Representations of the (2+1) Dimensional Galilean Group

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Abstract

Using the Mackey theory of induced representations, the unitary irreducible representations of the proper Galilean group in 2+1 space time dimensions are constructed. All representations, both true as well as the projective ones, are found. The concomitant representations of the Lie algebra are given.

I. Introduction

The projective unitary representations (PUR) of the Galilean group in 3+1 dimensional space time were constructed by Bargmann[1], in course of a general analysis of the PURs of a Lie group. Bargmann[1] found a one-parameter family of central extensions of the Galilean group, by the circle group, such that the desired PURs could be constructed from the linear (unitary) representations of the centrally-extended group. The central parameter, which had an associated superselection rule, could be interpreted as the non-relativistic mass of a particle. Secondly, the PURs were induced from representations of the universal covering group $SU(2)$ of the rotation group $SO(3)$ in particles rest frame. This procedure has its justification in the fact[2] that the universal central extension of a semi-simple group, such as the homogeneous Galilean group, is its universal covering group. The universal central extension of the Galilean group is the semidirect product of the universal central extension of its homogeneous subgroup with the subgroup of spacetime translations. In any event, the passage from $SO(3)$ to $SU(2)$ could be interpreted as a central extension of $SO(3)$ by the two-element group Z_2 . The PURs are thus characterized by the mass and spin of the non-relativistic particle. The ordinary representations of the Galilean group were considered earlier by Enönü and Wigner[2]. These do not correspond to physical systems that are localizable in space[2,3], or even states of definite velocity. A fuller description of the representations, both ordinary and projective, was later presented by Levy-Leblond[4], who explicitly constructed the group representations, as also those of the corresponding Lie algebra. The concept of a non-relativistic zero mass system is elucidated in this last reference. Representations based on a Hilbert bundle e.g., a vector bundle whose fiber is a Hilbert space, together with possible physical applications have been proposed by Borchers and Sen[5].

In view of the foregoing developments, it came as somewhat of a surprise to the present author that a similar analysis for the (2+1) dimensional Galilean group has not been attempted in

the literature. There is a strong physical motivation for doing this; since there exists physical systems that effectively live in the two spatial dimensions. The problem is also mathematically interesting. The Galilean groups in three and two spatial dimensions have significantly differing structures. For three spatial dimensions, the subgroup of homogeneous transformations has the structure of a semi-direct product of a simple group (subgroup of rotations) with a normal abelian subgroup (of boosts); it is, in fact, a perfect group (a group is perfect if it coincides with its commutator subgroup). In two spatial dimensions, the homogeneous group is the semidirect product of a compact abelian group (subgroup of rotations) with a normal vector subgroup of boosts; it has the structure of a solvable group.

The purpose of the present paper is to construct the unitary irreducible representations, both linear and projective, of the proper Galilean group in two spatial dimensions. As is well-known, the projective representations of a group can be constructed from a knowledge of the linear representations of the centrally-extended groups, by the circle group. For the present case, the problem of group extension was solved recently by us[6]. The results of ref. 6 will be utilized here. We shall classify the representations into four broad categories: i) the linear representations of the group; these correspond to zero mass and commutative boosts, and three classes of non-trivial PURs. These are characterized by ii) zero mass and non-commutative boosts, iii) finite mass with commutative boosts and iv) finite mass with non-commutative boosts.

This paper is organized as follows. In the next section 2 we summarize the properties of the Galilean group and its central extensions. We construct an universal central extension $G^{\tau\lambda}$ and show that it has a structure of a semidirect product of a normal abelian subgroup A^τ and the factor group $H^\lambda = G^{\tau\lambda}/A^\tau$. This is the desired setting to construct the induced representations. In section 3 we present a very brief summary of the procedure adopted in the inducing construction. The material of this section is well-known and is presented here to make this paper reasonably self-contained. The details concerning the construction of group representations are presented in

section 4. The representations of the Lie algebra that correspond to the group representations of section 4 are presented in section 5. In the last section 6, we make concluding remarks.

2. The Galilean group and its central extensions.

We write the typical element r of the Galilean group G in the fashion

$$r = (h, \eta, \mathbf{v}, \mathbf{u}) \quad (2.1)$$

where $\mathbf{v}$ and $\mathbf{u}$ are two-dimensional vectors signifying boosts and spatial translations, respectively, and η is a real number that denotes time translation; h is a complex number of unit magnitude that stands for rotation in a plane. The multiplication law in G is

$$r r' = (hh', \eta + \eta', \mathbf{v} + \rho(h) \mathbf{v}', \mathbf{u} + \rho(h) \mathbf{u}' + \eta' \mathbf{v}) \quad (2.2)$$

where $\rho(h)$ is the action of h on two-dimensional vectors. The inverse of the element r in G is

$$r^{-1} = (h^{-1}, -\eta, -\rho^{-1}(h) \mathbf{v}, \rho^{-1}(h) (\eta \mathbf{v} - \mathbf{u})) \quad (2.3)$$

and the identity of G is $(1, 0, 0, 0)$. The abelian subgroup A of all space time translations

$$A = \{(1, \eta, 0, \mathbf{u})\} \quad (2.4)$$

is normal in G . The subgroup H of homogeneous transformations

$$H = \{(h, 0, \mathbf{v}, 0)\} \quad (2.5)$$

is a semidirect product of the subgroup of rotations $\{(h, 0, 0, 0)\}$ and the abelian subgroup of boosts $\{(1, 0, v, 0)\}$; since every element of H admits the unique decomposition $(h, 0, v, 0) = (1, 0, v, 0)(h, 0, 0, 0)$. The first derived group[7] (commutator subgroup) of H is isomorphic with the abelian subgroup of boosts and the second derived group consists of the identity alone. Thus H is a solvable group of class 2. The Galilean group G is a semidirect product of A with H . Every element $r \in G$ admits the unique decomposition $r = ah$, $a \in A$ $h \in H$ given by

$$(h, \eta, v, u) = (1, \eta, 0, u)(h, 0, v, 0) \quad (2.6)$$

and the action of H on A is

$$(h, 0, v, 0)(1, \eta, 0, u)(h, 0, v, 0)^{-1} = (1, \eta, 0, \rho(h)u + \eta v) \quad (2.7)$$

This concludes our discussion of G and we now proceed to consider its central extensions by the circle group T .

As shown in ref. 6, the group G has two families of two-dimensional cohomology classes with values in T . The cohomology classes may be described by representative 2-cocycles $m_{1\tau}(r, r')$ and $m_{2\lambda}(r, r')$, where

$$m_{1\tau}(r, r') = \exp \frac{i\tau}{2} \{ \langle u, \rho(h)v' \rangle - \langle v, \rho(h)u' \rangle + \eta' \langle v, \rho(h)v' \rangle \} \quad (2.8)$$

and

$$m_{2\lambda}(r, r') = \exp \frac{i\lambda}{2} (v \wedge \rho(h)v') \quad (2.9)$$

In the above, τ and λ are non-zero real numbers, $\cdot$ denotes the Euclidean scalar product in two-dimensional space and for any pair $(\mathbf{a}, \mathbf{b})$ of two dimensional vectors $\mathbf{a} \wedge \mathbf{b} = a_1 b_2 - a_2 b_1$ in terms of their Cartesian components. We may note

$$m_{1\tau}(r, r^{-1}) = 1, \quad m_{2\lambda}(r, r^{-1}) = 1 \quad (2.10)$$

as follows from (2.3), (2.8) and (2.9). In the present context, it is appropriate to call $m_{1\tau}$ and $m_{2\lambda}$ multipliers and the expressions in the exponent of (2.8) and (2.9) as R-multipliers.

