FREE FIELD REALIZATIONS IN REPRESENTATION THEORY
AND CONFORMAL FIELD THEORY

EDWARD FRENKEL

Free field realizations are realizations of conformal algebras in terms of infinite-dimensional Heisenberg or Clifford algebras. From the physics point of view, this gives a representation of a two-dimensional conformal field theory via a free bosonic or free fermionic theory. Mathematically, this can be considered as “abelianization” of a complicated symmetry algebra. In recent years free field realizations have been found for a variety of conformal (super) algebras and they provided valuable insights on representation theory and quantum field theory associated to them.

In this report, which is based mainly on my joint works with Boris Feigin, we will focus on free field realizations of affine Kac-Moody algebras and $\mathcal{W}$-algebras. We will give two constructions of free field realizations: geometric and hamiltonian, and discuss their applications.

1. Preliminaries

1.1. Heisenberg algebras. We first introduce two types of Heisenberg algebras, $\hat{\mathfrak{u}}$ and $\mathcal{H}_S$.

Let $\mathfrak{g}$ be a finite-dimensional linear space with a scalar product $(\cdot, \cdot)$. We choose a basis $v_i$, $i = 1, \ldots, N$, of $\mathfrak{g}$. The Heisenberg Lie algebra $\hat{\mathfrak{u}}$ has generators $b_i(n), i = 1, \ldots, N, n \in \mathbb{Z}$, and $1$, with the commutation relations

$$[b_i(n), b_j(m)] = n(v_i, v_j)1, \quad [1, b_i(n)] = 0.$$

The second Heisenberg algebra, which we denote by $\mathcal{H}_S$, where $S$ is a set, has generators $a_\alpha(n), a^*_\alpha(n), \alpha \in S, n \in \mathbb{Z}$, and $1$, with the commutation relations

$$[a_\alpha(n), a^*_\beta(m)] = \delta_{\alpha,\beta}\delta_{n,-m}1, \quad [a_\alpha(n), a_\beta(m)] = 0, \quad [a^*_\alpha(n), a^*_\beta(m)] = 0,$$

and $1$ commutes with everything.

In physics terminology, $\hat{\mathfrak{u}}$ is the Heisenberg algebra of $N$ scalar fields, and $\mathcal{H}_S$ is a $\beta\gamma$-system. For more details, cf. [FF5].

---

Invited lecture at the International Congress of Mathematicians, Zürich, August 3-11, 1994
1.2. Fock representations. Let $\lambda$ be an element of $\mathfrak{a}^*$, the dual space to $\mathfrak{a}$, and $\nu \neq 0$ be a complex number. We define the Fock space representation $\pi_\nu^\lambda$ of $\hat{\mathfrak{a}}$ as a module freely generated by $b_i(n), i = 1, \ldots, N, n < 0$, from a vector $v_\lambda$, such that $b_i(n)v_\lambda = 0, n > 0; b_i(0)v_\lambda = \lambda(v_i)v_\lambda$; and on which the central element $1$ acts as $\nu$ times the identity. We put $\pi_\lambda = \pi_0^\lambda$.

We also define a Fock representation $M$ of the Lie algebra $\mathcal{H}_S$ as a module freely generated by $a_\alpha(n), \alpha \in S, n < 0$, and $a_\alpha^*(n), \alpha \in S, n \leq 0$, from a vector $v$, such that $a_\alpha(n)v = 0, n \geq 0; a_\alpha^*(n)v = 0, n > 0$. The central element $1$ acts on $M$ as the identity.

We introduce $\mathbb{Z}$-gradings on the Lie algebras $\hat{\mathfrak{a}}$ and $\mathcal{H}_S$ by putting $\deg b_i(n) = \deg a_\alpha(n) = \deg a_\alpha^*(n) = -n$, $\deg 1 = 0$. The modules $\pi_\lambda$ and $M$ inherit these gradings. These modules are always irreducible. They can be realized in a natural way in spaces of polynomials in infinitely many variables.

1.3. Vertex operator algebra structure. The modules $\pi_0^\nu$ and $M$, which we call vacuum modules, carry vertex operator algebra (VOA) structures. Recall [B, FLM] that a VOA structure is essentially a linear operation on a $\mathbb{Z}$-graded linear space $V$, which associates to any homogeneous vector $A \in V$, a formal power series, called a current, $Y(A, z) = \sum_{m \in \mathbb{Z}} A_m z^m$, where $A_m : V \to V$ is a linear operator of degree $\deg A + m$. These series satisfy certain axioms, cf. [B, FLM].

The VOA structure on $\pi_0^\nu$ and $M$ can be described explicitly. We will give here an explicit formula for $Y(\cdot, z)$ in the VOA $M$; the case of $\pi_0^\nu$ was treated in detail in [FF7], § 4.1.

Monomials $a_{\alpha_1}(m_1) \ldots a_{\alpha_k}(m_k) a_{\beta_1}^*(n_1) \ldots a_{\beta_l}^*(n_l)v, m_p < 0, n_q \leq 0$, form a linear basis in $M$. The series $Y(\cdot, z)$ associated to this monomial is given by

$$C : \partial_z^{-m_1-1} a_{\alpha_1}(z) \ldots \partial_z^{-m_k-1} a_{\alpha_k}(z) \partial_z^{-n_1} a_{\beta_1}^*(z) \ldots \partial_z^{-n_l} a_{\beta_l}^*(z),$$

where $C = [(-m_1 - 1)! \ldots (-m_k - 1)! (-n_1)! \ldots (-n_l)!]^{-1}$, and

$$a_i(z) = \sum_{n \in \mathbb{Z}} a_i(n) z^{-n-1}, \quad a_j^*(z) = \sum_{n \in \mathbb{Z}} a_j^*(n) z^{-n}. \quad (1)$$

The Fourier coefficients of currents form a Lie algebra, which lies in a certain completion of the universal enveloping algebra $U(\hat{\mathfrak{a}})$ or $U(\mathcal{H}_S)$ factored by the ideal generated by $(1 - \nu)$ or $(1 - 1)$, respectively. Following [FF6], we call this Lie algebra local completion of the universal enveloping algebra and denote it by $U(\mathfrak{a})_{\text{loc}}$ or $U(\mathcal{H}_S)_{\text{loc}}$, respectively. We also put $U(\mathfrak{a})_{\text{loc}} = U(\mathfrak{a})_{\text{loc}}$.

Let us also define bosonic vertex operators $V_\gamma^\nu(z) = \sum_{n \in \mathbb{Z}} V_\gamma^\nu(n) z^{-n+\gamma,\lambda} = T_\gamma z^{\gamma,\lambda} \exp \left( - \sum_{n < 0} \frac{\gamma(n) z^{-n}}{n} \right) \exp \left( - \sum_{n > 0} \frac{\gamma(n) z^{-n}}{n} \right), \quad (2)$

2
where $\gamma \in \mathfrak{a}^* \simeq \mathfrak{a}$ and $T_{\gamma}^\vee : \pi_\lambda^\vee \to \pi_{\lambda+\gamma}^\vee$ is such that $T_{\gamma} : \mathfrak{v}_1 = \mathfrak{v}_{\lambda+\gamma}$ and $[T_{\gamma}^\vee, h(n)] = 0, n < 0$. Thus, $V_{\gamma}^\vee(n), n \in \mathbb{Z}$, are well-defined linear operators acting from $\pi_\lambda^\vee$ to $\pi_{\lambda+\gamma}^\vee$. They appear naturally in the context of VOA [FF7], § 4.2.

1.4. Affine algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ of rank $l$. Recall that the affine Lie algebra associated to $\mathfrak{g}$ is the universal central extension $\hat{\mathfrak{g}}$ of the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. We denote by $K$ the central element of $\hat{\mathfrak{g}}$, and for $A \in \mathfrak{g}, n \in \mathbb{Z}$, we denote by $A(n)$ the element $A \otimes t^n \in \hat{\mathfrak{g}}$.

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the Cartan decomposition of $\mathfrak{g}$. The Lie algebra $\hat{\mathfrak{g}}$ has a Cartan decomposition: $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_-$, where $\hat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \otimes \mathbb{C}L \oplus \mathfrak{g} \otimes t^{\pm 1}[t, t^{-1}]$, and $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}L \oplus \mathbb{C}K$. We denote by $e_i, h_i, f_i, i = 0, \ldots, l$, the standard generators of $\hat{\mathfrak{g}}$ [K].

The Lie algebra $\hat{\mathfrak{g}}$ also has a loop decomposition: $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_-$, where $\hat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \otimes \mathbb{C}[t, t^{-1}]$, and $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$. Note that the latter subalgebra is the Heisenberg Lie algebra associated with $\mathfrak{h}$, where the scalar product is the restriction of the Killing form, and $K$ plays the role of $1$.

