Back Reaction on the Topological Degrees of Freedom in (1 + 2 ) Dimensional Spacetime

By

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An Autonomous Institution of the University Grants Commission
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ABSTRACT

We investigate the back-reaction effect of the quantum field on the topological degrees of freedom in (1+2)-dimensional toroidal universe, $\mathcal{M} \simeq \mathbb{R} \times T^2$. Constructing a homogeneous toroidal spacetime, we calculate the Casimir energy of a massless, conformally coupled scalar field, with a conformal vacuum. This Casimir energy is expressed as a function of the Teichmüller parameters $(\tau^1, \tau^2)$, which describe the topological degrees of freedom of a torus.

By performing the (1+2)-decomposition in the action for gravity, we can reduce the semiclassical Einstein equation, with this Casimir energy on the right-hand side, into a set of canonical equations for six variables --- $(\tau^1, \tau^2)$ and their conjugate momenta, a spatial volume $V$ and its conjugate momentum $\sigma$, which corresponds to a trace of the extrinsic curvature of a spatial surface. Then, we can explicitly examine the back-reaction effect.

Since the spatial section is a 2-torus, we can write down the partition function of this system, fixing the path-integral measure for gravity modes, by the use of techniques developed in string theories. We find out the non-trivial contribution of the measure for $(\tau^1, \tau^2)$ to their semiclassical dynamics. This indicates the importance of fixing the measure in semiclassical gravity.
1. Introduction

Topological considerations are necessary in many situations. Since physical laws are usually expressed in terms of local, differential equations, their importance is not prominent at first sight. However, once one proceeds to solve them, one has to take boundary conditions into account, which allow the topological information to enter the theory. In general relativity, which handles dynamics of spacetime itself, the topological properties acquire dynamical meaning and their consideration becomes more significant. The aim of this paper is to present an explicit, detailed investigation of the dynamics of topological degrees of freedom in spacetime, in the context of the back-reaction problem in semiclassical gravity. Here, we do not discuss the topology change.[1],[14] The term “topological degrees of freedom” indicates those global parameters, describing the global deformations of the spatial hypersurface, which have a topological origin (the moduli deformations), i.e. which are not subject to the conformal deformations nor to the spatial diffeomorphisms of the surface (§§3 - 6).

As a first preliminary study for the full quantum gravity, it is reasonable to consider the effect of the curvature of a fixed background spacetime on the behavior of quantum matter field, which is the subject of quantum field theory on a curved spacetime[2],[3]. Then, the next natural step is the investigation of the influence of such a quantum field on classical spacetime geometry, which is called the back-reaction problem in semiclassical gravity. Usually, one tries to describe this effect by the semiclassical Einstein equation,

\[ G_{\alpha \beta} = \alpha < T_{\alpha \beta} >, \]  

(1)

where \( T_{\alpha \beta} \) is some \( c \) number, obtained from the energy momentum tensor operator and the inner-product of some quantum states, and \( \alpha \) is an appropriate gravitational constant with physical dimension \([\alpha] = [L^{n-2}]\). (Here, \( n \) is the spacetime dimension. We take \([\hbar] = [1]\) and \( c = 1 \) in this paper.) There are several uncertain issues and technically complicated points about this treatment. First, it
is not so clear what kind of quantity should be chosen for $< T_{\alpha\beta} >$.[4] Here, we regard that

$< T_{\alpha\beta} >$ should be some expectation value, rather than the quantity

$< \text{out}|T_{\alpha\beta}|\text{in} >$, since the latter harms the reality and the causal nature of eq.(1).[5],[6],[7] Then, if one regards the path-integral formalism as fundamental for quantum gravity, the so-called in-in formalism[8],[5],[6] should be of more importance than the standard in-out formalism.[7] Second, the regularization of $< T_{\alpha\beta} >$ requires complicated, though well-established, techniques, which itself is one main topic of the quantum field theory on a curved spacetime.[2],[3] Third, eq.(1) in general becomes complicated, even though $< T_{\alpha\beta} >$ has been successfully computed, so that it is difficult to solve it and study the effect of the back-reaction in detail. Fourth, one can show that eq.(1) can be obtained from the first variation of the phase part in the in-in path-integral expression,[5],[7] in which matter part has been integrated out formally while gravity part is left unintegrated without the explicit fixation of the measure. If one wants to go one step further, however, one should also take care of the effect coming from the path-integral measure for the gravity part. It is usually difficult since a reasonable, general measure has not been fixed yet. Fifth, to speak rigorously, eq.(1) itself contains an inconsistency from the very beginning. Since gravity and matter couple, quantum fluctuations of matter cause corresponding quantum fluctuations of gravity. Thus, there is a limitation in principle to the semiclassical treatment, eq.(1), because we try to treat gravity classically while matter is treated by quantum theory.[7] Specifying the exact validity conditions for eq.(1) is one of the main topics of semiclassical gravity.[7],[9],[10]

In this paper, we consider a $(1 + 2)$-dimensional spacetime $\mathcal{M} \simeq \mathbb{R} \times \Sigma$, with $\Sigma \simeq T^2$, a torus. We choose, as a matter field, a massless conformally coupled scalar field with a conformal vacuum, and investigate explicitly the back-reaction effect, resulting from the Casimir energy of matter, on the topological degrees of freedom, i.e. the modular-deformations of the torus. As is stated above, the topological degrees of freedom is one of the essential ingredients of spacetime dynamics.
However, the back-reaction on topological modes has seldom been discussed so far, partially because such a finite number of degrees of freedom are hidden in infinite number of gravity modes in 4-dimensional spacetime. One advantage of the reduction of the number of dimension from 4 to 3 is that, only the finite topological modes plus a spatial volume remain dynamical in the gravity part, due to the dimensionality.\cite{11},\cite{12},\cite{13} One can understand this point as follows: When \( n = 3 \), the spatial metric \( h_{ab} \) has 3 independent components at each spatial point, while there are 3 constraints at each point. Thus, redundant infinite number of modes are gauged away and only a finite number of modes remains. Another advantage of the reduction of dimension in the discussion of topological aspects comes from the fact that 2-dimensional topology is completely classified in a simple manner so that it is easy to construct various topologies\cite{14}.

Another good point of this model is that some difficulties and complications stated above of the semiclassical Einstein equation, eq.(1), become simplified and tractable to a great extent in this case.

First, we choose a conformal vacuum \( |0> \) as a natural candidate for a vacuum state of matter in our case, and use \( <0|T_{\alpha\beta}|0> \) on the right-hand side of eq.(1). Second, since the background spacetime will be chosen as conformally flat and the matter field is conformally invariant, as will be presented in the following section, \( <T_{\alpha\beta}(g)> \) can be calculated from \( <T_{\alpha\beta}(\eta)> \) (\( \eta \): a flat metric) along with the trace-anomaly\cite{2}. Thus, the regularization of \( <T_{\alpha\beta}(g)> \) is reduced to that of \( <T_{\alpha\beta}(\eta)> \) and the estimation of the trace-anomaly \( <T^0_0(g)> \), which simplifies the manipulation. Furthermore, in our case, the spacetime dimension is odd, \( n = 3 \), so that there is no trace-anomaly\cite{2}. Thus, \( <T_{\alpha\beta}(g)> \) is simply related to \( <T_{\alpha\beta}(\eta)> \), which simplifies the calculation a lot. Third, because of the dimensionality, eq.(1) is reduced to a set of six first-order ordinary differential equations and we can investigate the effect of the back-reaction explicitly. Fourth, we restrict the metrics to a special class, with spatial part being the one for the locally flat metrics on a torus. Thus, we can fix the path-integral measure explicitly by the use of the techniques developed in string theories.\cite{13},\cite{16} Even though the
space of metrics chosen is much smaller than the full space of metrics on $M \simeq \mathbb{R} \times T^2$, one finds that the path-integral measure for the gravity part becomes highly non-trivial and it causes a significant influence on the semiclassical dynamics of gravity. Fifth, the (in-in) effective action for gravity, $W[g_+ : g_-]$, becomes relatively simple in our case, and this reduces to $W[\phi_+, \tau_+^1, \tau_+^2 : \phi_-, \tau_-^1, \tau_-^2]$, a functional of six functions of $t$, $(\phi_\pm, \tau_\pm^1, \tau_\pm^2)$, where $\phi_\pm$ indicate the conformal degree of freedom and $(\tau_\pm^1, \tau_\pm^2)$ are the Teichmüller parameters describing the topological degrees of freedom of a torus. Although the exact calculation of $W$ has already become difficult, we can still estimate its functional form to leading order in $\hbar$. In computing $W$, our model reveals explicitly the peculiarity of the semiclassical gravity, compared with the standard treatment of the quantum dissipative system, e.g. the Brownian motion[17]. There is no linear coupling between the sub-system (gravity) and the environment (matter field). Their coupling is put in the kinetic term of the matter field. This model might provide the simplest non-trivial example for the investigation of the quantum dissipative system including gravity.

In §2, we construct the quantum field theory on $M \simeq \mathbb{R} \times T^2$ and calculate the Casimir energy of a massless, conformally coupled scalar field with a conformal vacuum. In §3, we extract explicitly the topological degrees of freedom of a torus and reduce eq.(1) to a canonical system with a finite number of degrees of freedom. Then, we investigate explicitly the effect of the back-reaction of matter on the dynamics of the topological degrees of freedom. In §4, we calculate the partition function of this system, fixing the measure explicitly with the help of the techniques in string theories. Then, we examine the non-trivial effect of the path-integral measure on the semiclassical dynamics. We also estimate the functional form of $W$ to leading order in $\hbar$. Section 5 is for a conclusion.
2. Quantum field theory on a (1+2)-dimensional toroidal spacetime

(a) Scalar field on a torus

We consider a (1+2)-dimensional spacetime with topology \( \mathbb{R} \times T^2 \). We concentrate on the case when the geometry of the space \( \Sigma \simeq T^2 \) is locally flat. One can construct such a geometry on a 2-manifold \( \Sigma = [0, 1] \times [0, 1] / \sim \), where "/ \sim" stands for the point-identification on the boundary: \( (\xi^1, 0) \sim (\xi^1, 1) \) and \( (0, \xi^2) \sim (1, \xi^2) \) (Figure 1-a). A flat 2-geometry is endowed on \( \Sigma \) by giving a metric,

\[
dl^2 = h_{ab} d\xi^a d\xi^b,
\]

where

\[
h_{ab} = \frac{1}{\tau^2} \begin{pmatrix} 1 & \tau^1 \\ \tau^1 & |\tau|^2 \end{pmatrix} \tag{2-a}
\]