With the aid of the above multipliers, we can construct two group extensions of G by T . The first (extended) group G_1^τ consists of elements (r, ξ) with $r \in G$, $\xi \in T$ and with the multiplication law

$$(r, \xi)(r', \xi') = (r r', \xi \xi' m_{1\tau}(r, r')) \quad (2.11)$$

and the inverse

$$(r, \xi)^{-1} = (r^{-1}, \xi^{-1}) \quad (2.12)$$

Notice that eq. (2.10) has been utilized in obtaining the above result (2.12). The second group extension G_1^τ has exactly the same structure as above, except that in the multiplication rule (2.11) the multiplier $m_{1\tau}(r, r')$ is to be replaced by the multiplier $m_{2\lambda}(r, r')$. We shall now construct the group $G^{\tau\lambda}$ from which we shall be able to descend to G_1^τ and G_2^λ via homomorphisms. First of all, we will allow the parameters τ and λ that characterize the multipliers to run over the entire real line (including zero). A second question concerns the precise characterization of the rotation subgroup of $G^{\tau\lambda}$, and this hinges on the question of the unit in terms of which the angular momentum is quantized. If we choose the unit of quantization to be $\hbar/2$, then the rotation

subgroup of $G^{\tau\lambda}$ is the group of rotations in the plane – the group $SO(2)$. If, on the other hand, our unit for angular momentum is $\hbar$, then *we shall choose* the 'rotation' subgroup of $G^{\tau\lambda}$ to be the two-fold covering of $SO(2)$, which is the group $U(1)$. These two alternative descriptions are equivalent, e.g., lead to the same conclusions.

Note:

As Lie groups $SO(2)$ and $U(1)$ are isomorphic. Considered as transformation groups of the circle, they are different; they implement the 2π rotation of the circle differently.

The elements of $G^{\tau\lambda}$ are ordered triples (r, ζ, ξ) with $(\zeta, \xi) \in T \times T$:

$$G^{\tau\lambda} = \{(r, \zeta, \xi)\} \equiv \{(h, \eta, v, u; \zeta, \xi)\} \quad (2.13)$$

with the multiplication rule

$$(r, \zeta, \xi) \circ (r', \zeta', \xi') = (r r', m_{1\tau}(r, r') \zeta \zeta', m_{2\lambda}(r, r') \xi \xi') \quad (2.14)$$

where $\pi r'$ is given by eq. (2.2) and the multipliers by eqs. (2.8) and (2.9). The identity element is $(1, 1, 1) \equiv (1, 0, 0, 0; 1, 1)$ and the inverse

$$(r, \zeta, \xi)^{-1} = (r^{-1}, \zeta^{-1}, \xi^{-1}) \quad (2.15)$$

It is clear that there exists homomorphisms from $G^{\tau\lambda}$ onto G_1^{τ} and onto G_2^{λ} as well as onto the corresponding trivial extensions (direct products). In this sense, the group $G^{\tau\lambda}$ is universal. From a knowledge of the ordinary (unitary, irreducible) representations of $G^{\tau\lambda}$ we shall be able to construct the *projective* (unitary, irreducible) representations of the Galilean group; as will be seen in the sequel.

The structure of $G^{\tau\lambda}$ can be analyzed in more than one way, in terms of a chosen sequence of subgroups. For our purpose, the most convenient procedure is the one that we adopt below. Consider the subset

$$A^\tau = \langle (1, \eta, 0, \mathbf{u}; \zeta, 1) \rangle \quad (2.16)$$

Since $m_{1\tau}(r, r') = m_{2\lambda}(r, r') = 1$ whenever $\mathbf{v} = \mathbf{v}' = 0$, it follows from (2.2) and (2.14) that A^τ is an abelian subgroup of $G^{\tau\lambda}$. We shall sometimes write the element $(1, \eta, 0, \mathbf{u}; \zeta, 1)$ as $(\eta, \mathbf{u}, \zeta)$. Next, consider the subset

$$H^\lambda = \langle (h, 0, \mathbf{v}, 0; 1, \xi) \rangle \quad (2.17)$$

Since, $m_{1\tau}(r, r') = 1$ if $\mathbf{u} = \mathbf{u}' = \eta' = 0$, it follows that H^λ is a subgroup with

$$\begin{aligned} (h, 0, \mathbf{v}, 0; 1, \xi) (h', 0, \mathbf{v}', 0; 1, \xi') \\ = (hh', 0, \mathbf{v} + \rho(h)\mathbf{v}', 0; 1, m_{2\lambda}(r, r')\xi\xi') \end{aligned} \quad (2.18)$$

We may note the inverse

$$(h, 0, \mathbf{v}, 0; 1, \xi)^{-1} = (h^{-1}, 0, -\rho(h)^{-1}\mathbf{v}, 0; 1, \xi^{-1}) \quad (2.19)$$

We shall sometimes write $(h, \mathbf{v}, \xi)$ for $(h, 0, \mathbf{v}, 0; 1, \xi)$. The subgroup H^λ can be analyzed in terms of its subgroup of rotation consisting of elements of the form $(h, 0, 1)$ and the subgroup whose typical element is $(1, \mathbf{v}, \xi)$. This last subgroup is the central extension by T of a two-dimensional vector group. It is thus isomorphic with the Heisenberg group whose Lie algebra generators (q, p, e) satisfy the commutation relations $[q, p] = e$, $[q, e] = 0$ and $[p, e] = 0$. H^λ is a

semidirect product $(h, v, \xi) = (1, v, \xi)(h, 0, 1)$. The first derived group (commutator subgroup) of H^λ is the Heisenberg group and the second derived group (commutator subgroup of Heisenberg) is isomorphic with T . The third derived group of H^λ consists of identity alone. Thus H^λ is a solvable group of class 3.