There is a family of VOAs associated to $\hat{\mathfrak{g}}$, the vacuum representations of level $k, k \in \mathbb{C}$:

$$V_k = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])} \mathbb{C}_k,$$

where $\mathbb{C}_k$ stands for the trivial one-dimensional representation of the Lie subalgebra $\mathfrak{g} \otimes \mathbb{C}[t]$ of $\hat{\mathfrak{g}}$, on which $K$ acts by multiplication by $k$. Its $\mathbb{Z}$-grading is inherited from the standard $\mathbb{Z}$-grading on $\hat{\mathfrak{g}}$, such that $\deg A(n) = -n, \deg K = 0$.

The Fourier components of currents of $V_k$ form a Lie algebra $U_k(\hat{\mathfrak{g}})_{\text{loc}}$, which is called the local completion of the universal enveloping algebra of $\hat{\mathfrak{g}}$. It lies in a certain topological completion of $U(\hat{\mathfrak{g}})/(K - k)U(\hat{\mathfrak{g}})$, cf. [FF6].

1.5. Category $\mathcal{O}$ and Verma modules. Category $\mathcal{O}$ can be defined for an arbitrary Kac-Moody algebra using its Cartan decomposition. It consists of modules, on which (1) the upper nilpotent subalgebra $(\mathfrak{n}_+)$ in the case of $\hat{\mathfrak{g}}$) acts locally nilpotently, and (2) the Cartan subalgebra $(\mathfrak{h})$ in the case of $\hat{\mathfrak{g}}$) acts semi-simply [BGG, RW, DGK]. To motivate this definition in the affine case, it is worth mentioning that the Lie group of $\mathfrak{n}_+$ is an analogue of the compact subgroup of a simple Lie group $G$ over a local non-archimedean field. Elements of $\mathfrak{n}_+$ also annihilate the vacuum state of the corresponding quantum field theory.

The fundamental objects of the category $\mathcal{O}$ are Verma modules. Such a module is the induced representation

$$M_\lambda = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{n}_+ \oplus \mathfrak{h})} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional $\mathfrak{n}_+ \oplus \mathfrak{h}$-module, on which the first summand acts by $0$, and the second summand acts according to its character $\lambda \in \mathfrak{h}^*$, which is called the highest weight. We will write $\lambda = (\bar{\lambda}, k)$, where $\bar{\lambda} \in \mathfrak{h}^*$ is the restriction of $\lambda$ to
\[ \mathfrak{h} \subset \tilde{\mathfrak{h}}, \text{ and } k = \lambda(K), \text{ } k \text{ is called level. All irreducible objects in } \mathcal{O} \text{ can be obtained as quotients of Verma modules.} \]

2. Geometric approach to free field realizations

In this section we will give a construction of a family of free field representations of affine algebras, which we call Wakimoto modules. These modules were defined by Wakimoto [W] for the simplest affine algebra \( \widehat{\mathfrak{sl}}_2 \) and by Feigin and the author [FF1] for an arbitrary affine algebra.

As was said at the beginning, our goal is to construct explicitly an embedding of an affine algebra \( \widehat{\mathfrak{g}} \) into a Heisenberg algebra. A suitable Fock representation of the latter will then provide a family of \( \widehat{\mathfrak{g}} \)-modules from the category \( \mathcal{O} \), which have many nice properties. More precisely, we will construct an embedding of \( \widehat{\mathfrak{g}} \) into the local completion of the universal enveloping algebra of a Heisenberg algebra. This embedding yields a bosonization of the Wess-Zumino-Novikov-Witten (WZNW) model, which is a conformal field theory associated to \( \widehat{\mathfrak{g}} \), and allows to compute correlation functions of this model (see the next section).

Our geometric construction [FF1, FF4] of Wakimoto modules can be considered as a generalization of the Borel-Weil-Bott construction of representations of semi-simple Lie algebras, which we now recall.

2.1. Embeddings of \( \mathfrak{g} \) into differential operators. The universal enveloping algebra of a Heisenberg algebra, in which the central element is identified with 1, is nothing but the algebra of (algebraic) differential operators on an affine space (also called Weyl algebra). If \( X \) is a homogeneous space of the Lie group of \( \mathfrak{g} \), then \( \mathfrak{g} \) acts infinitesimally on \( X \) and hence on any open subspace of \( X \) by vector fields. If we choose an open subspace isomorphic to an affine space, we obtain an embedding of \( \mathfrak{g} \) into a Heisenberg algebra. We can even obtain a family of such embeddings by considering equivariant line bundles over \( X \).

Let \( \mathfrak{g} \) be a simple Lie algebra, and \( X \) be its flag manifold \( G/B_- \), where \( G \) is the Lie group of \( \mathfrak{g} \), and \( B_- \) is its Borel subgroup; the Lie algebra of \( \mathfrak{n}_- \oplus \mathfrak{h} \). As an open subspace of \( X \), we will take the big cell \( \mathcal{U} \), which is the orbit of the unit coset under the action of the Lie group \( N_+ \). Thus, we obtain an embedding \( \epsilon : \mathfrak{g} \to \text{Vect}\, \mathcal{U} \), where \( \text{Vect}\, \mathcal{U} \) is the Lie algebra of vector fields on \( \mathcal{U} \).

The big cell \( \mathcal{U} \) is isomorphic to the Lie group \( N_+ \) and hence to the Lie algebra \( \mathfrak{n}_+ \) via the exponential map. Therefore we can choose coordinates \( x_\alpha, \alpha \in \Delta_+ \), where \( \Delta_+ \) is the set of positive roots of \( \mathfrak{g} \), on \( \mathcal{U} \), such that \( x_\alpha \) has weight \( \alpha \) with respect to the action of the Cartan subgroup of \( G \) on \( N_+ \).

Recall that we have a standard filtration \( 0 \subset \mathcal{D}_0 \subset \mathcal{D}_1 \subset \ldots \) on the algebra \( \mathcal{D} \) of differential operators on \( \mathcal{U} \), where \( \mathcal{D}_i \) is the space of differential operators of order less than or equal to \( i \). We have the exact sequence of Lie algebras

\[
0 \to \mathcal{D}_0 \to \mathcal{D}_1 \to \text{Vect}\, \mathcal{U} \to 0.
\]
In order to construct a map from \( g \) to \( D \), we have to lift the map \( \epsilon \) to a map \( \epsilon': g \to D_1 \).

This can be done because the sequence (3) splits: \( H^2(\text{Vect } U, D_0) = 0 \). The inverse \( \text{Vect } U \to D_1 \) can be constructed by mapping vector fields to the differential operators of order 1, which annihilate constants.

However, such a lifting is not unique: the space of liftings is a torsor over \( H/1 \). The inverse map \( \text{Vect } U/2 \to D_0/1 \) can be constructed by mapping vector fields to the differential operators of order 1, which annihilate constants.

Proposition 1 The restriction of \( C[U] \) to the image of the homomorphism \( \epsilon' \) defines a \( g \)-module, which is isomorphic to the module \( M^{*}_{1} \) contragradient to the Verma module over \( g \) with highest weight \( \lambda \).

Example. The module \( M^{*}_{1}, \lambda \in \mathbb{C} \), over \( \mathfrak{sl}_2 \) can be realized in the space \( \mathbb{C}[x] \) as follows:

\[
\begin{align*}
\epsilon &= \frac{\partial}{\partial x}, & h &= -2x \frac{\partial}{\partial x} + \lambda, & f &= -x^2 \frac{\partial}{\partial x} + \lambda x.
\end{align*}
\]

2.2. Semi-infinite flag manifold. Our construction of Wakimoto modules essentially exploits the same idea: we should find an appropriate homogeneous space of the Lie group of \( \hat{g} \) and try to embed \( \hat{g} \) into the algebra of differential operators on a big cell. We should then choose a module over this algebra in such a way that its restriction to \( \hat{g} \) lies in the category \( \mathcal{O} \).

We can take as a homogeneous space, the standard flag manifold of \( \hat{g} \), i.e. the quotient of the Lie group of \( \hat{g} \) by the standard Borel subgroup – the Lie group of \( \hat{n} \oplus \hat{h} \). The construction of the previous section carries over to this case without any difficulties and gives a realization of contragradient Verma modules over \( \hat{g} \) in the space of functions on the big cell of this flag manifold.

However, there are other possibilities, which have no analogues in the finite-dimensional picture. The reason is that in the affine algebra there are many different Borel subalgebras, which are not conjugated to each other. One of them is \( \hat{n}_+ \oplus \hat{h} \oplus t \mathbb{C}[t] \), a Lie subalgebra of loops to the Borel subalgebra of \( g \). To this subalgebra there corresponds the semi-infinite flag manifold \( \hat{X} \), which is the quotient of the loop group \( LG \) by the connected component \( L \mathbb{P}_2 \) of the loop group of the Borel subgroup \( B_\infty \) of \( G \). One can also describe \( X \) as the universal covering space of the loop space of the flag manifold \( X \) of \( G \), cf. [FF4], § 4.