Here, \( (\tau^1, \tau^2) \) are parameters independent of spatial coordinates, \( (\xi^1, \xi^2) \), and \( \tau := \tau^1 + i\tau^2, \tau^2 > 0 \). (Throughout this paper, \( \tau^2 \) always indicates the second component of \( (\tau^1, \tau^2) \), and not the square of \( \tau \). The latter never appears in the formulae.) Note that \( \sqrt{\tau} := (\det h_{ab})^{1/2} = 1 \). In the next sub-section, we will construct a background spacetime \( \mathcal{M} \simeq \mathbb{R} \times \Sigma \) by giving a metric \( ds^2 = -dt^2 + \Omega^2(t) dl^2 \). Back-reaction from matter causes a temporal evolution of \( (\tau^1, \tau^2) \) as we will see later. The quantity \( (\tau^1, \tau^2) \) can then be interpreted as follows. On each time-slice \( t = \text{const} \) one can make a coordinate transformation

\[
\begin{align*}
x &= \frac{1}{\sqrt{\tau^2}} (\xi^1 + \tau^1 \xi^2), \\
y &= \sqrt{\tau^2} \xi^2
\end{align*}
\tag{3}
\]

Then, \( dl^2 \) turns out to be \( dl^2 = dx^2 + dy^2 \), and two generators of the torus becomes \( c_1 = (1/\sqrt{\tau^2}, 0) \) and \( c_2 = (\tau^1/\sqrt{\tau^2}, \sqrt{\tau^2}) \) (Figure 1-b). Thus, \( (\tau^1, \tau^2) \) are regarded as the Teichmüller parameters, describing the deformations of the global shape of a torus\(^{[15],[16]} \).
The fundamental modes on $\Sigma$ with the line element $dl^2$ (eq.(2 - $a$, $b$)) are

$$f_{n_1n_2}(\xi) = \exp i2\pi n_1 \xi^1 \exp i2\pi n_2 \xi^2 \quad (n_1, n_2 \in \mathbb{Z}) \quad (4)$$

Here, $f_{n_1n_2}(\xi)$'s satisfy,

$$\int_0^1 d\xi^1 \int_0^1 d\xi^2 \sqrt{f_{n_1n_2}(\xi)f_{n'_1n'_2}(\xi)} = \delta_{n_1n'_1}\delta_{n_2n'_2} \quad , \quad (5)$$

$$\sum_{n_1=\infty}^{\infty} \sum_{n_2=\infty}^{\infty} f_{n_1n_2}(\xi)f_{n_1n_2}(\xi') = \delta_1(\xi^1 - \xi'^1)\delta_1(\xi^2 - \xi'^2) \quad , \quad (6)$$

where $\delta_1(\xi) := \sum_{n=\infty}^{\infty} \exp i2\pi n \xi$ is a periodic $\delta$-function with a period 1. These $f_{n_1n_2}(\xi)$'s are eigenfunctions of the Laplacian,

$$\Delta := -1/\sqrt{\partial_a(h^{ab}\sqrt{\partial_b})} = -\frac{1}{\tau^2} (|\tau|^2\partial^2_1 - 2\tau^1\partial_1\partial_2 + \partial^2_2) \quad ,$$

with eigenvalues

$$\lambda_{n_1n_2} = \frac{4\pi^2}{\tau^2} (|\tau|^2n_1^2 - 2\tau^1n_1n_2 + n_2^2) \quad . \quad (7)$$

It is clear that only $n_1 = n_2 = 0$ corresponds to a zero-mode of $\Delta$.

Now, let us consider a spacetime $\mathcal{M} \simeq \mathbb{R} \times \Sigma$, with a line element $ds^2 = -dt^2 + h_{ab}d\xi^a d\xi^b$. The fundamental solutions for $\Box u(t, \xi^1, \xi^2) = 0$ are

$$\overline{u}_A(t, \xi) = \frac{1}{\sqrt{2\omega_A}} e^{-i\omega_A t} f_A(\xi) \quad , \quad (8)$$

where $A$ stands for $n_1n_2$ and $\omega_A := \sqrt{\lambda_{n_1n_2}} = \sqrt{\lambda_{n_1n_2}}$. They satisfy,

$$< \overline{u}_A, \overline{u}_B > = i \int_0^1 d\xi^1 \int_0^1 d\xi^2 \sqrt{u}_A \partial_0 \overline{u}_B - \overline{u}_B \partial_0 u_A = \delta_{AB} \quad ,$$

$$< \overline{u}_A, \overline{u}_B > = -\delta_{AB} \quad \text{and} \quad \overline{u}_A^* \overline{u}_B = 0 \quad .$$

We can expand a scalar field $\psi(t, \xi)$ satisfying $\Box \psi = 0$ in terms of $\overline{u}_A$,

$$\psi(t, \xi) = \sum_A (a_A \overline{u}_A + a_A^t \overline{u}_A^*) \quad ,$$

and we follow the standard procedure for the field quantization. [2],[3]
In the next section, we will investigate the evolution of $(\tau^1, \tau^2)$ as a function of $t$. It is notable that the $t$-dependence of $\tau$ does not change the form of the equation $\Box \psi = 0$, because of the form of the metric, $g_{\alpha \beta} = (-1, h_{ab})$ with $\det g_{\alpha \beta} = -1$.

(b) The model

We will investigate the back-reaction of the matter field on the topological degrees of freedom $(\tau^1, \tau^2)$. The most ideal treatment of the back-reaction described by eq.(1) may be the self-consistent determination of the geometry $g_{\alpha \beta}$ through eq.(1): $< T_{\alpha \beta} >$ depends on $g_{\alpha \beta}$ and this $g_{\alpha \beta}$ is self-consistently determined by eq.(1). However, it turns out that such a treatment becomes highly complicated even in our simple model. To make our analysis tractable, then, we treat the back-reaction in the following sense, which is usually adopted in the back-reaction problem: We prepare a background spacetime and calculate $< T_{\alpha \beta} >$ on it. Then, we discuss the modification of the background geometry due to the $< T_{\alpha \beta} >$ using eq.(1).

Now, as a background spacetime, we choose a solution of the vacuum Einstein equation, $G_{\alpha \beta} = 0$. More specifically, we prepare a locally flat spacetime, $ds^2 = -dt^2 + V dl^2 = V(-dt^2 + dl^2)$, where $dl^2$ is given by eqs.(2 - a, b), and $V$, $\tau^1$ and $\tau^2$ are chosen to be constant for the background spacetime. (Below, we occasionally treat this flat spacetime as conformally flat, for mathematical convenience.) As we will see in the next section, the back-reaction effect causes a time evolution of these variables. We choose as a matter field, a massless conformally coupled scalar field $\psi$,

$$S_m = \frac{-1}{2} \int (g^{\alpha \beta} \partial_\alpha \partial_\beta + \frac{1}{8} R \psi^2) \sqrt{g} \ d^3 x,$$  \hspace{1cm} (9)

where $\sqrt{g} = (-\det g_{\alpha \beta})^{1/2}$. The (improved) energy-momentum tensor operator$[3]$ becomes,

$$T_{\alpha \beta} = \frac{3}{4} \partial_\alpha \psi \partial_\beta \psi - \frac{1}{4} \partial_\gamma \psi \partial^\gamma g_{\alpha \beta} - \frac{1}{4} \psi \partial_\alpha \partial_\beta \psi + \frac{1}{12} \psi \Box \psi g_{\alpha \beta} + \frac{1}{8} \psi^2 (R_{\alpha \beta} - \frac{1}{3} g_{\alpha \beta} R)$$

We choose a conformal vacuum as a vacuum state for the matter field.
Before going ahead, it is appropriate to mention the advantage and the limitation of our model and our analysis. On account of the conformal invariance of the matter field, the time-evolution of the spatial volume $V$, which is caused by the back-reaction, does not directly set a limitation on the reliability of the analysis: In the case of a conformally invariant field on a conformally flat spacetime, the renormalized energy-momentum tensor $<T_{\alpha\beta}>$ is completely determined by $<T_{\alpha\beta}>$ calculated on a flat spacetime, and the trace-anomaly\textsuperscript{[1]}:

$$<T_{\alpha\beta}(g)> = \sqrt{3}^{-1} <T_{\alpha\beta}(\eta)> - 2\sqrt{3}^{-1} g^{\beta\gamma} \delta / \delta g_{\alpha\gamma} \int <T_{\mu\nu}(g)> \delta \Omega d^nx$$ ,

where $g_{\alpha\beta} = \Omega^2 \eta_{\alpha\beta}$, $\eta_{\alpha\beta}$: a flat metric. The spacetime dimension is odd, $n = 3$, in the present case, so that there is no trace-anomaly, $<T_{\mu\nu}(g)> = 0\textsuperscript{[1]}$. Thus, $<T_{\alpha\beta}(g)>$ is simply related to $<T_{\alpha\beta}(\eta)>$ as,

$$<T_{\alpha\beta}(g)> = \Omega^{-1} <T_{\alpha\beta}(\eta)> , \quad (11)$$

independent of the t-dependence of $\Omega$. Thus, as far as the spacetime can be treated as conformally flat, we can do almost all calculations on the flat spacetime, for which $ds^2 = -dt^2 + dl^2$. On flat spacetime, the field equation for $\psi$ becomes $\Box \psi = 0$, and eq.(8) can be used as fundamental solutions. The last term containing $R_{\alpha\beta}$ and $R$ in eq.(10) does not contribute to $<T_{\alpha\beta}>$.

However, the time-dependence of $(\tau^1, \tau^2)$ caused by the back-reaction harms the self-consistency of the analysis, which is inevitable if the tractability of the back-reaction problem, described by eq.(1), is to be maintained, as discussed at the beginning of this sub-section. When $(\tau^1, \tau^2)$ evolve in time, the functions in eq.(8) are no longer exact solutions for $(\Box + \frac{1}{8}R)\phi = 0$, because $R = \dot{K} + \dot{h} + \tilde{K}_{ab} + K^2 - K_{ab}K^{ab}$, where $K_{ab} = \frac{1}{2} \dot{h}_{ab}$, $K = \dot{h} + \tilde{K}_{ab}$ with eq.(2 - b), becomes non-vanishing, and because $\omega_A := \sqrt{\lambda_A}$ becomes t-dependent, through the t-dependence of $(\tau^1, \tau^2)$ (eq.(7)). Furthermore, the spacetime described by $ds^2 = -dt^2 + V(t)h_{ab}d\xi^a d\xi^b$ is no longer conformally flat when $(\tau^1, \tau^2)$ evolves, because of the t-dependence of $h_{ab}$. 
Thus, we should look at the results of the analysis in an adiabatic sense, i.e. valid
when terms including \( \tau^1 \) and \( \tau^2 \) are not dominant in the formulae prominently.
Such a conflict between self-consistency and the tractability of the analysis always
occurs in the back-reaction problem.