The group $G^{\tau\lambda}$ is a semidirect product of A^τ and H^λ . The only element that these two subgroups share in common is the identity and every element of $G^{\tau\lambda}$ can be written uniquely as a product of two elements, one belonging to A^τ and the other to H^λ . This decomposition is given by the following

$$(h, \eta, v, u; \zeta, \xi) = (1, \eta, 0, u; \zeta', 1)(h, 0, v, 0; 1, \xi) \quad (2.20)$$

where

$$\zeta' = \zeta \exp \left\{ -\frac{i}{2} \tau \langle u, v \rangle \right\} \quad (2.21)$$

Next problem is to figure out the action of H^λ on A^τ . Using eqs. (2.2), (2.3), (2.13) and (2.14) we derive

$$\begin{aligned} (h, 0, v, 0; 1, \xi) (1, \eta, 0, u; \zeta, 1) (h, 0, v, 0; 1, \xi)^{-1} \\ = (1, \eta, 0, \rho(h)u + \eta v; \zeta'', 1) \end{aligned} \quad (2.22)$$

where

$$\zeta'' = \zeta \exp \left\{ -\frac{i\tau}{2} (2 \langle v, \rho(h)u \rangle + \eta \langle v, v \rangle) \right\} \quad (2.23)$$

The above is the adjoint action of H^λ on A^τ ; $G^{\tau\lambda}$ is a semidirect product of A^τ and H^λ with respect to this action. In the sequel we propose to use a more economic notation for the adjoint action. We shall thus write eq. (2.22) in the following fashion

$$(h, v, \xi) : (\eta, u, \zeta) \rightarrow (\eta, \rho(h)u + \eta v, \zeta'') \quad (2.24)$$

The above notation will also be used to denote the adjoint action of H^λ on the dual group $\widehat{A}^\tau$ of A^τ (section 4).

3. Induced representations.

The Mackey theory of induced representations is the generalization, to the category of locally compact second countable groups, of the method of Frobenius for finite groups and of Wigner's celebrated construction of the unitary, irreducible representations of the inhomogeneous Lorentz group. The general theory is developed in many books[7-10]. The underlying intuitive content of the theory is beautifully explained by Mackey in his two memoirs[8,9]. A detailed presentation of the theory, including full workout of many examples, is given in the book by Varadarajan[10]. For our purpose, the Mackey theory in its full generality is not needed. Accordingly, we confine our attention to those aspects of the theory that we really need.

Let the Lie group G have the structure of a semidirect product of a normal, abelian subgroup A and the quotient group $H = G/A$. Then every element $g \in G$ can be written uniquely in the form $g = ah$ with $a \in A$ and $h \in H$. Let $\widehat{A}$ be the dual group of A ; its elements are the characters of A . Then the representations $U(g)$ of $g \in G$ is written

$$U(g) = \chi(a) V(h) \quad , \quad g = ah \quad (3.1)$$

where $\chi(a)$ is the character of a and it remains to figure out the representation $V(h)$ of $h \in H$. The group G acts on A and hence on $\hat{A}$ with the aid of the adjoint action. The H -space $\hat{A}$ breaks up into a union of H -orbits (H acts transitively on each orbit). On each orbit θ let us select a point x_0 and let H_0 be the stability subgroup of H at x_0 (the little group of x_0). The Mackey procedure constructs the desired representation V from a representation m of H_0 , as follows. Consider functions f defined on the orbit θ , which are square-integrable (in the Lebesgue sense) with respect to a measure μ that lives on θ . The desired representation V can then be written as

$$(V(h)f)(x) = \phi(h, \delta(h)^{-1}x) f(\delta(h)^{-1}x) \quad (3.2)$$

where $x \in \theta \subset \hat{A}$, δ is a homomorphism of H into the group of automorphisms of $\hat{A}$ ($\delta(h)x$ is the point on θ to which x is sent under the action of $h \in H$) and $\phi(h, x)$ is a (H, θ, M) cocycle that defines a representation m of the stability group H_0 at x_0 (see below). The representation U obtained in the foregoing manner is called an induced representation of G induced by the representation m of H_0 . Let us now take note of the following features of the general theory.

- (I) The induced representation U is irreducible if the inducing representation m is chosen to be irreducible.
- (II) The induced representation U will possess the required continuity properties (which is part of the definition of a representation) inherited from the continuity properties of the cocycle ϕ (see below).
- (III) The induced representation U can be made unitary if the measure μ on the H -orbit θ is invariant under the action of H . In the latter event, we define, on the linear space of the functions f , a norm $\|f\|$ by the formula

$$\|f\|^2 = \int_{\mathfrak{g}} f^*(x) f(x) d\mu(x) \quad (3.3)$$

The norm defined above satisfies the polarization identity and hence leads to an inner product and thus to a certain (separable) Hilbert space K . The representation U is unitary in K .

To complete the foregoing discussion, we wish now to explain what is meant by a cocycle. Consider the triple (G, X, M) , where G is a Lie group, X a G -space and M the unitary group in some Hilbert space H . A (G, X, M) cocycle f is a function on $G \times X$ taking values in M that satisfies the identities

$$(I) \quad f(e, x) = 1 \quad \forall x \in X$$

$$(II) \quad f(g_1 g_2, x) = f(g_1, g_2 x) f(g_2, x) \quad \forall g_1, g_2 \in G; \quad x \in X$$

and moreover has the property

$$(III) \quad f(g, x) \text{ is continuous in } g \text{ and in } x \text{ (it is enough to demand separate continuity). Here } M \text{ is given the strong topology of } H.$$

Remark: What we have called a cocycle above is called a strict cocycle in ref. 10. Moreover the requirement of continuity is phrased in a more general language e.g., involving the concept of a Borel map.

There is a standard procedure to construct a (G, X, M) cocycle. Let X be a transitive G -space (X a G -orbit). Let $x_0 \in X$ and G_0 the stability group at x_0 . Let c be a section i.e., a continuous map $X \rightarrow G$ that satisfies

$$c(x_0) = e, \quad c(x) x_0 = x \quad (3.4)$$

where e is the identity of G . Now, $c(gx)^{-1} g c(x)$ is an element of G_0 because

$$c(gx)^{-1} g c(x) x_0 = c(gx)^{-1} gx = x_0 \quad (3.5)$$

since $c(gx) x_0 = gx$. Now let m be a strongly continuous unitary representation of G_0 in some Hilbert space H . We set

$$f(g, x) = m(c(gx)^{-1} g c(x)) \quad (3.6)$$

It is now easy to check that $f(g, x)$ as given above is a (G, X, M) cocycle, where M is the unitary group in H .

A technical point to be touched upon is this. There is an equivalence relation on the set of all cocycles. Two cocycles f_1 and f_2 are cohomologous $f_1 \approx f_2$ if there exists a continuous map $b: X \rightarrow M$ such that $f_2(g, x) = b(gx) f_1(g, x) b(x)^{-1}$ for all $g \in G$ and $x \in X$. Under this equivalence, the cocycles fall into cohomology classes. Two members of the same class lead (via eq. (3.2)) to the same representation, up to unitary equivalence. Thus it is enough to select, out of each cohomology class, a typical representative cocycle.

A second point to consider is this. Under what condition does the inducing construction yield *all* of the irreducible unitary representations (IUR) of the group G ? The answer is this. Let the Lie group G be a semidirect product $G = AH$, A normal closed abelian and H the factor group. The inducing construction gives all the IURs of G if A is *regularly embedded* in G . The subgroup A is regularly embedded in G if the family of orbits of $\hat{A}$ (dual of A) under G is countably separated[9]. In the latter event we say that the orbit structure is *smooth*.

Thus far, we have discussed ordinary (linear) representations of a group and the method of constructing them via the inducing technique. Finally, a word on projective representations. A projective representation differs from a linear representation in just one way. Let x and y be any two elements of a group G , and U a projective representation of G . Then

$$U(xy) = m(x,y) U(x) U(y) \quad (3.7)$$

where $m(x,y)$ is a multiplier of G . These are constructible from the linear representations of the central extension of G (by T). We do not discuss this connection any further since explicit construction of projective representations will be presented in the sequel.