We take as the big cell on \( \hat{X} \), the orbit \( \hat{U} \) of the unit coset under the action of the loop group of \( N_+ \), \( L \mathbb{N}_+ \). This orbit is isomorphic to \( L \mathbb{N}_+ \), and hence to \( n_+ \otimes \mathbb{C}[t, t^{-1}] \), because \( N_+ \simeq n_+ \) via the exponential map. Hence we obtain coordinates \( x_\alpha(n) = x_\alpha \otimes t^n, \alpha \in \Delta_+, n \in \mathbb{Z} \), on the big cell \( \hat{U} \).
We can now identify the algebra of differential operators on $\hat{U}$ with $U(\mathcal{H}_{\Delta^+})$, where $\mathcal{H}_{\Delta^+}$ is the Heisenberg algebra introduced in § 1.1, factored by the relation $1 = 1$. Namely, we identify $a_\alpha(n)$ with $x_\alpha(-n)$ and $a_\alpha(n)$ with $\partial/\partial x_\alpha(n)$.

The loop algebra $L$ infinitesimally acts on $\hat{U}$ by vector fields. These vector fields are actually infinite sums, and therefore lie in a completion of the Lie algebra of vector fields on $\hat{U}$. If we could lift these vector fields to a completion of the algebra of differential operators, we would define an $L$-module from a module over the differential operators.

Of course, we could take as such a module, the space of functions on $\hat{U}$, i.e. the module generated by a vector $v$, such that $\partial/\partial x_\alpha(n) \cdot v = 0$, $n \in \mathbb{Z}$. But then the resulting module would not lie in the category $\mathcal{O}$, cf. [JK]. In order to obtain a module from the category $\mathcal{O}$, we should instead take the space of $\delta$-functions on $\hat{U}$ with support on its subspace $n_+ \otimes \mathbb{C}[t] \subset n_+ \otimes \mathbb{C}[t, t^{-1}] = \hat{U}$ of “semi-infinite dimension”. This module is therefore generated by a vector $v$, such that $\partial/\partial x_\alpha(n) \cdot v = 0$, $n \geq 0$, and $x_\alpha(n) \cdot v = 0$, $n < 0$. As an $\mathcal{H}_{\Delta^+}$-module, this is precisely the Fock module $M$, defined in § 1.2.

Thus, we want to make $M$ into an $L$-module. Therefore the completion of $U(\mathcal{H}_{\Delta^+})$, into which we should embed $L$, is the local completion $U(\mathcal{H}_{\Delta^+})_{\text{loc}}$, defined in § 1.3, because its action on $M$ is well-defined.

2.3. Wakimoto modules. There is a filtration of $U(\mathcal{H}_{\Delta^+})_{\text{loc}}$ by the powers of the generators $a_\alpha(n)$: $0 \subset D_{0, \text{loc}} \subset D_{1, \text{loc}} \subset \ldots$. We have the exact sequence:

\begin{equation}
0 \rightarrow D_{0, \text{loc}} \rightarrow D_{1, \text{loc}} \rightarrow \text{Vect} \tilde{U}_{\text{loc}} \rightarrow 0,
\end{equation}

and an embedding $\tilde{\epsilon} : L \rightarrow \text{Vect} \tilde{U}_{\text{loc}}$.

In order to make $M$ into a module over the loop algebra, we have to lift the map $\tilde{\epsilon}$ to a map $\epsilon' : L \rightarrow D_{1, \text{loc}}$. However, this can not be done, because in contrast to the finite-dimensional case, the exact sequence (4) does not split. Indeed, it defines a class in the cohomology group $H^2(\text{Vect} \tilde{U}_{\text{loc}}, D_{0, \text{loc}})$, which was shown to be one-dimensional [FF4], § 5.1.

This fact can be explained as follows. The Lie algebra $U(\mathcal{H}_{\Delta^+})_{\text{loc}}$ consists of infinite sums of monomials in $a_\alpha(n), a^*_\alpha(n)$. In order to make them act on the space $M$ we had to regularize them by means of normal ordering, cf. § 1.3. This normal ordering distorts commutation relations in such a way that in the commutator of two elements of $D_{1, \text{loc}}$ there appears an extra term lying in $D_{0, \text{loc}}$ (it is given by the sum of all double contractions in the Wick formula). This extra term defines a non-trivial extension (4). Note that we can not construct the inverse map $\text{Vect} \tilde{U}_{\text{loc}} \rightarrow D_{1, \text{loc}}$, because there are no “constants”, i.e. elements annihilated by all $\partial/\partial x_\alpha(n)$, in $M$.

Still, we can salvage the situation: it turns out that the extension of $L$ by $D_{0, \text{loc}}$ defined by (4) is cohomologically equivalent to its extension by $\mathbb{C} \subset D_{0, \text{loc}}$. It is possible therefore to lift $\tilde{\epsilon}$ to a map from the central extension $\hat{\mathfrak{g}}$ of $L$ to $D_{1, \text{loc}}$. 

6
Theorem 1 [W, FF1] (a) There exists a Lie algebra homomorphism $\hat{\mathfrak{g}} \to \mathcal{D}_{1,\text{loc}} \subset U(\mathcal{H}_{\Delta_k})_{\text{loc}}$, which maps $K$ to $-h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

(b) The space of homomorphisms $\hat{\mathfrak{g}} \to \mathcal{D}_{1,\text{loc}}$ is a principal homogeneous space over $\mathfrak{h} \otimes \mathbb{C}((z))dz$.

The Fock representation $M$ now provides a family of Wakimoto module over $\hat{\mathfrak{g}}$ of level $-h^\vee$, which is called the critical level. Such a module, $W_{\chi(z)}$ is attached to an arbitrary $H$-connection on the punctured formal disc, $\partial/\partial z + \chi(z)$, where $H$ is the dual group of the Cartan subgroup of $G$. We see that the category $\mathcal{O}$ at the critical level is much larger than at other levels. In fact, irreducible objects of this category (which are subquotients of Wakimoto modules) are parametrized by $G^L$-connections on the formal punctured disc satisfying a special transversality condition. These irreducible modules can be constructed as quotients of Verma modules by characters of the center of the local completion of $U(\hat{\mathfrak{g}})$, which are parametrized precisely by such connections, cf. [FF6] and § 4.5 below. Here $G^L$ is the Langlands dual group of $G$, and this fact can be used for constructing geometric Langlands correspondence (Drinfeld).

Example. Here we write down explicit formulas for the embedding of $\hat{\mathfrak{sl}}_2$ into $U(\mathcal{H})_{\text{loc}}$, depending on $\chi(z) = \sum_{n \in \mathbb{Z}} \chi_n z^{-n-1} \in \mathcal{O}(z))$. We will write $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$ for $x \in \mathfrak{sl}_2$. The currents $a(z)$ and $a^*(z)$ are defined by (1) (we omit unnecessary subscripts): $e(z) = a(z)$, $h(z) = -2 : a(z) a^*(z) : + \chi(z)$, $f(z) = - : a(z) a^*(z) a^*(z) : + \chi(z) a^*(z) - 2 \partial_z a^*(z)$. These formulas first appeared in [W]. Explicit formulas for $\hat{\mathfrak{sl}}_n$ first appeared in [FF1].

It is not difficult to generalize this construction to an arbitrary level.

Theorem 2 [W, FF1] There is a structure of $\mathfrak{g}$-module of level $k$ from the category $\mathcal{O}$ on $W_{\chi,k} = M \otimes \pi^k_{\chi+h^\vee}$.

2.4. Two-sided resolution. One can construct an analogue of the Bernstein-Gelfand-Gelfand (BGG) resolution [BGG, RW], which consists of Wakimoto modules. Any element $s$ of the Weyl group $W$ of $\hat{\mathfrak{g}}$ can be uniquely written as a product $\tilde{s} \cdot \gamma$, where $\tilde{s}$ is an element of the Weyl group of $\mathfrak{g}$, and $\gamma$ is an element of the root lattice of $\mathfrak{g}$. Put $lt(s) = l(\tilde{s}) + 2(\tilde{p}^\vee, \gamma)$, where $l(\tilde{s})$ is the usual length of $\tilde{s}$, and $\tilde{p}^\vee \in \mathfrak{h}^\vee$ is such that $(\tilde{p}^\vee, \alpha_i) = 1, i = 1, \ldots, l$; we call $lt(w)$ the modified length of $w$ [FF4]. Let $\lambda$ be a dominant integral weight.

Theorem 3 [FF4] There exists a complex of $\mathfrak{g}$-modules $R_{\lambda}^n$, where $R_{\lambda}^n = \oplus_{s \in W, lt(s) = n} W_{s(\lambda + \rho) - \rho}$, whose cohomology is non-zero only in dimension 0, where it is isomorphic to the irreducible $\mathfrak{g}$-module $L_\lambda$ of highest weight $\lambda$. 