We next need to get Hadamard’s elementary function\(^2,3\) \( G^{(1)}(x) \) for calculating \( < T_{\alpha\beta}(x) > \). We choose the conformal vacuum as a vacuum state of the matter.
Since our back-ground spacetime is described by \( ds^2 = -dt^2 + V dl^2 \), \( G^{(1)}(x) \) for
\( ds^2 = -dt^2 + dl^2 \) can be utilized. We first compute \( G^{(1)}(x) \) for \( \mathcal{M} \simeq \mathbb{R}^3 \), and
afterwards take care of the periodicity in \( \mathcal{M} \simeq \mathbb{R} \times T^2 \), using the mirror-image
method. For the n-dimensional Minkowski space, \( G^{(1)}(x) \) is,

\[
G^{(1)}(x) := < 0 | \{ \psi(x), \psi(y) \} | 0 > \\
= 2\pi \int \frac{d^n k}{(2\pi)^n} \delta(k^2) e^{-ik \cdot x} = \int_{-\infty}^{\infty} d\alpha \int \frac{d^n k}{(2\pi)^n} \exp i(\alpha k^2 - k \cdot x) \\
= \frac{\Gamma(\frac{n}{2} - 1)}{(2\pi)^{n/2}} \sigma^{-\frac{n}{2} + 1} 
\]

(12)

where \( \sigma := \frac{1}{2} \bar{x}^2 = \frac{1}{2} \eta_{\alpha\beta} x^\alpha x^\beta \), \( \frac{1}{2} \) times a square of a world distance. We have
assumed \( \sigma > 0 \) during the manipulation since this is the case which we need here.
For \( n = 3 \), we get,

\[
G^{(1)}(x) = \frac{1}{2\pi} (2\sigma)^{-1/2} \quad (\sigma > 0) 
\]

To get \( G^{(1)}(x) \) on \( \mathcal{M} \simeq \mathbb{R} \times T^2 \), all contributions from points identified with \( x \)
should be added (the mirror-image method):

\[
G^{(1)}(x) = \frac{1}{2\pi} \sum_{n_1, n_2 = -\infty}^{\infty} \left( 2\sigma_{n_1 n_2}(x) \right)^{-1/2} 
\]

(13)

where

\[
2\sigma_{n_1 n_2}(x) := -t^2 + \frac{1}{\tau^2} \left\{ (\xi^1 + n_1)^2 + 2\tau^1 (\xi^1 + n_1)\xi^2 + n_2 |\tau| \right\} \left( \xi^2 + n_2 \right)^2 \\
= -t^2 + \frac{1}{\tau^2} \left| (\xi^1 + n_1) + \tau (\xi^2 + n_2) \right|^2 
\]

From now on, the prime attached to the \( \Sigma \)-symbol, like in eq.(13), indicates that the
zero-mode \( n_1 = n_2 = 0 \) should be excluded from the summation, whenever it causes a divergence. This procedure corresponds to the point-splitting regularization\(^2\), since the divergence arising in the coincidence limit \( x \to 0 \) is removed.

Considering eq.(10),

\[
<T_{\alpha\beta}(g^{(0)})> = \frac{3}{8} \partial_\alpha \partial_\beta \mathcal{G}_x^{(1)} - \frac{1}{8} g^{(0)\gamma\delta} \partial_\gamma \partial_\delta \mathcal{G}_x^{(1)} g_{\alpha\beta}^{(0)} \\
- \frac{1}{8} \partial_\alpha \partial_\beta \mathcal{G}_x^{(1)} + \frac{1}{24} g^{(0)\gamma\delta} \partial_\gamma \partial_\delta \mathcal{G}_x^{(1)} g_{\alpha\beta}^{(0)},
\]

where \( \partial_\alpha \partial_\beta \mathcal{G}_x^{(1)} := \partial_{x^\alpha} \partial_{x^\beta} \mathcal{G}^{(1)}(x - x') \) etc., \( x^\alpha := (t, x^1, x^2) \) and \( g^{(0)}_{\alpha\beta} = (-\hbar, h_{ab}) \) with \( (2 - b) \). It turns out that \( \partial_\alpha \mathcal{G}_x^{(1)} = -\partial_\alpha \mathcal{G}_x^{(1)}, \; g^{(0)\gamma\delta} \partial_\gamma \partial_\delta \mathcal{G}_x^{(1)} = 0 \), so that,

\[
<T_{\alpha\beta}(g^{(0)})> = \frac{1}{2} \partial_\alpha \partial_\beta \mathcal{G}_x^{(1)} .
\]

Now it is straightforward to compute \( <T_{\alpha\beta}(g^{(0)})> \) explicitly. The result is,

\[
<T_{00}>=-\frac{(\tau^2)^{3/2}}{4\pi} \sum_{n_1 n_2} \frac{1}{|n_1 + \tau n_2|^3}, \tag{14 - a}
\]

\[
<T_{11}>=\frac{(\tau^2)^{1/2}}{4\pi} \sum_{n_1 n_2} \frac{1}{|n_1 + \tau n_2|^3} - \frac{3(\tau^2)^{1/2}}{4\pi} \sum_{n_1 n_2} \frac{(n_1 + \tau n_2)^2}{|n_1 + \tau n_2|^5}, \tag{14 - b}
\]

\[
<T_{22}>=\frac{(\tau^2)^{1/2} |\tau|^2}{4\pi} \sum_{n_1 n_2} \frac{1}{|n_1 + \tau n_2|^3} - \frac{3(\tau^2)^{1/2}}{4\pi} \sum_{n_1 n_2} \frac{(\tau^2 n_1 + |\tau|^2 n_2)^2}{|n_1 + \tau n_2|^5}, \tag{14 - c}
\]

\[
<T_{12}>=<T_{21}>=\tau \frac{(\tau^2)^{1/2}}{4\pi} \sum_{n_1 n_2} \frac{1}{|n_1 + \tau n_2|^3} - \frac{3(\tau^2)^{1/2}}{4\pi} \sum_{n_1 n_2} \frac{(n_1 + \tau n_2)(\tau^2 n_1 + |\tau|^2 n_2)}{|n_1 + \tau n_2|^5}, \tag{14 - d}
\]

\[
<T_{0a}>=<T_{a0}>=0 \quad (a = 1, 2) . \tag{14 - e}
\]

For a metric \( g_{\alpha\beta} = \Omega^2(-\hbar, h_{ab}) \), \( <T_{\alpha\beta}(g)> \) can be obtained by eq.(11). Since the Planck scale is the only scale which comes into our model, we understand that
a suitable power of $\alpha := l_{\text{Planck}}$ is multiplied to quantities like those in eq. (14 – a) – (14 – c), if necessary, in order to adjust their physical dimension. These non-vanishing contributions to $< T_{\alpha\beta} >$ in eq. (14 – a) – (14 – c), originating from a non-trivial spatial topology $\Sigma \simeq T^2$, are well-known as the Casimir effect[2],[3].

It is appropriate to briefly summarize what we have done and will do from a general viewpoint. We have prepared a background spacetime satisfying $G_{\alpha\beta} = 0$. $V$, $\tau^1$ and $\tau^2$ are constant in this background spacetime. Then, we have calculated $< T_{\alpha\beta} >$ on this background spacetime. Then, we will see in the next sub-section that the back-reaction equation, eq.(1), yields non-trivial solutions, describing the time-evolution of $V$, $\tau^1$ and $\tau^2$.

3. Back-reaction of the Casimir effect on the topological degrees of freedom

(a) (1+2)-decomposition

Having computed $< T_{\alpha\beta} >$ in the previous section, we will now investigate the back-reaction of $< T_{\alpha\beta} >$ on the evolution of the spacetime. We consider the Einstein gravity on $M \simeq \mathbb{R} \times T^2$ and a massless conformally coupled scalar field on it; $S = \frac{1}{\alpha} \int R \sqrt{g} + S_m$, where $\alpha := l_{\text{Planck}}$ and $S_m$ is given by eq.(9). The canonical formulation is suitable to investigate the temporal evolution of the spacetime. We thus perform the (1+2)-decomposition, but care should be taken because of the presence of the conformally coupled field. For the time being, let us consider the case of general $n$. (Then, the constant $\alpha$ should be understood as $[\alpha] = [L^{n-2}]$. Finally we set $n = 3$.)

It is convenient to express $S$ as $S = \int \mathcal{L}_G' + \int \mathcal{L}^\sim$, where $\mathcal{L}_G' := \frac{1}{\alpha'} R \sqrt{\lambda}$, $\frac{1}{\alpha'} := \frac{1}{\alpha} - \xi(n) \psi^2 (\xi(n) = \frac{4 n - 2}{4 n - 1})$ and $\mathcal{L}^\sim$ is the minimally coupled part of the matter Lagrangian. Following the standard manipulation[18], we get the Hamiltonian for the gravity part as,

$$H_G' := \pi^{ab} h_{ab} - \mathcal{L}_G'$$
\begin{equation}
= N \frac{1}{\alpha'} C_G + N_a \frac{1}{\alpha'} C^a_G + 2D_a(\sqrt{-1}N_b\pi^{ab}) \\
+ 2N\sqrt{(\nabla^\beta \frac{1}{\alpha'})}(n^\beta \nabla_\alpha n^\alpha - n^\alpha \nabla_\alpha n^\beta) - 2N_b(D_a \frac{1}{\alpha'})\alpha'\pi^{ab} .
\end{equation}

Here, \( N, N_a \) are the lapse and the shift functions and \( n^\alpha = (-1/N, N^\alpha/N) \) is the normal unit vector of the spatial surface. \( \nabla_\alpha, D_a \) are covariant derivatives w.r.t. \( g_{\alpha\beta}, h_{ab} \), respectively. \( \pi^{ab} := (K^{ab} - K h^{ab})/\alpha' \), \( K_{ab} \) is the extrinsic curvature of a spatial surface. \( C_G := (K_{ab} K^{ab} - K^2 - R)\sqrt{\alpha'}, C^a_G := -2D_b(K^{ab} - K h^{ab}) \). We will finally choose a coordinate system s.t. \( N^a = 0 \), so that \( n^\alpha = \left(-1/N, \vec{0}\right) \). Then, the second last term in eq.(15) vanishes. Thus, we discard this term at this stage.

In the back-reaction problem, we regard that \( \psi^2(x) \) is replaced by \( <\psi^2(x)> \), which turns out to be independent of spatial coordinates in our choice of coordinates, as we have seen in the last section. Thus, \( D_a \frac{1}{\alpha'} = 0 \) and the last term in eq.(15) vanishes. The third term in eq.(15) is a total derivative and can be discarded. Thus,

\[ H_G = N \frac{1}{\alpha'} C_G + N_a \frac{1}{\alpha'} C^a_G \]
\[ = N \frac{1}{\alpha} C_G + N_a \frac{1}{\alpha} C^a_G + N \frac{\xi}{2} \psi^2 2G_{\alpha\beta} n^\alpha n^\beta \sqrt{\alpha'} + N^a \frac{\xi}{2} \psi^2 (-G_{\alpha\beta} n^\beta) \sqrt{\alpha'} \]
\[ = N \frac{1}{\alpha} C_G + N_a \frac{1}{\alpha} C^a_G + N \{ -2\sqrt{\alpha'} \delta / \delta g^{\alpha\beta} (-\frac{1}{2} \xi \psi^2 R_{\alpha\beta}) n^\alpha n^\beta \sqrt{\alpha'} \}
\]
\[ + N^a \{ 2\sqrt{\alpha'} \delta / \delta g^{\alpha\beta} (-\frac{1}{2} \xi \psi^2 R_{\alpha\beta}) n^\beta \sqrt{\alpha'} \}
\]
\[ = N \frac{1}{\alpha} C_G + N_a \frac{1}{\alpha} C^a_G + NT_{\alpha\beta}^{(R)} n^\alpha n^\beta \sqrt{\alpha'} + N^a (-T_{\alpha\beta}^{(R)}) n^\beta \sqrt{\alpha'} ,
\]

where the identities \( C_G = -2G_{\alpha\beta} n^\alpha n^\beta \sqrt{\alpha'} \), \( C_{Ga} = G_{\alpha\beta} n^\beta \sqrt{\alpha'} \) have been used in the second line. \( T_{\alpha\beta}^{(R)} \) stands for the contribution to the energy-momentum tensor from the term \( -\frac{1}{2} \xi \psi^2 R_{\alpha\beta} \). On the other hand,

\[ H^\sim := \psi \frac{\partial L^\sim}{\partial \psi} - L^\sim \]
\[ = NT_{\alpha\beta}^{\sim} n^\alpha n^\beta \sqrt{\alpha'} + N^a (-T_{\alpha\beta}^{\sim}) n^\beta \sqrt{\alpha'} .
\]

\( T_{\alpha\beta}^{\sim} \) stands for the energy-momentum tensor coming from \( L^\sim \), the minimally cou-
pled part of the Lagrangian for matter. Then, we finally get the total Hamiltonian,

\[ H = H_G^I + H^\sim \]
\[ = N(\frac{1}{\alpha}C_G + < T_{a\beta} > n^\alpha n^\beta \sqrt{\gamma}) + N^a(\frac{1}{\alpha}C_{Ga} - < T_{a\beta} > n^\beta \sqrt{\gamma}) \]

where \( T_{a\beta} \) is the energy-momentum tensor determined by \( S_m \) in eq.(9) and we have replaced \( T_{a\beta} \) by \( < T_{a\beta} > \).