4. The Representations of the (2+1) Galilean Group.

We have shown in section 2 that the universal central extension group $G^{\tau\lambda}$ is a semidirect product $G^{\tau\lambda} = A^{\tau}H^{\lambda}$. This is the proper setting for induced representations. We shall construct the linear representations of $G^{\tau\lambda}$ and from these the projective representations of the Galilean group. Our representations will be irreducible since they will be induced from irreducible representations of the appropriate stability groups. Our representations will be unitary, since we will find an invariant measure on each orbit. Finally, the smoothness of the orbit structure will be verified so that we shall obtain all the representations via the inducing construction.

The first step in constructing the representations is to represent the abelian subgroup A^{τ} . We associate with each element a of A^{τ} a character $\chi(a)$ given by

$$\chi(\eta, \mathbf{u}, \zeta) = \zeta^n \exp i(\eta p_0 + \langle \mathbf{u}, \mathbf{p} \rangle) \quad (4.1)$$

where n is an integer, p_0 real and $\mathbf{p}$ a real two dimensional vector. Let $\hat{A}^\tau$ be the Pontryagin dual of A^τ . The elements of $\hat{A}^\tau$ are the characters of A^τ . We denote the typical element of $\hat{A}^\tau$ by $(n, p_0, \mathbf{p})$. Thus

$$\hat{A}^\tau = \{(n, p_0, \mathbf{p})\} \quad p_0 \in \mathbb{R}, \mathbf{p} \in \mathbb{R}^2, n \in \mathbb{Z} \quad (4.2)$$

Next problem is to figure out the adjoint action of H^λ on $\hat{A}^\tau$. This is found from eqs (2.24) and (4.1). A short calculation gives

$$(h, \mathbf{v}, \xi): (n, p_0, \mathbf{p}) \rightarrow \left(n, p_0 - \frac{n\tau}{2} \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{v}, \delta(h) \mathbf{p} \rangle, \delta(h) \mathbf{p} + n\tau \mathbf{v} \right) \quad (4.3)$$

Here $\delta(h)$ is the action of the rotation h on the two dimensional vectors. Technically, δ is a homomorphism of the rotation subgroup into the group of automorphisms of $\hat{A}^\tau$. Under the adjoint action (4.3) the following quantities

$$\langle \mathbf{p}, \mathbf{p} \rangle + 2n\tau p_0, \quad n \quad (4.4)$$

are invariant. Furthermore, under the action (4.3), the dual space $\hat{A}^\tau$ will break up into an union of H^λ -orbits. The resulting orbit structure will be analyzed by dividing the orbits into four separate classes.

$$4A. \quad \tau = 0, \lambda = 0.$$

In this case both τ and λ are set to zero. The resulting representations will be true representations of the Galilean group. The group G^{00} is simply the Galilean group (we shall set $\zeta=1, \xi=1$ here, thus also $n=0$).

The adjoint action is now

$$(h, v): (p_0, p) \rightarrow (p_0 - \langle v, \delta(h)p \rangle, \delta(h)p) \quad (4a.1)$$

The orbit structure is as follows. There are two types of orbits Z_{p_0} and X_r .

$$Z_{p_0} = (p_0, 0) \quad -\infty < p_0 < \infty \quad (4a.2)$$

$$X_r = \{(p_0, p)\} \quad \langle p, p \rangle = r^2 \quad r > 0 \quad (4a.3)$$

The orbit Z_{p_0} consists of one point; its stability group is H – the entire homogeneous Galilean group. The representations associated with Z_{p_0} are

$$U(h, \eta, v, u) = \exp(i p_0 \eta) \quad \pi(h, 0, v, 0) \quad (4a.4)$$

Where π is an arbitrary representation of H . The group H is isomorphic with the group of Euclidean motions on a plane, for which a complete system of (unitary, irreducible) representations have been given by Wigner[11].

Consider now the set X_r . This contains the point $x_0 = (0, p_*)$, where $p_* = (r, 0)$ in terms of its Cartesian components. Every point on X_r can be reached from x_0 under the action of a suitable element of H . Indeed, x_0 is carried to the typical point (p_0, p) of X_r under the action of (h_*, v) with $v = - (p_0/r) \hat{p}$, where $\hat{p}$ is a unit vector along p and h_* is such that $\delta(h_*)p_* = p$. Thus H action on X_r is transitive and X_r is an orbit. The stability group at x_0 is the abelian subgroup of boosts along the direction perpendicular to the vector p_* , as follows easily from eq. (4a.1).

To construct the desired cocycle, we first adopt the following convention. Let $x = (p_0, \mathbf{p})$, $x_0 = (0, \mathbf{p}_*)$, $\mathbf{p}_* = (r, 0)$ and g denote the element $(h, \mathbf{v})$ of H . This brings us in conformity with the notation used in section 3. We have the natural section

$$c(x) = \left(h_*, -\frac{p_0}{r} \hat{\mathbf{p}} \right) \quad (4a.5)$$

where h_* is such that $\delta(h_*)\mathbf{p}_* = \mathbf{p}$ and $\hat{\mathbf{p}}$ the unit vector along $\mathbf{p}$. Note that $c(x)$ takes x_0 to x , as shown before. The point gx is the point $(p_0 - \langle \mathbf{v}, \delta(h)\mathbf{p} \rangle, \delta(h)\mathbf{p})$ on X_r and

$$c(gx) = \left(hh_*, -\frac{p_0 - \langle \mathbf{v}, \delta(h)\mathbf{p} \rangle}{r^2} \delta(h)\mathbf{p} \right) \quad (4a.6)$$

From the above, we compute

$$c(gx)^{-1} g c(x) = (1, \tilde{\mathbf{v}}) \quad (4a.7)$$

where

$$\tilde{\mathbf{v}} = \delta(hh_*)^{-1} \mathbf{v} - \frac{\langle \delta(hh_*)^{-1} \mathbf{v}, \mathbf{p}_* \rangle}{r^2} \mathbf{p}_* \quad (4a.8)$$

Clearly, $\tilde{\mathbf{v}}$ is perpendicular to $\mathbf{p}_*$, thus (4a.7) belongs to the stability group at x_0 . Since the latter is abelian, all its irreducible representations are one-dimensional. Using eq. (3.6), we thus arrive at the following expression for our cocycle

$$\phi(g, x) = \exp \{ i a \langle \tilde{\mathbf{v}}, \hat{\mathbf{j}} \rangle \} \quad (4a.9)$$

where a is a real number and $\hat{\mathbf{j}}$ a unit vector along a direction perpendicular to the vector $\mathbf{p}_*$. With the aid of (4a.1) we derive from (4a.9)

$$\phi(g, g^{-1}x) = \exp \{ ia \langle \delta(h_*)^{-1} \mathbf{v}, \hat{\mathbf{j}} \rangle \} \quad (4a.10)$$

