}$ defined by

$$\begin{aligned} \langle \hat{O} \rangle &= \operatorname{tr} \hat{O} \rho \\ &= \int \mathcal{D}\phi \hat{O}(\phi) \rho(\phi, \phi). \end{aligned} \quad (10)$$

we can simply calculate the average values as,

$$\begin{aligned} \langle \phi(x) \phi(y) \rangle &= \frac{1}{2} \Omega^{-1} \coth\left(\frac{b\Omega}{2}\right)(x, y) \\ \langle \Pi(x) \Pi(y) \rangle &= \int \mathcal{D}\phi \frac{\partial^2 \rho(\phi, \phi')}{\partial \phi \partial \phi'} \Big|_{\phi=\phi'} \\ &= \frac{1}{2} \Omega \coth\left(\frac{b\Omega}{2}\right)(x, y). \end{aligned} \quad (11)$$

From Eqs.(8) and (11) it is easy to evaluate the entropy:

$$\begin{aligned} S &= - \langle \rho \ln \rho \rangle \\ &= - \langle \ln [\det(2 \sinh(\frac{b\Omega}{2}))] - \int_x \left\{ \frac{b}{2} \Pi^2(x) + \int_y \frac{1}{2} \phi(x) b\Omega^2(x, y) \phi(y) \right\} \rangle \\ &= -\operatorname{Tr} \ln [2 \sinh \frac{b\Omega}{2}] + \int_x \left[\frac{b\Omega}{2} \coth \frac{b\Omega}{2} \right](x, x), \end{aligned} \quad (12)$$

where $\int_x \equiv \int d^3x$. The Helmholtz free energy is defined by,

$$\beta F = \operatorname{tr} \rho \ln \rho + \beta \operatorname{tr} \rho H = \langle \ln \rho \rangle + \beta \langle H \rangle, \quad (13)$$

where β serves as a Lagrange multiplier. The first term, the entropy, was already computed in (12) and the average of the Hamiltonian is directly computed by using (11). The result is

$$\begin{aligned}\beta F = & Tr \ln[2 \sinh \frac{b\Omega}{2}] + b \int_x (\frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4) \\ & + \int_x \{ [\frac{\beta}{4} \Omega + \frac{\beta}{4} (-\nabla^2 + \mu^2 + \frac{\lambda}{2} \phi^2) \Omega^{-1} - \frac{1}{2} b\Omega] \coth \frac{b\Omega}{2}(x, x) \\ & + \frac{\beta\lambda}{32} [\Omega^{-1} \coth \frac{b\Omega}{2}](x, x) [\Omega^{-1} \coth \frac{b\Omega}{2}](x, x) \}.\end{aligned}\quad (14)$$

The finite temperature Gaussian effective potential is obtained from the free energy,

$$V_{eff} \equiv \frac{F}{\int d^3x}.\quad (15)$$

After Fourier transformation into momentum space, finite temperature Gaussian effective potential can be written as

$$\begin{aligned}\beta V_{eff} = & \int_p \ln[2 \sinh \frac{b\Omega(p)}{2}] + b \frac{1}{2} \mu^2 \phi^2 + b \frac{\lambda}{4!} \phi^4 \\ & + \int_p \{ [\frac{\beta}{4} \Omega(p) + \frac{\beta}{4} (p^2 + \mu^2 + \frac{\lambda}{2} \phi^2) \Omega^{-1}(p) - \frac{1}{2} b\Omega(p)] \coth \frac{b\Omega(p)}{2} \} \\ & + \frac{\beta\lambda}{32} [\int_p \Omega^{-1}(p) \coth \frac{b\Omega(p)}{2}] [\int_q \Omega^{-1}(q) \coth \frac{b\Omega(q)}{2}],\end{aligned}\quad (16)$$

where $\int_p = \int \frac{d^3p}{(2\pi)^3}$. We consider Ω and $b\Omega$ as variational parameters. Varying Ω at fixed $b\Omega$ produces a gap equation:

$$\Omega^2 = p^2 + \mu^2 + \frac{1}{2} \lambda \phi^2 + \frac{\lambda}{4} \int_p \Omega^{-1} \coth(\frac{b\Omega}{2}).\quad (17)$$

Variation with respect to $b\Omega$ at fixed Ω implies

$$b = \beta,\quad (18)$$

where Eq.(17) has been used. If we set $\Omega^2 = p^2 + m^2$, where m^2 is defined by

$$m^2 \equiv \mu^2 + \frac{1}{2} \lambda \phi^2 + \frac{\lambda}{4} \int_p \Omega^{-1} \coth(\frac{\beta\Omega}{2}),\quad (19)$$

the effective potential becomes,

$$\begin{aligned}V_{eff} = & \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + \frac{1}{2} \int_p \sqrt{p^2 + m^2} \\ & + \frac{1}{\beta} \int_p \ln(1 - e^{\beta\Omega}) - \frac{1}{4} (m^2 - \mu^2 - \frac{1}{2} \lambda \phi^2) \int_p \Omega^{-1} \coth(\frac{\beta\Omega}{2}) \\ & + \frac{\lambda}{32} (\int_p \Omega^{-1} \coth(\frac{\beta\Omega}{2})) (\int_q \Omega^{-1} \coth(\frac{\beta\Omega}{2})).\end{aligned}\quad (20)$$

If we set

$$\begin{aligned}G(x, x) = & \frac{1}{2} \int_p [\Omega^{-1} \coth(\frac{\beta\Omega}{2})] \\ = & \int_p [\frac{1}{2\Omega} + \frac{1}{\Omega} \frac{1}{e^{\beta\Omega} - 1}],\end{aligned}\quad (21)$$

where $G(x, x)$ corresponds to the composite operator, the effective potential can be written as

$$\begin{aligned}V_{eff} = & \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + \frac{1}{2} \int_p \sqrt{p^2 + m^2} \\ & + \frac{1}{\beta} \int_p \ln(1 - e^{\beta\Omega}) - \frac{1}{2} (m^2 - \mu^2 - \frac{1}{2} \lambda \phi^2) G(x, x) \\ & + \frac{\lambda}{8} G(x, x) G(x, x),\end{aligned}\quad (22)$$

which is the same as that of Amelino-Camelia and Pi[2]. Amelino-Camelia and Pi arrived at this result by using the self-consistent composite operator method. This is an example of the fact that the functional Schrödinger picture formulation of quantum field theory can provide the non-trivial non-perturbative physical informations in a simple straightforward manner.

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