(b) The extraction of dynamics of the modular deformations

Now the system in which we are interested is,

\[ S = \int \pi^{ab} h_{ab} - N \mathcal{H} - N^a \mathcal{H}_a \quad , \quad (16 - a) \]

where

\[ \mathcal{H} = \left\{ (K^{ab} - K^2 - R)/\alpha + < T_{a\beta} > n^\alpha n^\beta \right\} \sqrt{\gamma} \quad , \quad (16 - b) \]
\[ \mathcal{H}_a/\sqrt{\gamma} = -2D_b(K^{b}_{a} - \delta^b_a K)/\alpha - < T_{a\beta} > n^\beta = 0 \quad . \quad (16 - c) \]

We choose a coordinate system s.t. \( N^a = 0 \) so that \( n^a = (-1/N, 0) \). Thus, \( < T_{a\beta} > n^\beta = -1/N \cdot < T_{a0} >= 0 \) \((a = 1, 2)\) from eq.(14 - e). In our case, thus, (16 - c) becomes

\[ \mathcal{H}_a/\sqrt{\gamma} = -2D_b(K^{b}_{a} - \delta^b_a K)/\alpha = 0 \quad . \quad (16 - c') \]

Then, we can extract the moduli degrees of freedom (they correspond to the global deformations of a torus) by solving eq.(16 - c') explicitly\(^{[13]}\).

First, let us recall the fundamentals of the moduli space\(^{[15],[16]}\). Consider the 2-dimensional closed manifold \( \Sigma \) with genus \( g \). The moduli space \( \mathcal{M}_g \) of \( \Sigma \) is defined as a quotient space (a space of all equivalent classes) of \( Riem(\Sigma) \) (a space of all Riemannian metrics on \( \Sigma \)) with the Weyl group and the diffeomorphism group on.
\( \Sigma, Diff(\Sigma) \), as quotient groups;

\[ \mathcal{M}_g \simeq \text{Riem}(\Sigma)/\text{Weyl} \times Diff(\Sigma) \]

In other words, \( \mathcal{M}_g \) is a set constructed from sub-sets of \( \text{Riem}(\Sigma) \) which can neither be related to each other by the Weyl transformations nor by diffeomorphisms. It is helpful to investigate its tangent space, \( T(\mathcal{M}_g) \). Any variation of the spatial metric \( \delta h_{ab} \), i.e. any element of \( T(\text{Riem}(\Sigma)) \), can be decomposed into three parts: the Weyl deformation \( \delta_W h_{ab} \), the variation accompanying the diffeomorphism \( \delta_D h_{ab} \), and the moduli deformation \( \delta_M h_{ab} \):

\[
\delta h_{ab} = \delta_W h_{ab} + \delta_D h_{ab} + \delta_M h_{ab} = \delta_W h_{ab} + (P_1 v)_{ab} + T_{Aab} \delta \tau^A ,
\]

(17)

where

\[
\delta_W h_{ab} = \delta \phi h_{ab} \quad \text{for} \quad \exists \delta \phi , \\
(P_1 v)_{ab} = D_a v_b + D_b v_a - D_c v^c h_{ab} \quad \text{for} \quad \exists v^a , \\
T_{Aab} = \frac{\partial h_{ab}}{\partial \tau^A} - \frac{1}{2} h_{cd} \frac{\partial h_{cd}}{\partial \tau^A} h_{ab} .
\]

(18 - a) \hspace{2cm} (18 - b) \hspace{2cm} (18 - c)

Here \( \{ \tau^A \}_{A=1,\ldots,\dim \mathcal{M}_g} \) are parameters needed to specify a point in \( \mathcal{M}_g \), known as the Teichmüller parameters\(^{[15]}\),\(^{[16]}\), which we have already introduced in the beginning, eq.(2 - b). In short, the trace-part of \( \delta h_{ab} \) has been identified with \( \delta_W h_{ab} \) and the traceless part has been decomposed into \( \delta_D h_{ab} \) and \( \delta_M h_{ab} \).

We can introduce a natural inner-product on \( \text{Riem}(\Sigma) \),

\[
(A, B) := \frac{1}{\alpha^2} \int_{\Sigma} d^2 x \sqrt{g} h^{ac} h^{bd} A_{ab} B_{cd} \quad \text{for} \quad \forall A_{ab}, \forall B_{ab} \in T(\text{Riem}(\Sigma))
\]

(19)

As is seen from eq.(17), the tangent space of the modular space, \( T(\mathcal{M}_g) \), can be identified with the set of all symmetric, traceless tensors which are perpendicular
to $T(\text{Diff}(\Sigma))$, i.e. for $w \in T(M_g)$, $(w, P_1 v) = (P_1^* w, v) = 0$ for $\forall v^{a},$ where

$$
(P_1^* w)^{a} = -2 D_{b} w^{a b} .
$$

Thus,$^{15,16}$ \(dim_{\mathbb{R}} M_g = dim_{\mathbb{R}} T(M_g) = dim_{\mathbb{R}} \text{Ker} P_1^\dagger\), which is known as $= 0$ for $g = 0$, $= 2$ for $g = 1$, and $= 6g - 6$ for $g \geq 2$. For our case of a torus ($g = 1$), then, we need 2 Teichmüller parameters ($\tau^1, \tau^2$) to describe the modular deformations $\delta_M h_{ab} \in T(M_{g=1})$.

Now, let us return to eq.(16 $-$ c'). Our coordinate system $ds^2 = -dt^2 + V dt^2$ with $(2 - a, b)$ corresponds to York's time-slice$^{19}$, i.e. the time-slicing using spatial surfaces on which

$$
\sigma := -K/\alpha = \text{const} .
$$

By this choice of slice, eq.(16 $-$ c') becomes

$$
\alpha \mathcal{H}_a / \sqrt{\mathcal{H}} = -2 D_b \hat{K}_a^b = 0 ,
$$

(16 $-$ c'')

where $\hat{K}_a^b := K_a^b - \frac{1}{2} \delta_a^b K$, which is the traceless part of $K_a^b$. Comparing with eq.(20), eq.(16 $-$ c'') shows that $\hat{K}_{ab} \in \text{Ker} P_1^\dagger$. Accordingly, we can expand $\hat{K}_{ab}$ in terms of the basis of $\text{Ker} P_1^\dagger, \{\Psi^A_{ab}\}_{A=1,2}$;

$$
\hat{K}_{ab} = \frac{1}{\alpha} p_A \Psi^A_{ab} .
$$

(22)

In effect, we have solved the momentum constraint (16 $-$ c) explicitly$^{13}$.

Now, we can reduce our system eqs.(16 $-$ a, b, c) to a much simpler system in terms of $\tau^A, p_A, \sigma$ and $V$. 

16
(c) The evolution of the Teichmüller parameters caused by the back-reaction

By setting \( N(t) = 1 \), we get the canonical equations of motion described by the constraint function,

\[
\alpha H = \sum_{A,B=1}^{2} G^{AB} P_{A \phi B} - \frac{1}{2} \alpha^{2} \sigma^{2} V - (\tau^{2})^{3/2} f(\tau) \sqrt{V} / \alpha = 0 ,
\]

where

\[
f(\tau^{1}, \tau^{2}) = \frac{1}{4 \pi} \sum_{n_{1}, n_{2}=-\infty}^{\infty} \frac{1}{|n_{1} + \tau n_{2}|^{3}} = \frac{1}{4 \pi} \sum_{n_{1}, n_{2}=-\infty}^{\infty} \frac{1}{(n_{1}^{2} + 2 \tau^{1} n_{1} n_{2} + |\tau|^{2} n_{2}^{2})^{3/2}} .
\]

Clearly, \( f(\tau^{1}, \tau^{2}) = f(\tau^{1}, \tau^{2}) \), \( f(\tau^{1} + n, \tau^{2}) = f(\tau^{1}, \tau^{2}) \) (n: integer) and \( f(\tau^{1}, \tau^{2}) \) is singular at \( (\tau^{1}, \tau^{2}) = (n, 0) \). Figures (2 - a, b) show the behavior of the function \( f(\tau^{1}, \tau^{2}) \).