Writing out $\delta(h_*)$ as a 2×2 matrix that acts on two-dimensional column vectors, it is an elementary exercise to show that

$$\langle \delta(h_*)^{-1} \mathbf{v}, \hat{\mathbf{j}} \rangle = \frac{1}{r} (\mathbf{p} \wedge \mathbf{v}) \quad (4a.11)$$

With the form of the cocycle thus nailed down, we can now write down the final form of the irreducible representations.

$$\begin{aligned} (U(h, \eta, \mathbf{v}, \mathbf{u}) f)(p_0, \mathbf{p}) &= e^{i(\eta p_0 + \langle \mathbf{u}, \mathbf{p} \rangle)} \phi(g, g^{-1}x) f(g^{-1}x) \\ &= e^{i(\eta p_0 + \langle \mathbf{u}, \mathbf{p} \rangle)} e^{ia(\mathbf{p} \wedge \mathbf{v})/r} f(p_0 + \langle \mathbf{v}, \mathbf{p} \rangle, \delta(h)^{-1} \mathbf{p}) \end{aligned} \quad (4a.12)$$

where f is a square integrable function on X_r . The above representation is unitary because of the existence of the invariant measure on X_r , which is $d\mu(x) = dp_0 dt$, where dp_0 is the Lebesgue measure on $\mathbb{R}$ and dt the normalized measure on the circle. Note here that X_r is homeomorphic to the cylinder $\mathbb{R} \times \mathbb{T}$.

The orbit structure is evidently smooth; indeed, we have just two orbits. Thus we have found all the desired representations. These are given by (4a.4) and (4a.12). The IUR given by eq. (4a.12) is labeled by the pair (r, a) ; r is real, positive and a is real. More light on the significance of these two parameters will be shed in section 5. The representations are massless.

4B. $\tau = 0, \lambda \neq 0$

The group $G^{0\lambda}$ is a semidirect product of A – the spacetime translation group – with H^λ , which is a semidirect product of the rotation group with the Heisenberg group (section 2, eq. (2.17)). The adjoint action of H^λ on the dual $\hat{A}$ is here the same as in the previous case, section 4A (eq. (4a.1)). Consequently, the orbit structure is also the same as given by eqs. (4a.2) and (4a.3). The stability group at Z_{p_0} is the entire H^λ ; the corresponding representations are of the form given by eq. (4a.4), where now π stands for an arbitrary representation of H^λ . The latter are given in Bargmann[1]. We shall not discuss this case any further.

The orbit X_r contains the point $x_0 = (0, p_*)$, $p_* = (r, 0)$, as before. Let us find the stability group at x_0 . It consists of elements $\in H^\lambda$ of the form $(1, v, \xi)$, where v is perpendicular to p_* . Now a pair (v, v') of two-dimensional vectors, both of which are perpendicular to a vector p_* , must be parallel to each other and thus $v \wedge v'$ must vanish. The corresponding multiplier $m_{2\lambda}(r, r')$ will be unity. Thus the stability group has the multiplication rule

$$(1, v, \xi)(1, v', \xi') = (1, v + v', \xi\xi') \quad (4b.1)$$

We recognize the stability group at x_0 to be the direct product of an abelian subgroup of boosts along the direction perpendicular to p_* and the circle group.

To discuss the representations associated with X_r we adopt the following convention. Let x_0 and p_* be as before. The typical point (p_0, p) of X_r is denoted by x . The rotation h_* is defined by $\delta(h_*)p_* = p$. Let g here denote the element (h, v, ξ) of H^λ .

We have the sections

$$c(x) = \left(h^*, -\frac{p_0}{r^2} p, \xi^* \right) \quad (4b.2)$$

$$c(gx) = \left(hh^*, \frac{\langle v, \delta(h)p \rangle - p_0}{r^2} \delta(h)p, \xi^* \right) \quad (4b.3)$$

where ξ^* is a fixed element. The above expressions are practically identical with the corresponding ones of the previous case. The difference between the two cases will appear at the next step. We compute

$$gc(x) = \left(hh^*, v - \frac{p_0}{r^2} \delta(h)p, \xi \xi^* \exp \left(\frac{i\lambda}{2} \frac{p_0}{r^2} (\delta(h)p \wedge v) \right) \right) \quad (4b.4)$$

and from the above and eq. (4b.3), it follows

$$c(gx)^{-1} g c(x) = \left(1, \tilde{v}, \xi \exp \left(\frac{i\lambda}{2} \beta \right) \right) \quad (4b.5)$$

where

$$\beta = \frac{2p_0 - \langle v, \delta(h)p \rangle}{r^2} ((\delta(h)p) \wedge v) \quad (4b.6)$$

and $\tilde{v}$ is given by eq. (4a.8). Thus expression (4b.5) belongs to the stability group at x_0 . The latter is isomorphic with the direct product $R \times T$; its IURs are trivial to construct. We thus arrive at the cocycle

$$\phi(g, x) = \left(\exp \left\{ i\alpha \langle \delta(hh^*)^{-1} v, \hat{j} \rangle \right\}, \xi^m \exp \left(\frac{im\lambda\beta}{2} \right) \right) \quad (4b.7)$$

where a is real and m an integer. As a check on the calculation, we may verify directly that the above expression for $\phi(g, x)$ satisfies the cocycle identities (I)-(III) of section 3. This is straightforward. From (4b.7) we compute

$$\phi(g, g^{-1}x) = \left(\exp \left\{ \frac{ia}{r} \mathbf{p} \wedge \mathbf{v} \right\}, \exp \left\{ \frac{im\lambda}{2} \left(\frac{2p_0 + \langle \mathbf{v}, \mathbf{p} \rangle}{r^2} \mathbf{p} \wedge \mathbf{v} \right) \right\} \right) \quad (4b.8)$$

The desired representation now follow from eq. (4a.12) of the preceding case when we replace the cocycle of eq.(4a.10) by the cocycle (4b.8) of the present case. These representations are irreducible and unitary (the invariant measure on the orbit is the same as before), the orbit structure is also obviously smooth. These are the linear representations of the extended group $G^{0\lambda}$, from which we now proceed to construct the *projective* representations of the Galilean group.

First of all, it is not difficult to see that only the case $m = -1$ will lead to a projective representation, any other choice for m will not satisfy the defining condition eq. (3.7). Second, the typical element of the extended group $G^{0, \lambda}$ is of the form $(h, \eta, \mathbf{v}, \mathbf{u}; 1, \xi)$; and to descend to the Galilean group we have to set $\xi = 1$. We thus finally arrive at the explicit form of the PURs of the Galilean group

$$(U(h, \eta, \mathbf{v}, \mathbf{u})f)(p_0, \mathbf{p}) \quad (4b.9)$$

$$= \exp \{ i(\eta p_0 + \langle \mathbf{u}, \mathbf{p} \rangle) \} \exp \left\{ \frac{ia}{r} (\mathbf{p} \wedge \mathbf{v}) \right\} \exp \left\{ \frac{i\lambda}{2} \frac{2p_0 + \langle \mathbf{v}, \mathbf{p} \rangle}{r^2} (\mathbf{v} \wedge \mathbf{p}) \right\} f(p_0 + \langle \mathbf{v}, \mathbf{p} \rangle, \delta(h)^{-1} \mathbf{p})$$