7
In contrast to the usual BGG resolution of $\mathfrak{g}$, $R^*_f(\mathfrak{g})$ is two-sided. In the case of $\mathfrak{sl}_2$, these resolutions were constructed explicitly in [FF4], § 7.3 and in [BeF] (they are closely connected with similar resolutions over the Virasoro algebra constructed in [Fel]). In [BMP1, BMP2] a remarkable connection between $R^*_f$ and resolutions over the quantum group $U_q(\mathfrak{g})$ with $q = \exp 2\pi i/(k + h^\vee)$ was found, and explicit formulas for the differentials of $R^*_f$ were proposed.

2.5. Remarks. (1) The construction of Wakimoto modules is a semi-infinite version of the construction of induced and coinduced modules. In fact, it is not difficult to construct in a similar fashion a $\mathfrak{g}$-bimodule $\widehat{U}_k(\mathfrak{g})$ on which $\mathfrak{g}$ acts on the left with level $k$ and on the right with level $-2h^\vee - k$, so that $W_{i,k} \simeq \text{Tor}_{h^\vee/2}^\infty (\widehat{U}_k(\mathfrak{g}), \pi^k h^\vee)$, where $\text{Tor}_{h^\vee/2+}^\infty$ is the semi-infinite Tor functor [2] (compare with the construction of Verma modules from § 1.5).

(2) By construction, the modules $W_{i,k}$ and $W_{i,k}$ are free over the Lie algebra $\mathfrak{n}_+ \cap \mathfrak{n}_-$ and co-free over the Lie algebra $\mathfrak{n}_+ \cap \mathfrak{n}_+$. Therefore they are flat over $\mathfrak{n}_+$ in the sense of semi-infinite cohomology: $H^i_{\mathfrak{n}_+/\mathfrak{n}_+}(\mathfrak{n}_+, W_{i,k}) = \pi^k h^\vee$, if $i = 0$, and 0, if $i \neq 0$ (note that $H^i(\mathfrak{n}_+, M^*_\lambda) = \mathbb{C} \lambda$, if $i = 0$, and 0, if $i \neq 0$, where $H^i$ stands for the usual Lie algebra cohomology functor). Using this result and the two-sided resolution $R^*_f$, we computed in [FF4], Theorem 4, the semi-infinite cohomology of $\mathfrak{n}_+$ with coefficients in $L_\lambda$.

(3) In [FF4] a more general construction is given, which associates to an arbitrary parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, a “Borel subalgebra” of $\mathfrak{g}$. These “Borel subalgebras” are not conjugated with each other and therefore lead to different flag manifolds. Generalized Wakimoto modules can be defined as delta-functions on these manifolds. They are flat with respect to the corresponding “Borel subalgebra”. In particular, $M^*_{\lambda,k}$ corresponds to $\mathfrak{p} = \mathfrak{g}$ and $W_{i,k}$ corresponds to $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_*$.

(4) Wakimoto modules are related to the Shubert cell decomposition of the semi-infinite flag manifold in the same way as the Verma modules are related to the Shubert cell decomposition of the usual flag manifold, cf. [FF4]. In particular, the Floer cohomology of the semi-infinite flag manifold is the double of the semi-infinite cohomology of the Lie algebra $\mathfrak{n}_+$ (compare with the finite-dimensional case [Kos]).

(5) One can show that if $\chi(z)$ is of the form $\chi/z, \chi \in \mathfrak{h}^*$, and does not lie on certain hyperplanes, then $W_{\chi(z)}$ is isomorphic to the irreducible quotient of the Verma module $M_{\lambda,k}$. This gives a simple proof [FF2] of the Kac-Kazhdan conjecture [KK] on characters of irreducible modules.

(6) The character of the module $W_{i,k}$ coincides with the character of the Verma module $M_{i,k}$ and for generic values of $\chi$ and $k$ they are irreducible isomorphic to each other. When they are not irreducible, they may have different composition series, cf., e.g., [FF2] and [BeF] in the case of $\mathfrak{sl}_2$. A surprising fact [Fr2] is that if $k$ is real and $\leq k + h^\vee$, then $W_{i,k} \simeq M^*_{i,k}$ for positive $\chi$. 

8
3. Solutions of the Knizhnik-Zamolodchikov equation.

In this section we will outline the application of the Wakimoto realization to the computation of correlation functions (or conformal blocks) in the WZNW model. It is known that in genus zero they satisfy a system of PDE, which are called Knizhnik-Zamolodchikov (KZ) equations. Wakimoto realization allows one to express these correlation functions as integrals of much simpler correlation functions of free bosonic fields. This gives the Schechtman-Varchenko solutions of the KZ equations.

3.1. Genus zero conformal blocks. Let us recall the definition of the space of conformal blocks in the WZNW model. Consider the projective line $\mathbb{CP}^1$ with a global coordinate $t$ and $N$ distinct finite points $z_1, \ldots, z_N \in \mathbb{CP}^1$. In the neighborhood of each point $z_i$ we have the local coordinate $t - z_i$; denote $\mathfrak{g}(z_i) = \mathfrak{g} \otimes \mathbb{C}(t - z_i)$. Let $\mathfrak{g}_N$ be the extension of the Lie algebra $\bigoplus_{i=1}^N \mathfrak{g}(z_i)$ by one-dimensional center $\mathbb{C}K$, such that its restriction to any summand $\mathfrak{g}(z_i)$ coincides with the standard extension. The Lie algebra $\mathfrak{g}_N$ naturally acts on the $N$-fold tensor product of $\mathfrak{g}$-modules $\otimes_{i=1}^N M_i$ of a given level $k \neq -h^\vee$.

Let $C_N$ be the space $\mathbb{C}^N$ with coordinates $z_1, \ldots, z_N$ without the diagonals and $C'_N$ be the space $\mathbb{C}^N \times \mathbb{CP}^1$ with coordinates $z_1, \ldots, z_N$ and $t$ without the diagonals. Denote by $B(z)$ and $B'(z)$, where $z = (z_1, \ldots, z_N)$, the algebras of regular functions on $C_N$ and $C'_N$, respectively.

Introduce the Lie algebras $\mathfrak{g}_N(z) = \mathfrak{g}_N \otimes_{\mathbb{C}} B(z)$, and $\mathfrak{g}_z = \mathfrak{g} \otimes_{\mathbb{C}} B'(z)$. By expanding elements of $\mathfrak{g}_z$ in Laurent power series in the local coordinates $t - z_i$ at each point $z_i$, we obtain an embedding $\mathfrak{g}_z \to \mathfrak{g}_N(z)$.

The Lie algebra $\mathfrak{g}_N(z)$ acts on the space $X(M_1, \ldots, M_N; z) = \text{Hom}_{\mathbb{C}}(M_1 \otimes \ldots \otimes M_N, B(z))$. Denote by $H(M_1, \ldots, M_N; z)$ the $B(z)$-module of $\mathfrak{g}_z$-invariants of $X(M_1, \ldots, M_N; z)$. One has: $H(M_1, \ldots, M_N; z) \simeq H(M_1, \ldots, M_N) \otimes_{\mathbb{C}} B(z)$, where $H(M_1, \ldots, M_N)$ is called the space of conformal blocks.

Consider the operators $\nabla_i = \partial/\partial z_i - L_i^{(i)}$, $i = 1, \ldots, N$, on the space $X(M_1, \ldots, M_N; z)$, where $L_i^{(i)}$ denotes the operator acting as the dual of the Virasoro generator $L_{-1}$ (provided by the Sugawara construction) on the $i$th argument of $X(M_1, \ldots, M_N; z)$. One can check, cf., e.g., [FFR], Lemma 4, that the operators $\nabla_i$ commute with each other and normalize the action of the Lie algebra $\mathfrak{g}_z$. Therefore they define a flat connection on the trivial bundle over $C_N$ with the fiber $H(M_1, \ldots, M_N)$.

3.2. The KZ equations. Now let us choose as the modules $M_i$, the Wakimoto modules $W_{\lambda_i, k}$. Recall that $W_{\lambda_i, k}$ is the tensor product $M \otimes \pi_{\lambda_i}^{k,h^\vee}$. Consider its subspace $\mathbb{C}[a_0^-(0)]_{a \in \Delta_+} v \otimes v_{\lambda_i}$. As a module over the constant subalgebra $\mathfrak{g} \subset \mathfrak{g}$, it is isomorphic to the contragradient Verma module $M_{\lambda_i}^*$. For any $\Psi(z) \in H(W_{\lambda_1, k}, \ldots, W_{\lambda_N, k}; z)$ denote by $\psi(z)$ its restriction to the subspace $\otimes_{i=1}^N M_{\lambda_i}^* \subset \otimes_{i=1}^N W_{\lambda_i, k}$. Thus, $\psi(z)$ can be considered as a vector in $\otimes_{i=1}^N M_{\lambda_i}$.
Lemma 1 If $\nabla_i \Psi(z) = 0$ for $i = 1, \ldots, N$, then $\psi(z)$ satisfies the system of equations

$$\left(k + h^\vee\right) \frac{\partial \psi(z)}{\partial z_i} = H_i \cdot \psi(z), \quad i = 1, \ldots, N,$$

where $H_i = \sum_{j \neq i} I_a^{(j)} I_a^{(j)}/(z_i - z_j)$, and $I_a^{(j)}$ denotes an element of an orthonormal basis $\{I_a\}$ of $\mathfrak{g}$ acting on the $i$th factor of $\otimes_{i=1}^N M_{\lambda_i}$.