For the explicit investigation of the dynamics, let us first calculate \( G^{AB} \) according to eqs.(23) and (18 - c) with \( h_{ab} = \frac{V}{\alpha^{2} \tau^{1}} \left[ \begin{array}{cc} 1 & \tau^{1} \\ \tau^{1} & |\tau|^{2} \end{array} \right] \). (Note that \( ds^{2} = -dt^{2} + V d\ell^{2} \).) Then, we get

\[
T_{1ab} = \frac{V^{1/2}}{\sqrt{2 \alpha^{2} \tau^{1}}} \left( \begin{array}{cc} 0 & 1 \\ 1 & 2 \tau^{1} \end{array} \right) , \quad T_{2ab} = \frac{V^{1/2}}{\sqrt{2 \alpha^{2} (\tau^{2})^{2}}} \left( \begin{array}{cc} -1 & -\tau^{1} \\ -\tau^{1} & (\tau^{2})^{2} - (\tau^{1})^{2} \end{array} \right) ,
\]

where the factor \( \frac{V^{1/2}}{\sqrt{2 \alpha^{2} \tau^{1}}} \) has been multiplied to them for later convenience. Such a rescaling of the basis causes only a simple redefinition of \( \delta T^{A} \) (eq.(17)). \( \{ T_{Aa} \}_{A=1,2} \) are symmetric, traceless 2-tensors satisfying \( -2 D_{b} T_{Aa}^{b} = -2 \partial_{b} T_{Aa}^{b} = 0 \). Thus, \( \{ T_{Aa} \}_{A=1,2} \) can also be utilized to form a basis for \( K e r P_{1}^{+} \), \( \{ \psi_{A}^{+} \}_{A=1,2} \). By
normalizing them to satisfy \((\Psi^A, T_B) = \delta^A_B\), we obtain,

\[
\Psi_{ab}^1 = \frac{\tau^2 V^{1/2}}{\sqrt{2\alpha^2}} \begin{pmatrix} 1 & \tau^1 \\ \tau^1 & 2\tau^1 \end{pmatrix}, \quad \Psi_{ab}^2 = \frac{V^{1/2}}{\sqrt{2\alpha^2}} \begin{pmatrix} 1 & -\tau^1 \\ -\tau^1 & (\tau^2)^2 - (\tau^1)^2 \end{pmatrix}.
\] (27 - b)

Thus, the Weil-Petersson metric becomes,

\[
G_{AB} = (T_A, T_B) = \frac{1}{(\tau^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
G^{AB} = (\Psi^A, \Psi^B) = (\tau^2)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (28)

This metric is known as the Poincaré metric. Hence, the Poincaré geometry\(^{(20)}\) (negative constant curvature geometry) is endowed on the Teichmüller space (a space of \((\tau^1, \tau^2)\), a universal covering space of \(\mathcal{M}_{g=1}\)), which is equivalent to the upper half-plane \(H_+ ((\tau^1, \tau^2) \in \mathbb{R} \times \mathbb{R}_+ )\). Then, the system has been finally reduced to the constraint system \(((V, \sigma), (\tau^1, p_1), (\tau^2, p_2); H = 0)\) with

\[
\alpha H = (\tau^2)^2(p_1^2 + p_2^2) - \frac{1}{2} \alpha^2 \sigma^2 V - (\tau^2)^{3/2} f(\tau) V^{1/2} / \alpha = 0.
\] (29)

The equations of motion for \((V, \sigma)\) are

\[
\dot{V} = -\alpha \sigma V \quad (30 - a)
\]

\[
\dot{\sigma} = \frac{\alpha}{2} \sigma^2 + \frac{1}{2\alpha^2} (\tau^2)^{3/2} f(\tau) V^{-1/2} \quad (30 - b)
\]

The equations of motion for \((\tau^1, p_1)\) and \((\tau^2, p_2)\) are

\[
\dot{\tau}^1 = \frac{2}{\alpha} (\tau^2)^2 p_1, \quad (31 - a)
\]

\[
\dot{p}_1 = \frac{1}{\alpha^2} (\tau^2)^{3/2} \frac{\partial f(\tau)}{\partial \tau^1} \sqrt{V}, \quad (31 - b)
\]

\[
\dot{\tau}^2 = \frac{2}{\alpha} (\tau^2)^2 p_2, \quad (32 - a)
\]
\[ \dot{p}_2 = -\frac{2}{\alpha} \tau^2 (p_1^2 + p_2^2) + \frac{3}{2\alpha} (\tau^2)^{1/2} f(\tau) \sqrt{V} + \frac{1}{\alpha} (\tau^2)^{3/2} \frac{\partial f(\tau)}{\partial \tau^2} \sqrt{V}. \] (32 - b)

First, we should note that the time evolution becomes trivial when there is no matter field, \( f(\tau) \equiv 0 \), in the following sense: In this case, eqs.(30 - a, b) allow a solution, \( \sigma \equiv 0, \ V = \text{const} \). It turns out that eqs.(29), (31 - a, b) and (32 - a, b) do not allow any solution, compatible with \( \sigma \equiv 0, \ V = \text{const} \), other than \( \tau^1 = \text{const}, \ \tau^2 = \text{const} \). This corresponds to the 3-dimensional Minkowski space \((T, X^1, X^2)\) with suitable identifications in spatial section \((X^1, X^2)\) described by \((\tau^1, \tau^2)\). The above unique solution shows that there is no time evolution with respect to the standard time-slice, \( T = \text{const} (\sigma = 0) \). This configuration is what we have chosen as a background spacetime. We can choose other York slices \( \sigma \neq 0 \) (then \( \sigma \) increases w.r.t. \( t \) according to (30 - b)). In this case, the constraint equation eq.(29) \((f(\tau) \equiv 0)\) allows the evolution of \((\tau^1, \tau^2)\) along the geodesic of the Poincaré geometry on \( H_+ \) (upper half-plane).[13] This evolution, however, should be regarded as only a kinematical one rather than a dynamical one, caused simply by a non-trivial choice of the time-slice, \( \sigma \neq 0 \).

The back-reaction of the quantum field causes a non-trivial evolution of \((\tau^1, \tau^2)\), i.e. global deformations of a torus. It is clear from eq.(29) that even when \( \sigma \approx 0 \) so that the term \(-\frac{1}{2} \alpha^2 \sigma^2 V\) in eq.(29) can be neglected, a non-trivial evolution of \((\tau^1, \tau^2)\) occurs, because of the negativity of the term \(-\tau^2)^{3/2} f(\tau) V^{1/2}/\alpha\) in eq.(29). The choice of the solution \( \sigma \equiv 0, \ V = \text{const} \) is no more allowed within our framework of the analysis, as is seen from eqs.(30 - a, b). This point will be analyzed again in the next subsection, from a more general viewpoint.

Figures 3 - a, b, c show a typical example of the evolution of \((\tau^1, \tau^2), (p_1, p_2)\) and \((V, \sigma)\), respectively. We have set \( \alpha = 1 \), i.e. the length is measured in the unit of \( \alpha \). We have set the initial conditions for \((\tau^1, \tau^2), p_1, \sigma \) and \( V \). The initial condition for \( p^2 \) has been decided using the constraint equation eq.(29). Irrespective of the initial conditions, we can observe the universal behaviors of the system due to the back-reaction: The back-reaction makes the system unstable, and drives the
trajectory into the direction corresponding to a thinner torus, i.e. \( \tau^2 \to 0 \) while \( \tau^1 \to \infty \). At the same time, the 2-volume \( V \) of the torus approaches to zero.

\[(d) \text{ The consistency between the York slice and the lapse choice } N = N(t)\]

Our analysis relies on the choice of the lapse function, \( N(t, \xi^1, \xi^2) = N(t) \). We should show that this choice is compatible with the York's time-slicing, which is also essential in the analysis. The compatibility can be shown by investigating the time evolution of \( \sigma = -K/\alpha \) in a general coordinate system. To make the analysis simple, it is good to note that energy density of the matter field

\[H_m = \langle T_{ab} \rangle n^a n^b,\]

which entered into the Hamiltonian constraint, is related to the one calculated in the flat spacetime \( H_m(0) \), as \( H_m = H_m(0)\sqrt{1/2} \) (eq.(11)).

When we write the Hamiltonian as \( H = \int (N\mathcal{H} + N^a\mathcal{H}_a)d^3x \), \( N\mathcal{H} \) and \( N^a\mathcal{H}_a \) becomes

\[N\mathcal{H} = N\sqrt{\left\{ (\pi_{ab}\pi^{ab} - \pi^2)\sqrt{-2} - R + H_m \right\}} \]

\[= N'\left\{ (\pi_{ab}\pi^{ab} - \pi^2)\sqrt{-3/2} - R\sqrt{1/2} + H_m(0) \right\} \]

\[= N'\mathcal{H}', \]

where \( N' := N\sqrt{1/2} \), and

\[N^a\mathcal{H}_a = N^a(-2D_c\pi_a^c + p_\psi \partial_a \psi) \]

where \( p_\psi \) is a conjugate momentum of \( \psi \). Then, we calculate the quantity \( \dot{\sigma} \):

\[\dot{\sigma} = (\pi \sqrt{-1})' = \dot{\pi}\sqrt{-1} - \sqrt{-2}\sqrt{-1} \pi \]

\[= (\pi^{ab}h_{ab} + \pi^{ab}\dot{h}_{ab})\sqrt{-1} - \frac{1}{2}\sqrt{-1}h^{ab}\dot{h}_{ab}\pi \]

We apply the canonical equations of motion for \( h_{ab} \) and \( \pi^{ab} \) induced by \( H = \int N'\mathcal{H}' + N^a\mathcal{H}_a \), in almost the same way as the standard treatment\[18\], noting
that $h_{ab} \partial H_m(0)/\partial h_{ab} = \frac{1}{2} \sqrt{\partial H_m(0)/\partial \sqrt{}} = 0$. The result is

$$\dot{\sigma} = 3N \tilde{K}_{ab} \tilde{K}^{ab} - N\sigma^2/2 + 3/2.N H_m + \Delta N + N^c D_c \sigma,$$

where $\Delta := -D_a D^a$. It is known that $\tilde{K} \in Ker P_1^\dagger$ has $4g - 4$ zeros. Thus, in the case of a torus ($g = 1$) we can choose the basis of $Ker P_1^\dagger$, $\{\Psi^A_{ab}\}$, as spatially constant tensors like eq. (27 - b). Furthermore, $H_m$, the Casimir energy in our model, has turned out to be constant w.r.t. $\xi^1$ and $\xi^2$. Then, it is clear from above equation that the constancy of $\sigma$ w.r.t. $\xi^1$ and $\xi^2$ is preserved once $\sigma =$const and $N =$const w.r.t. $\xi^1$ and $\xi^2$ on one York's slice. It is also clear that the York's time-slicing and the choice $N = N(t)$ are not compatible with each other for $g \geq 2$ because of the zeros of $\tilde{K}_{ab}$, i.e. because of the non-constancy of $\tilde{K}_{ab}$.[13] Eliminating $\tilde{K}_{ab} \tilde{K}^{ab}$ by the use of the Hamiltonian constraint $\mathcal{H} = 0$, the above equation can also be expressed as

$$\dot{\sigma} = N\sigma^2 + 3NR - \frac{3}{2}N H_m + \Delta N + N^c D_c \sigma.$$

If there were no matter field, then $H_m = 0$ and the solution $\sigma \equiv 0$ is allowed. Thus, we can choose the standard time-slicing $\sigma = 0$. There is no non-trivial evolution in $(\tau^1, \tau^2)$ w.r.t. this time-slicing as discussed in §§(c). Once matter field is taken into account, the solution $\sigma \equiv 0$ is no more allowed. At the same time, non-trivial evolutions in $(\tau^1, \tau^2)$ are induced as we have seen in §§(c).
4. The effective action for the modular degrees of freedom

(a) Partition function

We have treated so far the back-reaction of the quantum field on the modular
degrees of freedom, in the sense that the semiclassical Einstein equation, eq.(1),
has been solved, with $< T_{\alpha \beta} >$ on the right-hand side being calculated in the
background spacetime. We can handle the same problem in a more systematic
manner by the path-integral approach. The significance of this investigation is
as follows: First, we know that we can derive eq.(1) by formally taking the first
variation of the phase w.r.t. $g_{\alpha \beta}$ in the in-in path-integral expression for $g_{\alpha \beta}$ and
$\psi$. However, when we discuss the semiclassical gravity in more detail, it is
preferable to take into account the effects coming from the path-integral measure
of $g_{\alpha \beta}$. Usually, we cannot discuss much about this effect since we cannot fix
the measure in a reasonable manner. Fortunately, our model is simple enough to
investigate the measure to a great extent, by making use of techniques developed
in string theories.