To see the projective character of the above representation, we proceed as follows. We compute the representation of a pure boost $\mathbf{v}_1 \equiv (1, 0, \mathbf{v}_1, 0)$ by setting $\eta = 0, \mathbf{u} = 0$ in (4b.9). Next we calculate the effect of two successive boosts $\mathbf{v}_1$ and $\mathbf{v}_2$. We obtain thus

$$(U(\mathbf{v}_1) U(\mathbf{v}_2)f)(p_0, \mathbf{p}) = \exp \left\{ \frac{ia}{r} (\mathbf{p} \wedge \mathbf{v}) \right\} \exp \left\{ \frac{i\lambda}{2} \frac{\sigma}{r^2} \right\} f(p_0 + \langle \mathbf{v}, \mathbf{p} \rangle, \mathbf{p}) \quad (4b.10)$$

where

$$\sigma = (2p_0 + \langle \mathbf{v}_1, \mathbf{p} \rangle) \mathbf{v} \wedge \mathbf{p} + \langle \mathbf{v}, \mathbf{p} \rangle \mathbf{v}_2 \wedge \mathbf{p} \quad (4b.11)$$

and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ (not to be confused with the generic $\mathbf{v}$ in (4b.9)). With the aid of the identity

$$\langle \mathbf{a}, \mathbf{b} \rangle \mathbf{a} \wedge \mathbf{c} - \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{a} \wedge \mathbf{b} = \langle \mathbf{a}, \mathbf{a} \rangle \mathbf{b} \wedge \mathbf{c} \quad (4b.12)$$

that is satisfied by any three two-dimensional vectors, we may put eq. (4b.11) into the form

$$\sigma = (2p_0 + \langle \mathbf{v}, \mathbf{p} \rangle) \mathbf{v} \wedge \mathbf{p} - r^2 (\mathbf{v}_1 \wedge \mathbf{v}_2) \quad , \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \quad , \quad \langle \mathbf{p}, \mathbf{p} \rangle = r^2 \quad (4b.13)$$

From eqs. (4b.9), (4b.10) and (4b.13), we thus find

$$(U(\mathbf{v}_1) U(\mathbf{v}_2) f)(p_0, \mathbf{p}) = \exp \left\{ -\frac{i\lambda}{2} (\mathbf{v}_1 \wedge \mathbf{v}_2) \right\} (U(\mathbf{v}) f)(p_0, \mathbf{p}) \quad , \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \quad (4b.14)$$

Comparison of the above with eq. (3.7) shows that our representation is projective.

Remark

If we do not put $m = -1$, then the parameter λ in eq. (4b.14) would be replaced by $-m\lambda$. This proves the statement, made earlier, that only $m = -1$ leads to a projective representation.

The PURs (4b.9) are labeled by (a, r, λ) ; where a is real, r real positive and λ real non-zero.

$$4C. \lambda = 0, \tau \neq 0$$

The group $G^{\tau 0}$ is a semidirect product $A^{\tau}H$. The adjoint action of H on the dual $\hat{A}^{\tau}$ is given by

$$(h, v) : (n, p_0, p) \rightarrow \left(n, p_0 - \frac{n\tau}{2} \langle v, v \rangle - \langle v, \delta(h)p \rangle, \delta(h)p + n\tau v \right) \quad (4c.1)$$

Under the above action

$$\langle p, p \rangle + 2n\tau p_0 = \rho \quad (4c.2)$$

remains invariant. Here, n is an integer and ρ an arbitrary real number except when $n = 0$; in the latter event ρ is positive real. Let us introduce sets $Z_{n,\rho} = \{(n, p_0, p)\}$ with the variables restricted by eq.(4c.2). First consider the cases $n = 0$. The one point set $Z_{0,0}$ is not an orbit. $Z_{0,\rho}$ is an orbit which is identical in structure with X_r of section 4A. We have already discussed the representations associated with this case.

The set $Z_{n,\rho}$ ($n \neq 0$) is an orbit. It contains the point $\left(n, \frac{1}{2n\tau} \rho, 0\right)$ from which every point on the orbit can be reached under the action of a suitable element of H . Indeed

$$\left(1, \frac{1}{n\tau} p\right) : \left(n, \frac{1}{2n\tau} \rho, 0\right) \rightarrow (n, p_0, p) \quad (4c.3)$$

Thus $Z_{n,\rho}$ is an orbit for each fixed n . Let us adopt the notation; x denotes the typical point (n, p_0, p) on the orbit, g the typical element (h, v) of the subgroup H and x_0 the distinguished point $\left(n, \frac{1}{2n\tau} \rho, 0\right)$. The stability subgroup of H at x_0 consists of all elements of H of the form $(h, 0)$; it is the group of rotations in the rest frame. We construct the section $c: Z_{n,\rho} \rightarrow H$

$$c(x) = \left(1, \frac{1}{n\tau} \mathbf{p}\right) \quad (4c.4)$$

The point gx on the orbit is the point

$$gx = \left(n, p_0 - \frac{n\tau}{2} \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{v}, \delta(h) \mathbf{p} \rangle, \delta(h) \mathbf{p} + n\tau \mathbf{v}\right) \quad (4c.5)$$

and thus

$$c(gx) = \left(1, \frac{1}{n\tau} (\delta(h) \mathbf{p} + n\tau \mathbf{v})\right) \quad (4c.6)$$

From (4c.4)-(4c.6) we compute

$$c(gx)^{-1} g c(x) = (h, 0) \quad (4c.7)$$

which is an element of the stability group H_0 at x_0 . We thus have the cocycle

$$\phi(g, x) = s(h) \quad (4c.8)$$

where s is a (irreducible) representation of H_0 . Let $f(\mathbf{p})$ be square-integrable functions (taking values in some Hilbert space K on which s acts) of momenta $\mathbf{p}$. Recalling the decomposition (see eqs. (2.20) and (2.21))

$$(h, \eta, \mathbf{v}, \mathbf{u}; \zeta, 1) = (1, \eta, 0, \mathbf{u}; \zeta', 1) (h, 0, \mathbf{v}, 0; 1, 1) \quad (4c.9)$$

and using eqs. (4.1) and (4c.8) in the general expressions (3.1) and (3.2) we have

$$\begin{aligned} (U(h, \eta, \mathbf{v}, \mathbf{u}; \zeta) f)(\mathbf{p}) \\ = \zeta^n e^{i(\eta p_0 + \langle \mathbf{u}, \mathbf{p} \rangle)} e^{-i n \tau \langle \mathbf{u}, \mathbf{v} \rangle / 2} s(h) f(\delta(h)^{-1} (\mathbf{p} - n \tau \mathbf{v})) \end{aligned} \quad (4c.10)$$