For the proof of Lemma 1, cf., e.g., [FFR], § 6. The equations (5) are called the KZ equations [KZ].

3.3. Bosonic correlation functions. In the same way as in § 3.1 we can define spaces of conformal blocks with respect to the Heisenberg algebra $\mathfrak{h} \oplus \mathcal{H}_{\Delta^+}$ [FFR]. Denote by $J_p(x)$ the space associated to the tensor product of Wakimoto modules $\otimes_{i=1}^N W_{\chi_i, k}$, where $x = (x_1, \ldots, x_p)$ [FFR], § 6. This is a free $\mathcal{B}(x)$-module with one generator $\psi_p$, such that $\psi_p(v_p) = 1$, where $v_p$ is the tensor product of the highest weight vectors of $W_{\chi_i, k}, i = 1, \ldots, p$.

In the same way as in the previous section one can show that the operators $\nabla_i = \partial/\partial x_i - L_1^{(i)}$, $i = 1, \ldots, p$, act on the space $J_p(x)$. If $\Phi \in J_p(x)$ is such that $\nabla_i \Phi = 0$, then $\varphi = \Phi(v_p) \in \mathcal{B}(x)$ satisfies the system of equations

$$\left(k + h^\vee\right) \frac{\partial \varphi}{\partial x_i} = \sum_{j \neq i} \left(\chi_i, \chi_j\right) \varphi, \quad i = 1, \ldots, p.$$ 

These equations are the analogues of the KZ equations for the Heisenberg algebra, but they are much simpler: the unique up to a constant factor solution is $\varphi_p = \prod_{i < j} (x_i - x_j)^{\left(\chi_i, \chi_j\right)/(k + h^\vee)}$. Denote $\tau_p = \varphi_p \cdot v_p \in J_p(x)$. This is the correlation function of the scalar bosonic filed.

3.4. Solutions. Now put $p = N + m$, $x_i = z_i, \chi_i = \lambda_i, i = 1, \ldots, N$, and $x_{N+j} = w_j, \chi_{N+j} = -\alpha_i, j = 1, \ldots, m$. Then $\varphi_{N,m} = \prod_{i < j} (z_i - z_j)^{(\lambda_i, \lambda_j)/(k + h^\vee)} \prod_{i} (z_i - w_j)^{-(\lambda_i, \alpha_j)/(k + h^\vee)} \prod_{i < j} (w_i - w_j)^{(\alpha_i, \alpha_j)/(k + h^\vee)}$.

Denote by $C_{m,x}$ the space $\mathbb{C}^n$ with coordinates $w_1, \ldots, w_m$ without all diagonals $w_j = w_i$ and all hyperplanes of the form $w_j = z_i$. The multi-valued function $\varphi_{N,m}$ defines a one-dimensional local system $\mathcal{L}$ on the space $C_{m,x}$.

We can define another action of the Lie algebra $\hat{\mathfrak{n}}_+$ on the module $W_{\chi, k}$, which commutes with the action defined by the Wakimoto realization. It comes from the right action of the Lie algebra $\hat{\mathfrak{n}}_+$ on the big cell $\mathcal{U}$ of the semi-infinite flag manifold, [FF2, FF3]. We denote the operator of the right action of $\mathcal{L}(n) \in \hat{\mathfrak{n}}_+$ on $W_{\chi, k}$ by $e_i^R(n)$.

The restriction of $\tau_{N,m} = \varphi_{N,m} v_{N,m} \in J_{N,m}(z, w)$ to the subspace $\otimes_{i=1}^N W_{\lambda_i} \otimes e_1^R(-1)v_{-\alpha_1} \otimes \ldots \otimes e_m^R(-1)v_{-\alpha_m}$.
defines an element of \( X(W_{\lambda_1,k},\ldots,W_{\lambda_N,k}; z) \) depending on \( w_1,\ldots,w_m \). We denote this restriction by \( \Phi_{N,m} \). For any \( \omega \in W_{\lambda_1,k} \otimes \cdots \otimes W_{\lambda_N} \), the multi-valued form \( \Phi_{N,m}(\omega) \, dw_1 \cdots dw_m \) can be considered as an \( m \)-form on \( C_{m,x} \) with coefficients in the local system \( \mathcal{L} \). It can be integrated over an \( m \)-cycle \( \Delta \) with coefficients in the dual local system \( \mathcal{L}^* \), cf. [SV1, SV2].

Special “screening” properties of the vectors \( e_i^R(-1)v_{-\alpha_i} \in W_{-\alpha_i,k} \) make them “invisible” for the affine algebra after integration and lead to the following result.

**Theorem 4** \( \int_{\Delta} \Phi_{N,m} \, dw_1 \cdots dw_m \) lies in \( H(W_{\lambda_1,k},\ldots,W_{\lambda_N,k}; z) \) and \( \nabla_i \int_{\Delta} \Phi_{N,m} \, dw_1 \cdots dw_m = 0 \) for \( i = 1,\ldots,N \).

By Lemma 1 we can obtain solutions of the KZ equations by restricting this integral to the subspace \( \otimes_{\lambda_i=1}^N M_{\lambda_i}^* \) of \( \otimes_{\lambda_i=1}^N W_{\lambda_i,k} \). We refer the reader to [ATY] and [FFR] for this computation and only give the final result.

Introduce the vector \( |w_{i_1}^1,\ldots,w_{i_m}^m\rangle \in \otimes_{\lambda_i=1}^N M_{\lambda_i} \) by the formula

\[
|w_{i_1}^1,\ldots,w_{i_m}^m\rangle = \sum_{\pi=(i',\ldots,i^N)} \prod_{j=1}^N \left( \frac{f_{i_1}^{(j)} f_{i_2}^{(j)} \cdots f_{i_{j-1}}^{(j)}}{w_{i_1}^j - w_{i_2}^j} (w_{i_2}^j - w_{i_3}^j) \cdots (w_{i_{j-1}}^j - z_j) \right) 0, 
\]

where the summation is taken over all ordered partitions \( I^1 \sqcup I^2 \sqcup \cdots \sqcup I^N \) of the set \( \{1,\ldots,m\} \), \( I^j = \{i_1^j, i_2^j, \ldots, i_{\lambda_j}^j\} \), \( f_{i_1}^{(j)} \) denotes the generator \( f_i \in \mathfrak{g} \) acting on the \( j \)-th component of \( \otimes_{\lambda_i=1}^N M_{\lambda_i} \), and \( |0\rangle = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_N} \).

**Corollary** Let \( \Delta \) be an \( m \)-dimensional cycle on \( C_{m,x} \) with coefficients in \( \mathcal{L}^* \). The \( \otimes_{\lambda_i=1}^N M_{\lambda_i} \)-valued function

\[
\int_{\Delta} \varphi_{N,m} |w_{i_1}^1,\ldots,w_{i_m}^m\rangle \, dw_1 \cdots dw_m
\]

is a solution of the KZ equation.

Thus, we obtained solutions of the KZ equations in terms of generalized hypergeometric functions using the Wakimoto modules. These solutions were first derived by Schechtman and Varchenko by other methods [SV1] (cf. also [I, DJMM]).

### 3.5. Remarks.

1. One can derive these solutions in a slightly different way as matrix elements of compositions of primary fields and screening operators, which can be constructed explicitly [FF2, FF3, BeF, BMP1, BMP2, ATY].

2. The results of this section mean that a complicated \( D \)-module on the space \( \mathbb{C}^N \setminus \{\text{diagonals}\} \) defined by the KZ equations (5) can be embedded into the direct image of a much simpler \( D \)-module on a larger space \( \mathbb{C}^{N+m} \setminus \{\text{diagonals}\} \) defined by the equations (6). Wakimoto realization provides a natural explanation of this remarkable fact.
Using Wakimoto modules at the critical level in a similar fashion, it was shown in [FFR] that the vector (7) is an eigenvector of the Gaudin Hamiltonians $H_i$, if $w_j$'s satisfy a system of Bethe ansatz equations.

4. Free field realizations from the theory of non-linear equations

Local integrals of motion of non-linear integrable equations form Poisson algebras, which in many cases can be naturally embedded into larger Heisenberg-Poisson algebras. By quantizing this embedding we can obtain an embedding of the algebra of quantum integrals of motion into a Heisenberg algebra. This provides another source for free field realizations. We will describe this construction in the case of Toda equations.