Second, when we need to investigate validity conditions of the semiclassical
treatment described by eq.(1), then, we have to study the second variation of the
effective action $W[V_+, \tau_+; V_-, \tau_-]$. Thus, we need to estimate $W[V_+, \tau_+; V_-, \tau_-]$ by the use of the in-in path-integral formalism.

We first discuss within the framework of the standard in-out path-integral
formalism and later generalize it to the in-in formalism. For convenience, almost
all calculations will be done with the metric of the Riemannian signature.

The metric in our model has the form $(\S 2 - b)$

$$ ds^2 = dt'^2 + e^{2\phi(t')} \hat{g}_{\alpha \beta} d\xi^\alpha d\xi^\beta = e^{2\phi(t)}(dt^2 + \hat{h}_{ab} d\xi^a d\xi^b) $$

$$= g_{\alpha \beta} dx^\alpha dx^\beta,$$  \hspace{1cm} (33)

where

$$ g_{\alpha \beta} = e^{2\phi(t)} \bar{g}_{\alpha \beta} = e^{2\phi(t)}(1, \hat{h}_{ab}), \quad \hat{h}_{ab} = \frac{1}{\tau^2} \begin{pmatrix} 1 & \tau^1 \\ \tau^1 & |\tau|^2 \end{pmatrix}. $$  \hspace{1cm} (34)
Noting that there is no conformal invariance in the total action \( S = \frac{1}{\alpha} \int R \sqrt{\gamma} + S_m \) (although it is there in the matter action \( S_m \)), we should take care only of the spatial diffeomorphism invariance, \( Diff(\Sigma) \) in defining the path-integral measure. Now,

\[
\delta g_{\alpha\beta} = g_{\alpha\beta} - \bar{g}_{\alpha\beta} = 2 \delta \phi \bar{g}_{\alpha\beta} + e^{2\phi} \delta \bar{h}_{\alpha\beta} = 2 \delta \phi \bar{g}_{\alpha\beta} + e^{2\phi} \left\{ (\dot{\bar{\cal{P}}}_v)_{\alpha\beta} + \dot{\bar{\cal{T}}}_{\alpha\beta} \right\}, \tag{35}
\]

where \( \delta \bar{h}_{\alpha\beta} := (0, \delta \dot{h}_{ab}), (\dot{\bar{\cal{P}}}_v)_{\alpha\beta} := (0, (\dot{\bar{\cal{P}}}_v)_{ab}) \) and \( \dot{\bar{\cal{T}}}_{\alpha\beta} := (0, \dot{\bar{\cal{T}}}_{ab}) \) ("\( \dot{\ } \)" for \( (\dot{\bar{\cal{P}}}_v)_{ab}, \dot{\bar{\cal{T}}}_{ab} \) and \( \dot{\Psi}_{ab}^A \) indicates that these quantities are calculated by using \( \dot{h}_{ab} \)).

Introducing a natural inner-product on 3-tensors, \( (\ , \ )_* \),

\[
(A, B)_* = \frac{1}{\alpha^3} \int d^3 x \sqrt{\gamma} \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} A_{\alpha\beta} B_{\gamma\delta}, \tag{36}
\]

then,

\[
||\delta g||^2 = 4(\delta \phi, g, \delta \phi, g)_* + (e^{2\phi}(\dot{\bar{\cal{P}}}_v + \dot{\bar{\cal{T}}}_A \delta \tau^A), e^{2\phi}(\dot{\bar{\cal{P}}}_v + \dot{\bar{\cal{T}}}_B \delta \tau^B))_*
= \frac{12}{\alpha} \int dt e^{3\phi}(\delta \phi)_*^2 + \frac{1}{\alpha} \int dt e^{3\phi}(\dot{\bar{\cal{P}}}_v + \dot{\bar{\cal{T}}}_A \delta \tau^A, \dot{\bar{\cal{P}}}_v + \dot{\bar{\cal{T}}}_B \delta \tau^B),
\]

Here, the natural inner-product \( (\ , \ ) \) on 2-tensors is given by eq.(19) with \( h_{ab} \) replaced by \( \dot{h}_{ab} \). Note that \( (A, B)_* = \frac{1}{\alpha} \int dt e^{-\phi}(A, B) \), for \( A = (0, \dot{\bar{A}}_{ab}), B = (0, \dot{\bar{B}}_{ab}) \).

To modify the above expression for \( ||\delta g||^2 \), we choose

\( \{(\dot{\bar{\cal{P}}}_v)_{ab}\}_{\forall w^a \notin Ker \tilde{P}_1}, \{\dot{\Psi}_{ab}^A\}_{A=1,2} \) as the basis of traceless 2-tensors, where

\( \{\dot{\Psi}_{ab}^A\}_{A=1,2} \) is the orthogonal basis for \( Ker \tilde{P}_1 \) w.r.t. the inner product eq.(19) (with \( h_{ab} \) there replaced by \( \dot{h}_{ab} \)). We assume that \( (\dot{\bar{\cal{P}}}_v)_{ab} \) (\( \forall w^a \notin Ker \tilde{P}_1 \)) have already been made orthogonal to each other by the formal Schmidt orthogonalization, starting from \( (\dot{\bar{\cal{P}}}_v)_{ab} \) in eq.(35). Then, we can expand the traceless 2-tensor
\[ \hat{T}_{Aab} \delta \tau^A \] in terms of these basis,

\[ \hat{T}_{Aab} \delta \tau^A = \sum_{A,B=1}^{2} \frac{\hat{\psi}^B}{(\hat{\psi}_B, \hat{\psi}_B)}(\hat{\psi}_B, \hat{\psi}_B) (\hat{T}_{A} \delta \tau^A) + \sum_{A=1}^{2} \sum_{w \notin \ker \hat{P}_1} (\hat{P}_1 w)_{ab} \frac{(\hat{P}_1 w, \hat{T}_{A}) \delta \tau^A}{(\hat{P}_1 w, \hat{P}_1 w)}. \]

Thus,

\[ (\hat{P}_1 v + \hat{T}_A \delta \tau^A, \hat{P}_1 v + \hat{T}_B \delta \tau^B) = (\hat{P}_1 v', \hat{P}_1 v') + \sum_{A,B,C,D=1}^{2} \frac{(\hat{\psi}_C, \hat{T}_A) (\hat{\psi}_D, \hat{T}_B)}{(\hat{\psi}_C, \hat{\psi}_C) (\hat{\psi}_D, \hat{\psi}_D)} (\hat{\psi}_C, \hat{\psi}_C) \delta \tau^A \delta \tau^B, \]

where \( v' = v + \sum_A \sum_{w \notin \ker \hat{P}_1} (\hat{P}_1 w, \hat{T}_A) \delta \tau^A \).

Thus,

\[ ||\delta g||^2 = \frac{12}{\alpha} \int dte^{3\phi} (\delta \phi)^2 + \frac{1}{\alpha} \int dte^{3\phi} (\hat{P}_1 v', \hat{P}_1 v') + \int dte^{3\phi} \sum_{A,B,C,D=1}^{2} \frac{(\hat{\psi}_C, \hat{T}_A) (\hat{\psi}_D, \hat{T}_B)}{(\hat{\psi}_C, \hat{\psi}_C) (\hat{\psi}_D, \hat{\psi}_D)} (\hat{\psi}_C, \hat{\psi}_C) \delta \tau^A \delta \tau^B. \]

Fixing the measure by

\[ 1 = \int [d\delta g \exp -||\delta g||^2] = \int [d\phi][dv^1][dv^2][d\tau^1][d\tau^2] \exp -||\delta g||^2, \]

we get

\[ J = \prod_t J_t, \quad J_t = e^{3\phi} \text{Det}^{1/2}(e^{3\phi} \hat{P}_1 \hat{P}_1) \frac{\text{Det} e^{3\phi} (\hat{\psi}_C, \hat{T}_D)}{\text{det}^{1/2}(\hat{\psi}_A, \hat{\psi}_B)}. \]

where, "t" attached to \( \text{Det}'(e^{3\phi} \hat{P}_1 \hat{P}_1) \) indicates that \( \ker \hat{P}_1 \) should be removed from the domain of \( \hat{P}_1 \hat{P}_1 \) for the computation of this determinant. Trivial numerical factors in the measure have been removed using the freedom of \( \mathcal{N} \). Thus,

\[ Z = \mathcal{N} \int [d\phi][dv^1][d\tau][dv^2] \text{Det}^{1/2}(e^{3\phi} \hat{P}_1 \hat{P}_1) \frac{\text{det}(\hat{\psi}_C, \hat{T}_D)}{\text{det}^{1/2}(\hat{\psi}_A, \hat{\psi}_B)} e^{3\phi} \exp -S[\hat{g}, \psi] \]

\[ = \left( \prod_t V_{Diff} \right) \mathcal{N} \int [d\phi][d\tau][dv^1] \frac{\text{Det}^{1/2}(e^{3\phi} \hat{P}_1 \hat{P}_1) \text{det}(\hat{\psi}_C, \hat{T}_D)}{\text{det}^{1/2}(\hat{\chi}^A, \hat{\chi}^B) \text{det}^{1/2}(\hat{\psi}_A, \hat{\psi}_B)} e^{3\phi} \exp -S[\hat{g}, \psi]. \]

(37)

Here \( e^{3\phi} \) in \( \text{det} e^{3\phi} (\hat{\psi}_C, \hat{T}_D) \) has been moved outside of \( \text{det} \), noting that \( (\hat{\psi}_C, \hat{T}_D) \)
forms $2 \times 2$ matrix. The factor $e^{3\phi}$ in $Det^{1/2}(e^{3\phi} \hat{P}_1^a \hat{P}_1^b)$ will be also handled later. In the last line in eq.(37), $\{\hat{X}^A \}_{A=1,2}$ is the basis of $\text{Ker} \hat{P}_1$, i.e. a space of conformal Killing vectors on $\Sigma$ w.r.t. $\hat{h}_{ab}$ (see eq.(18 - b)). It is known that $\text{dim}_R \text{Ker} \hat{P}_1 = \text{dim}_R \text{Ker} \hat{P}_1 + 6 - 6g$ (Riemann-Roch theorem[15],[16]), so that $\text{dim}_R \text{Ker} \hat{P}_1 = 2$ for a torus. The factor $\text{det}^{-1/2}(\hat{X}^A, \hat{X}^B)$ has appeared by factoring out the volume of the spatial diffeomorphism group (homotopic to 1) $V_{Diff_0} = V_{Diff_0}' \cdot V_{\text{Ker} \hat{P}_1}$ ($V_{Diff_0}' := \int_{\psi' \in \text{Ker} \hat{P}_1} [d\psi']$).