where it is understood that (the now redundant) variable p_0 is to be expressed as $(\rho - \langle \mathbf{p}, \mathbf{p} \rangle) / 2n\tau$, following eq. (4c.2). The above representation is unitary because of the existence of the invariant measure $d\mu = d^2p$ on the orbit. Defining the norm as in eq. (3.3) with this measure, we thus have a certain (separable) Hilbert space K' on which the unitary representation acts. It is irreducible since we have chosen s to be irreducible. The orbit structure is evidently smooth. From the above linear representation of $G^{\tau 0}$ we construct the corresponding projective representations of the Galilean group as follows. First, we note without difficulty that only the case $n = -1$ will lead to a projective representation. Second, we put $\zeta = 1$. Third, we drop the factor $\exp(-i n \rho / 2\tau)$ in the resulting expression for U . This last step is mathematically permissible and is also taken recourse to in the construction of PURs of the (3+1) dimensional Galilean group[4]. Physically, it corresponds to no more than selecting a suitable zero for the scale of energy such that the relation between the energy and the momentum is the usual one of a non-relativistic theory (see eq. (5c.1)). The final form of PURs of the Galilean group is

$$\begin{aligned} (U(h, \eta, \mathbf{v}, \mathbf{u}) f)(\mathbf{p}) \\ = \exp \left\{ i \left(\langle \mathbf{u}, \mathbf{p} + \frac{1}{2} \tau \mathbf{v} \rangle + \frac{1}{2\tau} \eta \langle \mathbf{p}, \mathbf{p} \rangle \right) \right\} s(h) f(\delta(h)^{-1} (\mathbf{p} + \tau \mathbf{v})) \end{aligned} \quad (4c.11)$$

The above PURs are characterized by the non-zero real number τ and the IUR s of the 'rotation group in the rest frame' which is labeled by an integer (in units of $\hbar/2$). It is clear that (a positive) τ has the physical interpretation as the mass of a particle, whose momentum-space wave functions are the above functions $f(\mathbf{p})$.

4D. $\tau \neq 0, \lambda \neq 0$

The group $G^{\tau\lambda}$ is a semidirect product $A^{\tau}H^{\lambda}$ and the action of H^{λ} on the dual $\hat{A}^{\tau}$ is given by eq. (4.3). The resulting orbit structure is the same as in the preceding sub-section 4C. Using the same notation as in 4C to label point sets, we have, $Z_{0,0}$ is not an orbit; $Z_{0,p}$ is an orbit. The representations associated with this are precisely those of subsection 4B, lifted to the present case.

We consider the orbit $Z_{n,p}$ with $n \neq 0$. Let us agree to use the *same notation, as in the preceding subsection*, to denote the points x and x_0 on the orbit. Let g denote the typical element (h, v, ξ) of H^{λ} . The stability subgroup of H^{λ} at x_0 consists of elements $(h, 0, \xi)$; thus the stability group is isomorphic with a direct product – a 2-torus. We have the natural expression for the section $c : Z_{n,p} \rightarrow H^{\lambda}$ given by

$$c(x) = \left(1, \frac{1}{n\tau} p, 1\right) \quad (4d.1)$$

and proceeding exactly as in the preceding case, we obtain

$$c(gx)^{-1} g c(x) = \left(h, 0, \xi \exp \left\{ \frac{i\lambda}{2n\tau} (v \wedge \delta(h)p) \right\}\right) \quad (4d.2)$$

Thus the above belong to the stability group of x_0 , as it must. From the above, we construct the cocycle (via eq. (3.6))

$$\phi(g, x) = \left(s(h), \xi^m \exp \left\{ \frac{i\lambda m}{2n\tau} (v \wedge \delta(h)p) \right\}\right) \quad (4d.3)$$

and from the above

$$\phi(g, g^{-1}x) = \left(s(h), \xi^m \exp \left\{ \frac{i\lambda m}{2n\tau} (v \wedge p) \right\}\right) \quad (4d.4)$$

where m is an integer. The desired IU representations of $G^{\tau\lambda}$ are now easy to write down. They act on functions f of momenta $\mathbf{p}$; the invariant measure on the orbit is $d\mu(x) = d^2p$ as before, and s is the IR of the 'rotation group at the rest frame'. From these linear IUs of $G^{\tau\lambda}$ we construct the corresponding PURs of the Galilean group by setting $\zeta = 1$, $\xi = 1$, $n = -1$, $m = -1$; and after legislating away the factor $\exp(-i\eta p/2\tau)$. The final form of the PURs, thus obtained, is

$$\begin{aligned} & (U(\mathbf{h}, \eta, \mathbf{v}, \mathbf{u}) f)(\mathbf{p}) \\ &= \exp \left\{ i \left(\left\langle \mathbf{u}, \mathbf{p} + \frac{1}{2} \tau \mathbf{v} \right\rangle + \frac{1}{2\tau} \eta \langle \mathbf{p}, \mathbf{p} \rangle - \frac{\lambda}{2\tau} \mathbf{v} \wedge \mathbf{p} \right) \right\} s(\mathbf{h}) \\ & \quad \times f(\delta(\mathbf{h})^{-1}(\mathbf{p} + \tau \mathbf{v})) \end{aligned} \quad (4d.5)$$

The representations are characterized by the pair (τ, λ) of real numbers and by s , which is labeled by an integer (in units of $\hbar/2$).

5. Representations of the Lie Algebra.

Consider the Lie algebra of the universal central extension group $G^{\tau\lambda}$. We can choose a basis consisting of basis vectors M, N_i, H, P_i, d and m that satisfy[6] the commutation relations

$$\begin{aligned} [M, N_i] &= \varepsilon_{ij} N_j & [N_i, N_j] &= \varepsilon_{ij} d \\ [H, P_i] &= 0 & [P_i, P_j] &= 0 \\ [M, P_i] &= \varepsilon_{ij} P_j & [N_i, P_j] &= \delta_{ij} m \\ [M, H] &= 0 & [N_i, H] &= P_i \end{aligned} \quad (5.1)$$

and the central generators d and m commute with every generator of the Lie algebra. The indices i and j take up values 1 and 2, ϵ_{ij} is the antisymmetric symbol $\epsilon_{ji} = -\epsilon_{ij}$ with $\epsilon_{12} = 1$. A repeated index implies summation.

We note the existence of the invariant element c_2 belonging to the universal enveloping algebra

$$c_2 = m M + \epsilon_{ij} P_i N_j - dH \quad (5.2)$$

which commutes with all the generators of the Lie algebra. Note also the element $c_1 = p_1^2 + p_2^2$, which commutes with all generators except N_i , for which

$$[N_i, c_1] = 2 m P_i \quad i = 1, 2 \quad (5.3)$$

Thus c_1 becomes an invariant for those representations for which $m = 0$.

What are the representations of the Lie algebra that correspond to the group representations found in section 4? Before proceeding to answer this question, let us sketch the technique of descending from the group representation to the corresponding algebra representation, by means of an example. Consider the group representation given by eq. (4a.12). Construct from it the effect of a pure boost $(1, 0, \mathbf{v}, 0)$, with $\mathbf{v}$ infinitesimal. We have

$$(U(\mathbf{v})f)(p_0, \mathbf{p}) \approx \left(1 + \frac{\mathbf{a}}{r} \mathbf{p} \wedge \mathbf{v}\right) \left\{ f(p_0, \mathbf{p}) + \langle \mathbf{v}, \mathbf{p} \rangle \left(\frac{\partial f}{\partial p_0} \right)(p_0, \mathbf{p}) \right\} \quad (5.4)$$

where we expanded f in Taylor series and retained only terms linear in $\mathbf{v}$. Setting now $U(\mathbf{v})f \approx (1 + v_1 N_1 + v_2 N_2)f$ and comparing with the above we get

$$N_1 = p_1 \frac{\partial}{\partial p_0} - \frac{ia}{r} p_2, \quad N_2 = p_2 \frac{\partial}{\partial p_0} + \frac{ia}{r} p_1 \quad (5.5)$$

Repeated use of the above technique will yield the representations of all the generators of the Lie algebra. We proceed to discuss the four separate cases of the representations.

$$5A. \tau = 0, \lambda = 0$$

This case corresponds to the group representations of section 4A. The representations of the algebra that correspond to the group representation given by eq. (4a.4) are well-known and will not be repeated here.