4.1. Classical Toda field theory. Let $\mathfrak{g}$ be a simple Lie algebra and $\alpha_1, \ldots, \alpha_l \in \mathfrak{h}^*$ be the set of simple roots of $\mathfrak{g}$. The Toda equation associated to $\mathfrak{g}$ reads

$$\partial_t \phi_i(t, \tau) = \sum_{j=1}^{l} (\alpha_i, \alpha_j) \exp[-\phi_j(t, \tau)], \quad i = 1, \ldots, l,$$

where each $\phi_i(t, \tau)$ is a family of functions on the circle with a coordinate $t$, depending on the time variable $\tau$.

We will now review the hamiltonian formalism of the Toda equations following our papers [FF7, FF8], in which turn used technique developed in [GD, KW].

Let $\pi_0 = \mathbb{C}[u_i^{(n)}]_{1 \leq i \leq l, n \geq 0}$ be the algebra of differential polynomials in $u_1, \ldots, u_l$, where $u_i \equiv u_i^{(0)}$. It is $\mathbb{Z}$-graded according to $\text{deg} u_i^{(n)} = n+1$, and there is a derivation $\partial$ on $\pi_0$, such that $\partial u_i^{(n)} = u_i^{(n+1)}$. Let us formally introduce variables $\phi_i, i = 1, \ldots, l$. For any element $\lambda = \sum_{1 \leq i \leq l} \lambda_i \alpha_i$ of the root lattice of $\mathfrak{g}$, define the linear space $\pi_\lambda = \pi_0 \otimes e_\lambda$, where $\lambda = \sum_{0 \leq i \leq l} \lambda_i \phi_i$. We extend the action of the operator $\partial$ to $\pi_\lambda$ by putting $\partial e_\lambda = \sum_{0 \leq i \leq l} \lambda_i u_i e_\lambda$. In other words, we put $\partial \phi_i = u_i$.

Denote by $F_\lambda$ the quotient of $\pi_\lambda \otimes \mathbb{C}[t, t^{-1}]$ by the subspace of total derivatives (and constants, if $\lambda = 0$), where the action of $\partial$ on $\pi_\lambda \otimes \mathbb{C}[t, t^{-1}]$ is given by $\partial \otimes 1+1 \otimes \partial_t$. We denote by $f$ the projection $\pi_\lambda \rightarrow F_\lambda$. The space $F_\lambda$ can be viewed as the space of functionals in $u_1(t), \ldots, u_l(t) \in \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$ of the form $\int P(u(t), \partial_t u(t), \ldots; t)e^{\lambda}(t)dt$, where $P$ is a differential polynomial. We call $F_0$ the space of local functionals in $u_1(t), \ldots, u_l(t)$.

There is a unique partial Poisson bracket $\{\cdot, \cdot\} : F_0 \times F_\lambda \rightarrow F_\lambda$, such that:

$$\{ \int u_i^{(n)} t^m, \int u_j^{(n)} t^m \} = n(\alpha_i, \alpha_j) \delta_{m,-m}, \quad \{ \int u_i^{(n)} t^m, \int e^{\lambda}(t) \} = (\alpha_i, \lambda) \int e^{\lambda}(t) n+m,$$

cf. [FF7], § 2.2. The restriction of this bracket to $F_0$ makes it into a Lie algebra.

The equation (8) can be presented in the hamiltonian form as $\partial_t u_i(t, \tau) = \{ H, u_i(t, \tau) \}, \quad i = 1, \ldots, l$, where $H = \sum_{i=1}^l \int e^{-\phi_i}$. This motivates the definition of the space $I(\mathfrak{g})$ of local integrals of motion of the Toda theory associated to $\mathfrak{g}$ as the intersection of
4.2. Hidden nilpotent action. Introduce linear operators

$$\tilde{Q}_i = \sum_{1 \leq r \leq m \geq 0} (a_i, a_j) \partial^n e^{-\phi} \cdot \frac{\partial}{\partial u_j^{[n]}}, \quad i = 1, \ldots, l,$$

acting from $\pi_0$ to $\pi_{-n}$. Put $Q_i = e^\phi \tilde{Q}_i : \pi_0 \rightarrow \pi_i, i = 1, \ldots, l$. The following crucial statement was proved in [FF7], (2.2.8).

Lemma 2 The operators $Q_i, i = 1, \ldots, l$, and $\tilde{Q}_i, i = 1, \ldots, l$, generate the nilpotent Lie subalgebra $n_+ \subset \mathfrak{g}$. The operators $\tilde{Q}_i$ commute with $\partial$ and the corresponding operators $F_\partial = \partial F_i \colon F_0 \rightarrow F_{-n}$. We will extend this complex further to the right using the BGG resolution of $\mathfrak{g}$ and then reduce the computation to the cohomology of $n_+$ with coefficients in $\pi_0$, cf. [FF7], §3 2.3-2.4.

Recall [BGG] that the BGG resolution of the trivial $\mathfrak{g}$-module is a complex $B_\ell(g)$, such that $B_\ell(g) = \bigoplus_{\ell(s) = j} M_{\ell(\rho) - \rho}$, where $M_\lambda$ denotes the Verma module $\mathfrak{g}$ with highest weight $\lambda$ and $w$ runs over the Weyl group of $\mathfrak{g}$. The differential $d_j : B^j_\ell(g) \rightarrow B^{j+1}_\ell(g)$ is an alternating sum of embeddings of Verma modules $i_{s',s} : M_{\ell(\rho) - \rho} \rightarrow M_{\ell(\rho) - \rho}$, where $\ell(s) = j - 1, \ell(s') = j$ and $s \leq s'$. Under this embedding, the highest weight vector $1_{s'(\rho) - \rho}$ maps to a unique singular vector $P_{s',s} 1_{s'(\rho) - \rho} \in M_{\ell(\rho) - \rho}$, where $P_{s',s}$ is an element of $U(n)$.

We now define a complex $F^\ast(\mathfrak{g})$, such that $F^\ell(\mathfrak{g}) = \bigoplus_{\ell(s) = j} \pi_{s(\rho) - \rho}$. Let $P_{s',s}(Q) : \pi_{s'(\rho) - \rho} \rightarrow \pi_{s(\rho) - \rho}$ be the map, obtained by inserting into $P_{s',s} \in U(n)$ the operators $Q_i$ instead of $F_i$. Introduce the differential $\delta^\ast : F^{\ell-1}(\mathfrak{g}) \rightarrow F^\ell(\mathfrak{g})$ of our complex as the alternating sum of the appropriate $P_{s',s}(Q)$’s. Since $Q_i$’s generate $n_+$, this differential is nilpotent.

Moreover, this differential commutes with the action of $\partial$ on $\pi_{s(\rho) - \rho}$ [FF7], (2.4.9). Therefore we can define a new complex $F^\ast(\mathfrak{g})$ as the quotient of $F^\ast(\mathfrak{g}) \otimes \mathbb{C}[t, t^{-1}]$ by the total derivatives and constants. We have $F^\ell(\mathfrak{g}) = \bigoplus_{\ell(s) = j} F_{s(\rho) - \rho}$. By definition, $H^\ell(\mathfrak{g})$ is the $\ell$th cohomology of $F^\ast(\mathfrak{g})$.

Theorem 5 [FF7] (a) There exist elements $W_i \in \pi_0$ of degrees $d_i + 1, i = 1, \ldots, l$, where $d_i$ is the $i$th exponent of $\mathfrak{g}$, such that the $0$th cohomology $H^0(\mathfrak{g})$ of the complex $F^\ast(\mathfrak{g})$ is isomorphic to the algebra of differential polynomials in $W_1, \ldots, W_l$. $H^0(\mathfrak{g}) = \mathbb{C}[W_i^{[m]}]_{i=1,\ldots,l,m\geq0}$. All higher cohomologies of the complex $F^\ast(\mathfrak{g})$ vanish.

(b) The space $I^0(\mathfrak{g})$ is isomorphic to the quotient of $W(\mathfrak{g}) \otimes \mathbb{C}[t, t^{-1}]$ by the total derivatives and constants, i.e. the space of local functionals in $W_1(t), \ldots, W_l(t)$.
Example \( W(\mathfrak{sl}_2) = \mathbb{C}[W^{(m)}]_{m \geq 0} \), where \( W = \frac{1}{2} u^2 - \partial v \). Thus, for \( \mathfrak{g} = \mathfrak{sl}_2 \) integrals of motion are local functionals in \( W(t) = \frac{1}{2} u(t) - \partial v(t) \).

Let \( \mathcal{L} \) be the Virasoro algebra, which is the central extension of the Lie algebra \( \mathbb{C}[f, t^{-1}] \partial t \) of vector fields on the circle. We can consider \( W(t) \) as an element of the hyperplane in its dual \( \mathcal{L}^* \), which consists of the linear functionals taking value 1 on the central element. This hyperplane is equipped with a canonical Kirillov-Kostant hyperplane in its dual \( \mathcal{L}^* \). The space of local functionals on this hyperplane is isomorphic to \( I(\mathfrak{sl}_2) \), cf. [FF7], § 2.1, for more details.