We can examine each factor appearing in (37) in detail. To begin with, let us investigate the action (in the Riemannian signature) $S[\bar{g}, \psi] = -\frac{1}{a} \int \hat{R} \sqrt{\bar{g}} + S_m[\psi, \bar{g}]$. From eq.(33), $\hat{g}_{\alpha \beta} = e^{2\phi(t)} \hat{g}_{\alpha \beta}$ where $\hat{g}_{\alpha \beta} = (1, \hat{h}_{ab})$. Thus, according to the transformation formula of the scalar curvature under conformal transformations[18],

$$1/a. \int \hat{R} \sqrt{\bar{g}} = 1/a. \int e^{\phi} \left( \hat{R} - 4(1/2. \dot{\phi}^2 + \ddot{\phi}) \right) = \alpha \int dt e^{\phi} \dot{\phi}^2 + 1/a. \int dt e^{\phi} \int d^2 \xi (-K_{ab} \hat{R}^{ab} + \hat{R}^2 + R) \sqrt{\bar{g}}$$

$$= \alpha \int dt e^{\phi} \dot{\phi}^2 - \frac{e^{-\phi}}{4(\tau^2)^2} ((\dot{\tau}^1)^2 + (\dot{\tau}^2)^2)$$

Here, the last term in the first line has been integrated by parts and the surface term has been discarded. Eqs.(22), (28) and $\int d^2 \xi R \sqrt{\bar{g}} = 4\pi \chi(T^2) = 0$ have been used in the middle line and eqs.(30 - a), (31 - a) and (32 - a) have been used in the last line. As is clear from the discussion in §§3-a, this analysis is valid only for the case in which the quantum effect of matter $\psi$ does not harm the homogeneous, and locally isotropic nature of the space $\Sigma \simeq T^2$, so that the momentum constraint has the form eq.(16 - c'). We assume that the matter and its vacuum state are so chosen as to respect this nature, at least approximately. Following our previous analysis, we choose a massless conformally coupled scalar field with the conformal vacuum.

Then, the matter part of the action $S_m[\psi, \bar{g}]$ is conformally invariant, i.e. in-
variant under the transformation \( \tilde{g}_{\alpha \beta} \rightarrow \tilde{g}_{\alpha \beta} = e^{-2\phi} \tilde{g}_{\alpha \beta}, \psi \rightarrow \tilde{\psi} = e^{\phi/2} \psi \). Thus,

\[
S_m[\psi, \tilde{g}] = S_m[e^{\phi/2} \psi, \tilde{g}]
\]

\[
= \frac{1}{2} \int dt d^2 \xi \left( \tilde{g}^{\alpha \beta} \partial_\alpha (e^{\phi/2} \tilde{\psi}) \partial_\beta (e^{\phi/2} \tilde{\psi}) + \frac{1}{8} \tilde{R}(e^{\phi/2} \tilde{\psi})^2 \right)
\]

where \( \tilde{g}_{\alpha \beta} = (1, \tilde{h}_{ab}) \) (det\( \tilde{g}_{\alpha \beta} = -1 \)). By introducing a variable \( \tilde{\psi} = e^{\phi/2} \psi \), we get

\[
\int [d\psi] \exp -S_m[\psi, \tilde{g}]
\]

\[
= \int [d\tilde{\psi}] \text{Det}(e^{-\phi/2} 1) \exp \left( - \frac{1}{2} \int d^3 x \left( \tilde{g}^{\alpha \beta} \partial_\alpha \tilde{\psi} \partial_\beta \tilde{\psi} + \frac{1}{8} \tilde{R}(\tilde{\psi})^2 \right) \right)
\]

We can also express \( \tilde{R} \) in terms of \( \tau^1, \tau^2 \) and \( \phi \) in the same manner as for the gravity part. For the later convenience, however, we keep it in this form. In the next sub-section, we will drop this term regarding that we are interested in the back-reaction effect on the conformally flat background spacetime. As discussed at the beginning of sections 2–b, the most idealistic treatment of the back-reaction might be the self-consistent determination of \( g_{\alpha \beta} \) using eq.(1). Since such a self-consistent treatment becomes too complicated to be tractable in practice, we usually use eq.(1) to estimate the modification of the background geometry due to \( < T_{\alpha \beta} > \).

Following this spirit, we have prepared a conformally flat space time as a background and calculated \( < T_{\alpha \beta} > \) on it. In the present context, neglecting \( \tilde{R} \)-term in the above \( S_m \) corresponds to this treatment. In fact, we will see in the next sub-section that the equations of motion for \( V, \tau^1 \) and \( \tau^2 \) (eqs.(30–32)) are exactly reproduced from the first variation of the effective action calculated by dropping the \( \tilde{R} \)-term in \( S_m \).

Thus, a part of eq.(37) becomes

\[
\int [d\tilde{\psi}] \exp -S[\tilde{g}, \tilde{\psi}] = \int [d\tilde{\psi}] \text{Det}(e^{-\phi/2} 1) \exp -(S_g + S_{\tilde{\psi}}), \quad (38 \text{a})
\]

where

\[
S_g = \alpha \int dt \{-2e^\phi \dot{\phi}^2 + \frac{e^{-\phi}}{4(\dot{\tau})^2} ((\ddot{\tau}^1)^2 + (\ddot{\tau}^2)^2) \}
\]

(38-b)
\[ S_\psi = \frac{1}{2} \int d^3 x \left( \bar{\psi} \gamma^\alpha \partial_\alpha \psi + \frac{1}{8} \tilde{R} \psi^2 \right) \]  

(38 - c).

Now, \( \text{det}(\hat{\Psi}^C, \hat{T}_D) \), \( \text{det}^{-1/2}(\hat{\chi}^A, \hat{\chi}^B) \) and \( \text{det}^{-1/2}(\hat{\Psi}^A, \hat{\Psi}^B) \) in eq.(37) can easily be estimated. Adapting the choice eq.(27 - a, b), we get,

\[
\frac{\text{det}(\hat{\Psi}^C, \hat{T}_D)}{\text{det}^{1/2}(\hat{\Psi}^A, \hat{\Psi}^B)} = \text{det}^{-1/2}(\hat{\Psi}^A, \hat{\Psi}^B) = \text{det}^{1/2} G_{AB} = \frac{1}{(\tau^2)^2}. 
\]

A set of vectors \( \{\hat{\chi}^A\}_{\hat{A}=1,2} \) is the basis for \( \text{Ker} \hat{P}_1 \), a space of conformal Killing vectors. Fortunately, we can choose as a basis, constant vectors, \( \hat{\chi}^1 = (1, 0) \) and \( \hat{\chi}^2 = (0, 1) \) for the case of a torus, without introducing any singular point. Then,

\[
det^{1/2}(\hat{\chi}^A, \hat{\chi}^B) = det \hat{h}_{ab} = 1. 
\]

The factor \( \text{Det}^{1/2}(e^{3\phi} \hat{P}_1 \hat{P}_1) \) can also be estimated explicitly. Note that

\[
(\hat{P}_1 \hat{P}_1 v)_a = -2D^a(\hat{P}_1 v)_a = 2 \hat{\Delta} v_a - 2 \hat{R}^a c v_c = (2 \Delta \delta^a_b) v_b, 
\]

where \( \hat{\Delta} = -1/\sqrt{\lambda}. \) \( \partial_a (\hat{h}^{ab} \sqrt{\lambda} \partial_b) = -\hat{h}^{ab} \partial_a \partial_b. \) Thus, \( \text{Det}^{1/2}(e^{3\phi} \hat{P}_1 \hat{P}_1) \)

\[
= \text{Det}'(2e^{3\phi} \hat{\Delta}) = \frac{1}{2} e^{-3\phi} \text{Det}(2e^{3\phi}) \cdot \text{Det}' \hat{\Delta}. 
\]

First, we can estimate \( \text{Det}(2e^{3\phi}) \) by the heat-kernel method \cite{21},\cite{15}. Noting that this determinant has come from the determinant \( \text{Det}' e^{3\phi} \hat{P}_1 \hat{P}_1 \), a spatial (2-dimensional) operator, we should use the 2-dimensional heat-kernel. The result is \( \text{Det}(2e^{3\phi}) = \exp -\lambda' \int d^2 \xi \sqrt{\lambda} \), where \( \lambda' = \lim_{\epsilon, \delta \rightarrow 0} \Gamma(\epsilon)/\delta. \) Note that

\[
\int [d\tau(\cdot)] \text{Det}^{1/2}(e^{3\phi} \hat{P}_1 \hat{P}_1) = \prod_t \int d\tau_t e^{-3\phi} \text{Det}'(\hat{\Delta}(\tau)) \exp -\lambda' \int d^2 \xi \sqrt{\lambda} 
\]

\[
= \int [d\tau(\cdot)] e^{-3\phi} \text{Det}'(\hat{\Delta}(\tau)) \exp -\lambda'' \int d^3 x \sqrt{\lambda}. 
\]

Thus, \( \text{Det}2e^{3\phi} \) can be absorbed into the cosmological constant. Here, for simplicity, we understand that \( \lambda_{bare} \) is so chosen as to cancel exactly the contributions of the form \( \exp - \int \sqrt{\lambda} \) arising from quantum effects.
The estimation of $\text{Det}' \hat{\Delta}$ can be explicitly done by the use of the eigenvalues of $\hat{\Delta}$ in eq. (7).

$$\text{Det}' \hat{\Delta} = \prod_{n_1, n_2 = -\infty}^{\infty} \frac{4\pi^2}{\tau^2} (n_2 - n_1 \tau)(n_2 - n_1 \tilde{\tau}) = \tau^2 \text{Det}(\frac{1}{\tau^2} 1) \prod_{n_1, n_2} 2\pi (n_2 - n_1 \tau) 2\pi (n_2 - n_1 \tilde{\tau}) ,$$

where the relation $\text{Det}'(\frac{1}{\tau^2} 1) = \tau^2 \text{Det}(\frac{1}{\tau^2} 1)$ has been used. Again, $\text{Det}(\frac{1}{\tau^2} 1)$ can be absorbed into the cosmological constant. There is a standard technique to convert the above double infinite product to a single infinite product.\[15\] The result is

$$\text{Det}' \hat{\Delta} = \tau^2 (\exp 4\pi^2 \sum_{n=1}^{\infty} n ) \cdot \prod_{n=1}^{\infty} (1 - \exp i2\pi n \tau)^4 = \tau^2 (\exp -\frac{\pi}{3} \tau^2 ) \cdot \prod_{n=1}^{\infty} (1 - \exp i2\pi n \tau)^4 ,$$

where the regularization $\sum_{n=1}^{\infty} n = \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} ne^{-\epsilon n} = (\lim_{\epsilon \to 0} e^{-2}) - \frac{1}{12}$ (or $\zeta(1) = -\frac{1}{12}$ by the analytic continuation from $\zeta(z), Rez > 1$) has been implemented. Thus, we get,

$$Z = N' \int [d\phi][d\tau^1 d\tau^2][dv]$$

$$\left(\exp \frac{9\phi}{2}\right) \frac{1}{\tau^2} \cdot (\exp -\frac{\pi}{3} \tau^2 ) \cdot \prod_{n=1}^{\infty} (1 - \exp i2\pi n \tau)^4 \exp -(S_g + S_\psi),$$

with \((38 - b, c)\).