We consider the group representation given by eq. (4a.12). We obtain the following for the infinitesimal generators of the Lie algebra eq. (5.1). The operators for P_i and H are multiplication by p_i and p_0 , the central generators vanish $d = 0$, $m = 0$ and

$$M = -p_1 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_1} \quad (5a.1)$$

while N_i are given by eq. (5.5). One may check that eq. (5.1) is satisfied. Notice that P_i , H are hermitian whereas M , N_i are skew-hermitian. The enveloping algebra in this case ($m = 0$) contains two invariants (Casimir operators), c_1 and c_2 . The eigenvalues of these invariants are $c_1 = r^2$, $c_2 = iar$.

$$5B. \tau = 0, \lambda \neq 0$$

This case corresponds to the group representations given in eq. (4b.9) subsection 4B. As before, the operators P_i and H are operators of multiplication by p_i and p_0 , respectively. The central generator m is zero. The remaining operators are

$$M = -p_1 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_1}, \quad d = -i\lambda \quad (5b.1)$$

$$N_1 = p_1 \frac{\partial}{\partial p_0} - i \left(\frac{a}{r} - \frac{\lambda p_0}{r^2} \right) p_2, \quad N_2 = p_2 \frac{\partial}{\partial p_0} + i \left(\frac{a}{r} - \frac{\lambda p_0}{r^2} \right) p_1 \quad (5b.2)$$

The invariant operators c_1 and c_2 belonging to the enveloping algebra have eigenvalues r^2 and iar , respectively. These, together with the eigenvalue λ of the central generator characterize the representations.

$$5C. \tau \neq 0, \lambda = 0$$

This case corresponds to the group representations given by eq. (4c.11) of subsection 4C. The operators P_i are operators of multiplication by p_i . The central generator d is now zero. The remaining generators of the Lie algebra are given by

$$H = \frac{1}{2\tau} (p_1^2 + p_2^2), \quad m = \tau \quad (5c.1)$$

$$M = iS + p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2}, \quad N_i = \tau \frac{\partial}{\partial p_i} \quad i = 1, 2 \quad (5c.2)$$

The space rotation operator M has two (mutually-commuting) pieces. The hermitian operator S is the generator of the spin group (group of rest-frame rotations); it arises from the representation $s(h)$ of the spin group in eq. (4c.11). The remaining term in M corresponds to the orbital angular momentum. The operator c_1 is no longer an invariant. The eigenvalue of the invariant c_2 is its , where s is the eigenvalue of S . The representations are characterized by the pair (τ, s) . In case τ is positive, it has the interpretation of being the non-relativistic mass. The case of a negative τ can also be given some meaning in terms of the mass of an antiparticle, along the same lines as is done in the corresponding case of the (3+1) dimensional Galilean group[4].

$$5D. \tau \neq 0, \lambda \neq 0$$

This case corresponds to the group representation of eq. (4d.5) in subsection 4D. As before, P_i are multiplication by p_i and the remaining operators are

$$H = \frac{1}{2\tau} (p_1^2 + p_2^2) \quad , \quad m = \tau \quad , \quad d = -i\lambda \quad (5d.1)$$

$$M = iS + p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2} \quad (5d.2)$$

$$N_1 = \tau \frac{\partial}{\partial p_1} + \frac{i\lambda}{2\tau} p_2 \quad , \quad N_2 = \tau \frac{\partial}{\partial p_2} - \frac{i\lambda}{2\tau} p_1 \quad (5d.3)$$

The invariant c_2 of the enveloping algebra has the eigenvalue its , where s is the eigenvalue of the spin operator S . The representations are labeled by the triple (τ, λ, s) .

This concludes our discussion of the Lie algebra representations.

6. Concluding Remarks.

We constructed the complete system of unitary, irreducible representations of the connected Galilean group in two spatial dimensions. The representations are classified into four classes. The first (sec. 4A) corresponds to true representation of the Galilean group and the remaining three are projective representations. The cases 4A and 4B are massless representations while 4C and 4D correspond to the massive case. The last two correspond to localizable systems; the cases 4A and 4B do not. This can be proved by a verbatim repetition of the arguments used in connection with the representations of the (3+1) dimensional Galilean group[10]. The case 4C is the one that represents a non-relativistic Schrödinger particle, as is evident. The case 4D corresponds to a Schrödinger-like system for which the representation of the boost operators is non-commutative. This gives a quantum-mechanical departure from the classical velocity addition law of non-relativistic physics. Finally, we wish to discuss the apparently controversial [12-13] question concerning the spectrum of angular momentum in two spatial dimensions.

At the level of the representations of the *Lie algebra* there is a difference between rotations in three and two spatial dimensions. In three spatial dimensions, the representations of the Lie algebra of the rotation group already leads to a quantization of the angular momentum (since $2J+1$ is the dimensionality of the vector space of representations), whereas no such conclusion is derivable from considerations of representations of the Lie algebra of the rotation group in two spatial dimensions. However, one should not conclude from this situation that the spectrum of angular momentum in two spatial dimensions is unrestricted. The restriction to integer quantization for the latter arises from representations of the *group*; that is, from the way a finite rotation (say a 2π rotation) is implemented.

The Pontryagin dual (the group of characters) of $SO(2)$ is $\mathbb{Z}$ – the additive group of integers. Thus the spectrum of planar angular momentum is integer-valued. However, one does not know, from this argument, as to what the unit is in terms of which the angular momentum is integer-quantized. The choice of a unit is really a question of physics and not at all a question of group representation. Exactly the same question may also be raised for three-dimensional rotations. For the latter, it is an experimental fact that angular momentum is quantized in units of $\hbar/2$, where $\hbar$ is Planck's reduced constant. (By convention, we describe this situation by the statement that the quantization unit is $\hbar$ and the lowest admissible angular momentum is $\hbar/2$.) The restriction of three-dimensional rotations to two dimensions leads to quantization in units of $\hbar/2$ for planar angular momentum. There is an additional piece of evidence for this last conclusion. This concerns the zero-mass, discrete spin representations of the Poincaré group. These are induced from (appropriate) representations of the stability group which is isomorphic with the group of Euclidean motions on a plane. The corresponding helicity of the massless particle just reflects the representation of the two-dimensional rotation subgroup of the planar Euclidean group[11]. In view of the existence of the neutrino, it would appear settled that the quantization unit for planar angular momentum is $\hbar/2$.

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