**Remark** The Lie algebra \( I(\mathfrak{g}) \) is called the classical \( \mathcal{W} \)-algebra associated to \( \mathfrak{g} \). It can be identified with the Poisson algebra of local functionals on an infinite-dimensional hamiltonian space obtained from the dual space to \( \mathfrak{g} \) by the Drinfeld-Sokolov reduction [DS]. For \( \mathfrak{g} = \mathfrak{sl}_N \), it was first defined by Adler and Gelfand-Dickey.

### 4.3. Quantum integrals of motion

In the previous section we obtained the space of integrals of motion of a Toda theory as a Lie subalgebra of \( \mathcal{F}_0 \), which lies in the kernel of the operators \( \mathcal{Q}_I \). Now we want to quantize this embedding. In order to do that, we have to quantize the Lie algebra \( \mathcal{F}_0 \) and the operators \( \mathcal{Q}_I \).

The Lie algebra \( \mathcal{F}_0^\nu := U(\mathfrak{h})_{0\nu} \) defined in § 1.3 is a quantum deformation of the Lie algebra \( \mathcal{F}_0 \), in the following sense. We have a natural map \( \mathcal{F}_0^\nu \to \mathcal{F}_0 \), which sends a Fourier component of a current, \( f : P(h_i(z), \partial_i h_i(z), \ldots) : z^m dz \in \mathcal{F}_0^\nu \) to the local functional \( \int P(u_i(t), \partial_i u_i(t), \ldots) t^m dt \in \mathcal{F}_0 \) and sets \( \nu \) to 0\(^1\). Denote by \( \mathcal{A} \) the image of \( A \in \mathcal{F}_0^\nu \) in \( \mathcal{F}_0 \) under this map. The commutator of any two elements of \( \mathcal{F}_0^\nu \) has the form: \([A, B] = \nu \cdot C + \nu^2(\ldots)\), where \( C = \{\mathcal{A}, B\} \) is the bracket of \( \mathcal{A} \) and \( B \) in \( \mathcal{F}_0 \), cf. [FF7], § 4.2 (\( \nu \) was denoted by \( \beta \) in [FF7]).

By definition, the Lie algebra \( \mathcal{F}_0^\nu \) is the quotient of \( \pi_0^\nu \otimes \mathbb{C}[z, z^{-1}] \) by the total derivatives and constants, where \( \pi_0^\nu \) is the VOA of the Heisenberg algebra \( \mathfrak{h} \). Denote by \( \mathcal{F}_0^\nu \) the quotient of \( \pi_0^\nu \otimes \mathbb{C}[z, z^{-1}] \) by the total derivatives. We have a map \( \pi_0^\nu \to \pi_\lambda \), where \( \pi_\lambda \) was defined in § 4.1, which sends \( h_1(n_1) \ldots h_m(n_m) v_\lambda \in \pi_0^\nu \) to \((-n_1 - 1)! \ldots (-n_m - 1)! u_1^{-(n_1 - 1)} \ldots u_m^{-(n_m - 1)} \otimes e^1 \in \pi_\lambda \), and sets \( \nu \) to 0.

The quantizatum deformation of the operator \( \mathcal{Q}_I : \pi_0 \to \pi_{-\alpha} \), is the operator \( \mathcal{Q}_I^\nu := V_\nu^\alpha(1) : \pi_0^\nu \to \pi_{-\alpha}^\nu \), where \( V_\nu^\alpha(n) \) was defined by formula (2). Indeed, one can check, cf. [FF7], (4.2.4), that \( \mathcal{Q}_I^\nu = \nu \cdot \mathcal{Q}_I + \nu^2(\ldots) \). Further, we can check that \( \mathcal{Q}_I^\nu \) commutes with the derivative \( \partial \). The corresponding operators \( \mathcal{Q}_I^\nu : \mathcal{F}_0^\nu \to \mathcal{F}_{-\alpha}^\nu \) are quantum deformations of \( \mathcal{Q}_I \), \( i = 1, \ldots, l \).

We can now define the space \( I_\nu(\mathfrak{g}) \) of quantum integrals of motion of Toda theory associated to \( \mathfrak{g} \) as

\[
I_\nu(\mathfrak{g}) = \bigcap_{i=1}^l \text{Ker} \mathcal{Q}_I^\nu.
\]

\(^1\)To be more precise, we should consider \( \nu \) as a formal parameter and \( \mathcal{F}^\nu_0, \pi_\lambda^\nu \) as free modules over \( \mathbb{C}[[\nu]] \). Then setting \( \nu \) to 0 means taking the quotient by \( \nu \mathbb{C}[[\nu]] \)
One can check that \( I_{\nu}(\mathfrak{g}) \) is a Lie subalgebra of \( \mathcal{F}_\nu^0 \), \([FF7], (4.2.8)\). Thus, we define it through its embedding into \( \mathcal{F}_\nu^0 \), i.e. through its free field realization.

We also define the space \( \mathcal{W}_\nu(\mathfrak{g}) \) as

\[
\mathcal{W}_\nu(\mathfrak{g}) = \bigcap_{i=1}^{l} \ker_{\pi_0} \tilde{Q}_i^\nu.
\]

One can check that \( \mathcal{W}_\nu(\mathfrak{g}) \) is a VOA \([FF7], (4.2.8)\).

### 4.4. \( \mathcal{W} \)-algebras

We will now deform the complex \( F^\nu(\mathfrak{g}) \). Vanishing of higher cohomologies of \( F^\nu(\mathfrak{g}) \) will imply their vanishing for the deformed complex. This will allow us to prove that the 0th cohomology \( \mathcal{W}_\nu(\mathfrak{g}) \), and hence \( I_{\nu}(\mathfrak{g}) \), are quantum deformations of \( \mathcal{W}(\mathfrak{g}) \) and \( I(\mathfrak{g}) \), respectively.

The construction of the quantum complex is based on the fact that while the operators \( \tilde{Q}_i \) generate \( U(n_+) \), their quantum deformations, operators \( \tilde{Q}_i^\nu \), generate the quantized enveloping algebra \( U_q(n_+) \) with \( q = \exp \pi i v \) in a certain sense, cf. \([FF7], \S \, 4.5\), for a precise statement. This was discovered in \([BMP]\), cf. also \([SV2]\), where a more general connection between local systems on configuration spaces and quantum groups was established.

We can then use a quantum deformation of the BGG resolution, cf. \([FF7], \S \, 4.4\), to construct the deformed complex \( F^\nu_q(\mathfrak{g}) \). As a linear space, \( F^\nu_q(\mathfrak{g}) = \oplus_{i=1}^{l} \pi_{-\rho} \). The differentials are constructed using the operators \( \tilde{Q}_i^\nu \), cf. \([FF7], \S \, 4.5\). The differential \( \delta_\nu^i : \pi_0^\nu \rightarrow \oplus_{i=1}^{l} \pi_{-\rho} \) is given by the sum of the operators \( \tilde{Q}_i^\nu \), \( i = 1, \ldots, l \), so that the 0th cohomology of \( F^\nu_q(\mathfrak{g}) \) is \( \mathcal{W}_\nu(\mathfrak{g}) \).

Now we have a family of complexes \( F^\nu_q(\mathfrak{g}) \), depending on \( \nu \). From vanishing of higher cohomologies for \( \nu = 0 \), cf. Theorem 5, we obtain the following result.

**Theorem 6** \([FF7]\) (a) For generic \( \nu \) higher cohomologies of the complex \( F^\nu_q(\mathfrak{g}) \) vanish. The 0th cohomology, \( \mathcal{W}_\nu(\mathfrak{g}) \), is a VOA, in which there exist elements \( W_i^\nu \) of degrees \( d_i + 1 \), \( i = 1, \ldots, l \), where \( d_i \) is the \( i \)th exponent of \( \mathfrak{g} \), such that \( \mathcal{W}_\nu(\mathfrak{g}) \) has a linear basis of lexicographically ordered monomials in the Fourier components \( W_i^\nu(n) \), \( 1 \leq i \leq l, n_i < -d_i \), of the currents \( Y(W_i^\nu, z) = \sum_{n \in \mathbb{Z}} W_i(n) z^{-n-d_i-1} \).

(b) The Lie algebra \( I_{\nu}(\mathfrak{g}) \) of quantum integrals of motion of the Toda theory associated to \( \mathfrak{g} \) consists of all Fourier components of currents of the VOA \( \mathcal{W}_\nu(\mathfrak{g}) \).

The Lie algebra \( I_{\nu}(\mathfrak{g}) \) is the \( \mathcal{W} \)-algebra associated to \( \mathfrak{g} \). Such \( \mathcal{W} \)-algebras are, along with affine algebras, the main examples of algebras of symmetries of conformal field theories \([BS]\). Part (a) of Theorem 6 means that its VOA \( \mathcal{W}_\nu(\mathfrak{g}) \) is “freely generated” by the currents \( Y(W_i^\nu, z) \).

Note that \( \mathcal{W}_0(\mathfrak{sl}_1) \) is the VOA of the Virasoro algebra. Its embedding into \( \pi_0^\nu \) has been known for a long time \([FCT]\). In was used by Feigin-Fuchs \([FeFu]\) to study representations of the Virasoro algebra, and by Dotsenko-Fateev \([DF]\) to obtain correlation functions of the minimal models, in the same way as in \( \S \, 3 \).
The VOA $\mathcal{W}_e(\mathfrak{g}_e)$ was first constructed by Zamolodchikov [Z1], and the VOA $\mathcal{W}_e(\mathfrak{g}_h), n > 3$, was first constructed by Fateev-Lukyanov [FL] (cf. also [BG]). The existence of $\mathcal{W}_e(\mathfrak{g})$ as a VOA “freely” generated by currents of degrees $d_i + 1$ for an arbitrary $\mathfrak{g}$ was an open question until [Fr1, FF7].

To summarize, we defined the VOA of a $\mathcal{W}$–algebra as a vertex operator subalgebra of a VOA of free fields, subject to a set of constraints. The constraints satisfy certain algebraic relations, which make it possible to describe the structure of the $\mathcal{W}$–algebra: the operators $Q_i^\pm$ generate the nilpotent part of $U_q(\mathfrak{g})$, so that $U_q(\mathfrak{g})$ and $\mathcal{W}_e(\mathfrak{g})$ form a “dual pair”. The classical origin of these constraints is a non-linear integrable equation; the Toda equation, and therefore the classical limit of a $\mathcal{W}$–algebra consists of local integrals of motion of this equation. In our forthcoming joint work with Feigin we will derive the Wakimoto realization in a similar fashion.

4.5. Quantum Drinfeld-Sokolov reduction. $\mathcal{W}$–algebras can also be defined through the quantum Drinfeld-Sokolov reduction [FF6, Fr1]. Let $\mathcal{C}$ be the Clifford algebra with generators $\psi_\alpha(n), \psi_\alpha^+(n), \alpha \in \Delta_+, n \in \mathbb{Z}$, and anti-commutation relations
\[
[\psi_\alpha(n), \psi_\beta(m)]_+ = [\psi_\alpha^+(n), \psi_\beta^+(m)]_+ = 0, \quad [\psi_\alpha(n), \psi_\beta^+(m)]_+ = \delta_{\alpha, \beta} n - m.
\]
Denote by $\Lambda$ its Fock representation, generated by vector $v$, such that $\psi_\alpha(n)v = 0, n \geq 0, \psi_\alpha^+(n)v = 0, n > 0$. This is the super-VOA of $\mathcal{C}$. Introduce a $\mathbb{Z}$–grading on $\mathcal{C}$ and $\Lambda$ by putting $\deg \psi_\alpha(n) = -\deg \psi_\alpha(n) = 1, \deg v = 0$.

Now consider the complex $(V_k \otimes \Lambda, d)$, where $V_k$ is the VOA of $\mathfrak{g}$ of level $k$, and $d = d_{\text{st}} + \chi$. Here $d_{\text{st}}$ is the standard differential of semi-infinite cohomology of $\hat{\mathfrak{h}}$ with coefficients in $V_k$ [F], and $\chi = \sum_{i=1}^r \psi_\alpha^+(1)$ corresponds to the Drinfeld-Sokolov character of $\hat{\mathfrak{h}}$ [DS]. The cohomology $H^k_*(\mathfrak{g}) = \bigoplus_{i \geq 0} H^i_k(\mathfrak{g})$ of this complex is a VOA [FF6]. This cohomology can be computed using the spectral sequence, in which the 0th differential is $d_{\text{st}}$ and the first differential is $\chi$.

**Proposition 2** [FF6, Fr1] For generic $k \neq -h^\vee$ the spectral sequence degenerates into the complex $F^*_{1/(k + h^\vee)}(\mathfrak{g})$. Thus, $H^0_k(\mathfrak{g}) \cong \mathcal{W}_{1/(k + h^\vee)}(\mathfrak{g})$ and $H^i_k(\mathfrak{g}) = 0, i \neq 0$.

The second part of Proposition 2 was proved for an arbitrary $k$ in [dBT] using the opposite spectral sequence.

For any module $M$ from the category $\mathcal{O}$ of $\mathfrak{g}$, the cohomology of the complex $(M \otimes \Lambda, d)$ is a module over the $\mathcal{W}$–algebra $I_{1/(k + h^\vee)}(\mathfrak{g})$. This defines a functor, which was studied in [FKW].

The limit of the $\mathcal{W}$–algebra $I_{1/(k + h^\vee)}(\mathfrak{g})$ when $k = -h^\vee$ is isomorphic to the center $Z_{-h^\vee}(\mathfrak{g})_{\text{loc}}$ of $U_{-h^\vee}(\mathfrak{g})_{\text{loc}}$ [FF6, Fr1]. It can also be identified with $I(\mathfrak{g}^L)$, where $\mathfrak{g}^L$ is the Langlands dual Lie algebra to $\mathfrak{g}$ [FF6, Fr1]. This proves Drinfeld’s conjecture that $Z_{-h^\vee}(\mathfrak{g})_{\text{loc}} \cong I(\mathfrak{g}^L)$, which can be used in the study of geometric Langlands correspondence.
4.6. **Affine Toda field theories.** An analogue of the complex $F^*_\nu(g)$ can be constructed for an arbitrary Kac-Moody algebra. In the case of an affine algebra its first cohomology can be identified with the space of local integrals of motion of the corresponding affine Toda field theory [FF7].

Let us first consider the classical case [FF7], § 3, [FF8]. The Toda equation associated to an affine algebra $\hat{g}$ is given by formula (8), in which the summation is over $i = 0, \ldots, l$, and $\phi_0(t) = -1/a_0 \sum_{i=0}^l a_i \phi_i(t)$, where $a_i$'s are the labels of the Dynkin diagram of $\hat{g}$ [K]; $\phi_0(t)$ corresponds to the extra root $a_0$ of $\hat{g}$. Following the scheme of § 4.1, we define the space $I(\hat{g})$ of local integrals of motion as

$$I(\hat{g}) = \bigcap_{i=0}^l \text{Ker}_{\pi_0} \hat{Q}_i.$$  

Using the BGG resolution of the affine algebra $\hat{g}$ [RW], we can construct a complex $F^*(\hat{g})$ in the same way as in § 4.2. Now the operators $\hat{Q}_i : \pi_0 \to \pi_{-a_i}, i = 0, \ldots, l$, generate the nilpotent subalgebra $\hat{n}_+$ of $\hat{g}$. The cohomology of the complex $F^*(\hat{g})$ coincides with the cohomology of $\hat{n}_+$ with coefficients in $\pi_0$, $H^*(\hat{n}_+, \pi_0)$.  

**Theorem 7** [FF7, FF8] $H^*(\hat{n}_+, \pi_0) \simeq \Lambda^*(\hat{a}^*_+)$, and $I(\hat{g}) \simeq H^*(\hat{n}_+, \pi_0) = \hat{a}^*_+$, where $\hat{a}^*_+$ is the dual space to the principal abelian subalgebra of $\hat{n}_+$.  

Theorem 7 implies that local integrals of motion of the Toda theory associated to $\hat{g}$ have degrees equal to the exponents of $\hat{g}$ modulo the Coxeter number. The corresponding hamiltonian equations form the modified KdV hierarchy [DS, KW]. In [FF8] we gave a geometric interpretation of these equations.

We can now define the space $I_\nu(\hat{g})$ of quantum integrals of motion as

$$I_\nu(\hat{g}) = \bigcap_{i=0}^l \text{Ker}_{\pi_0} \hat{Q}_i^\nu.$$  

Using the BGG resolution over $U_\nu(\hat{g})$ we proved in [FF7] that all classical integrals of motion can be quantized, so that $I_\nu(\hat{g}) \simeq I(\hat{g})$. Note that for $\hat{g} = \hat{sl}_2$ this was conjectured in [G]. The quantum integrals of motion form an abelian subalgebra of the $\mathcal{W}$–algebra $I_\nu(\hat{g})$. They can be viewed as conservation laws of integrable perturbations of conformal field theories associated to $\mathcal{W}_\nu(g)$ [Z2, EY, HM].

Free field realization, which at first appeared as a technical tool for computing correlation functions of conformal field theories, has evolved in the last few years to a powerful method of representation theory of conformal algebras. There is every indication that similar ideas are applicable to a much broader class of models of quantum field theory and statistical mechanics, and to related representations.
References


605-609.


*Department of Mathematics, Harvard University, Cambridge, MA 02138, USA*