As is indicated in eq.(37), $\text{Diff}_0(\Sigma)$ (the diffeomorphism group on $\Sigma$ homotopic to 1) has been factored out from the path-integral. What is needed to be factored out really is the whole diffeomorphism group on $\Sigma$, $\text{Diff}(\Sigma)$. Note that\[15,16\]

$$\mathcal{M}_g \simeq \text{Riem}(\Sigma)/\text{Weyl} \times \text{Diff}(\Sigma) \simeq (\text{Riem}(\Sigma)/\text{Weyl} \times \text{Diff}_0(\Sigma)) / \text{MCG} \simeq H_+ / \text{PSL}(2, \mathbb{Z}) \simeq D(\mathbb{H}_+)/ \sim .$$

Here, $\text{MCG} := \text{Diff}(\Sigma)/\text{Diff}_0(\Sigma)$ is the mapping-class group for $\Sigma$, and $\text{MCG} \simeq$
$\text{PSL}(2, \mathbb{Z})$ for $\Sigma \cong T^2$ (i.e. a group of $2 \times 2$ unimodular matrix with integer coefficients, modulo sign). $D(H_+)$ is the fundamental region in $H_+$ (upper half-plane) w.r.t. the action of $\text{PSL}(2, \mathbb{Z})$ (e.g. the Dirichlet region $D = \{z \in H_+ | |Rez| \leq 1/2, |z| \geq 1\}$) and $/ \sim$ indicates the identification $(\tau^1, \tau^2) \sim -(\tau^1, \tau^2)$ on the boundary of $D$.\textsuperscript{[15],[16]} Thus, the integral region for $(\tau^1, \tau^2)$ in eq.(39) should be understood as over $\mathcal{M}_{g=1}$ rather than over $H_+$, considering that we have factored out the volume of the mapping-class group $\text{MCG} \simeq \text{PSL}(2, \mathbb{Z})$ as well as $Diff_0(\Sigma)$ from the path-integral.

(b) Estimation of the functional determinant for matter

Now, we estimate the path-integral for matter $\psi$ in eq.(39). Our aim is to obtain the effective action of the form $W[\phi, \tau^1(\cdot), \tau^2(\cdot)]$ by integrating out quantum fluctuations of matter. Generalizing the framework to the in-in formalism and getting $W[\phi_+, \tau_+; \phi_-, \tau_-]$, one can discuss the validity conditions for the semiclassical treatment\textsuperscript{[7]}, eq.(1). At this stage, the peculiarity of the system including gravity is prominent. In the standard treatment of a dissipative system, like a quantum Brownian motion\textsuperscript{[17]}, the interaction between the sub-system and the environment is described by a weak, linear coupling. In our case, however, there is no such interaction term between gravity (analogous to the sub-system) and matter (analogous to the environment). Rather, the interaction is bilinear in $\psi$ and non-linear in $(\tau^1, \tau^2)$ and $\phi$, as is seen in eq.(38 – c) with eq.(34). Thus, it requires a new treatment for a deeper analysis. Here, we should be content with only a rough estimation of the effect of the nonlinear coupling. We want to estimate the partition function for matter,

$$Z_\psi = \int [d\psi] \exp -\frac{1}{2\hbar} \psi(\partial^2 + \frac{\mathcal{R}}{2}) \psi = \text{Det}^{-1/2} \left\{ \frac{1}{2\hbar} (-\partial^2 + \frac{\mathcal{R}}{2}) \right\}$$

$$= \exp -\frac{1}{\hbar} \mathcal{W}[\tau(\cdot)] \ .$$

Here, " $\cdot$ " denotes the Riemannian signature quantity. We calculate using the metric $\tilde{g}_{\alpha\beta} = (1, \hat{h}_{ab})$ (eq.(34)). It is difficult to estimate the above functional determinant exactly for a general function $(\tau^1(\cdot), \tau^2(\cdot))$. From the viewpoint of
the quantum dissipative system, this difficulty comes from the peculiarity of the interaction between gravity and matter. As discussed in the beginning of §§2 - b and in the previous sub-section, we treat the back-reaction problem in the sense that we investigate the modification of the background geometry due to matter, i.e. due to $< T_{\alpha \beta} >$ calculated on the background spacetime. We have chosen as a background, a conformally flat spacetime. Thus, for the lowest order approximation, we treat $\tau^1$ and $\tau^2$ as constant, so that we can set $\dot{R} = 0$. This treatment corresponds to the lowest order estimation of the functional form of the effective potential in standard quantum field theory.[21]

Thus, we need to estimate the determinant of the operator

$$\dot{A} := -\frac{\alpha^2}{2\pi \hbar} \partial^2 = -\frac{\alpha^2}{2\pi \hbar} (\partial_0^2 + \hat{h}^{ab} \partial_a \partial_b)$$

where $\alpha^2$ has been inserted to make $\dot{A}$ non-dimensional. The Planck constant $\hbar$ should be here regarded as $[\hbar] = [1]$. It has been inserted for the convenience of recovering a formula for pseudo-Riemannian signature. In the present context, it may be more convenient to regard $[\hat{h}_{ab}] = [\alpha^2]$, $[x^0] = [1]$ rather than $[\hat{h}_{ab}] = [1]$, $[x^0] = [\alpha]$. Now, we need to solve the heat equation[21],

$$\begin{cases}
\dot{A} \rho = -\frac{\partial}{\partial s} \rho \\
\rho(x, y, s = 0) = \delta^{(3)}(x - y)
\end{cases}$$

Here, $x := (x^0 = t, \xi^1, \xi^2)$. Taking care of the periodicity in space, the solution is given by

$$\rho(x, x', s) = \left(\frac{\hbar}{2\alpha^2 s}\right)^{3/2} \sum_{n_1, n_2} \exp -\frac{\pi \hbar}{2\alpha^2 s} \{(x^0 - x'^0)^2 + \hat{h}_{ab}(\xi - \xi'^0)^a (\xi - \xi'^0)^b\}$$

especially,

$$\rho(x, x, s) = \left(\frac{\hbar}{2\alpha^2 s}\right)^{3/2} \sum_{n_1, n_2} \exp -\frac{\pi \hbar}{2\alpha^2 s} (n, n)$$

where $(n, n) := \hat{h}^{ab} n^a n^b = \frac{1}{\tau^2}(n_1^2 + 2\tau^1 n_1 n_2 + |\tau|^2 n_2^2)$. Thus, the $\zeta$-function asso-
\[ \zeta_A(z) = \frac{1}{\Gamma(z)} \int_0^\infty ds s^{z-1} \text{Tr}\rho(s) \]
\[ = \frac{\Omega}{\Gamma(z)} \sum_{n_1,n_2} \int_0^\infty ds s^{z-1} \left( \frac{\hbar}{2\alpha^2 s} \right)^{3/2} \exp \left( -\frac{\pi\hbar}{2\alpha^2 s} (n,n) \right) \]
\[ = \left( \frac{\hbar}{2\alpha^2} \right)^{3/2} \frac{z\Gamma(\frac{3}{2} - z)}{\Gamma(z+1)} \Omega \sum_{n_1,n_2} \left( \frac{\pi\hbar}{2\alpha^2} (n,n) \right)^{z-3/2}, \quad (40) \]

where \( \Omega = \int d^3x \sqrt{g} \) and a transformation of variable \( s (z := \frac{\pi\hbar}{2\alpha^2} (n,n)s^{-1}) \) has been done to get the formula in the last line from the middle line. Noting that
\[ \frac{d}{dz}|_{z=0} \left( \frac{z\Gamma(\frac{3}{2} - z)}{\Gamma(z+1)} C^{z-1} \right) = \frac{\sqrt{\pi}}{2} C^{-1} \text{ for } \forall C \text{ when } C \text{ is independent of } z, \]
we get
\[ \zeta_A'(0) = \frac{\Omega}{2\pi} \sum_{n_1,n_2} (n,n)^{-3/2}. \]

Thus,
\[ \hat{W} = \frac{\hbar}{2} \ln \text{Det} \hat{A} = -\frac{\hbar}{2} \zeta_A'(0) \]
\[ = -\frac{\hbar\Omega}{4\pi\alpha^3} (\tau^2)^{3/2} \sum_{n_1,n_2} \frac{1}{(n_1^2 + 2\tau n_1 n_2 + |\tau|^2 n_2^2)^{3/2}}. \quad (41-a) \]

To recover \( W \) for the pseudo-Riemannian signature, we replace \( \hbar \rightarrow i\hbar, \alpha \rightarrow i\alpha \) (no change in \( \Omega, dz^0 dx^1 dx^2 dx^3 \rightarrow dx^0 dx^1 dx^2 dx^3 \)). This replacement comes from the comparison between \( W = i\frac{\hbar}{2} \ln \text{Det} \left( \frac{\alpha^2}{2\pi\hbar} (-\partial^2) \right) \) and \( \hat{W} = \frac{\hbar}{2} \ln \text{Det} \left( \frac{\alpha^2}{2\pi\hbar} (-\partial^2) \right) \).

Thus,
\[ W[\tau^1, \tau^2] = \frac{\hbar\Omega}{4\pi\alpha^3} (\tau^2)^{3/2} \sum_{n_1,n_2} \frac{1}{(n_1^2 + 2\tau n_1 n_2 + |\tau|^2 n_2^2)^{3/2}}. \quad (41-b) \]

Since we have used the expectation value of the energy-momentum tensor for matter, \( < T_{\alpha\beta} > \), to couple with gravity (eq.(16-b) or eq.(24)), we need to use
the in-in path-integral formalism, rather than the standard in-out formalism.\footnote{5},\footnote{6},\footnote{7}

Then the matter part of the action (pseudo-Riemannian) in eq.(39) with eq.(38 - b) should be reinterpreted as,

\[
S_{\psi} = -\frac{1}{2} \int_c (\hat{g}^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi + \frac{1}{8} \hat{R} \psi^2) \sqrt{\hat{g}} \\
= -\frac{1}{2} \int_+ (\hat{g}^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi + \frac{1}{8} \hat{R} \psi^2) \sqrt{\hat{g}} + \frac{1}{2} \int_- (\hat{g}^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi + \frac{1}{8} \hat{R} \psi^2) \sqrt{\hat{g}},
\]

where "c" stands for the closed-time contour and "+" and "−" stand for, respectively, the + -branch and the - -branch of the time-contour. Then,

\[
S_{\psi} = \int_+ \sqrt{\hat{g}} \psi \frac{1}{2\hbar} (\partial^2 \psi - \frac{1}{8} \hat{R} \psi^2) - \int_- \sqrt{\hat{g}} \psi \frac{1}{2\hbar} (\partial^2 \psi - \frac{1}{8} \hat{R} \psi^2) \\
= \int (\psi_+ \psi_-) \begin{pmatrix} \frac{1}{2\hbar} (\partial^2 - \frac{1}{8} \hat{R} \psi^2) \sqrt{\hat{g}} & 0 \\
0 & -\frac{1}{2\hbar} (\partial^2 - \frac{1}{8} \hat{R} \psi^2) \sqrt{\hat{g}} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}
\]

Since + and − components are separated completely, it is enough to look at only the + -sector (or − -sector).

(c) The contribution of the path-integral measure to the semiclassical dynamics

Now, let us investigate the effective action, \( S[\phi, \tau^1, \tau^2] = S_g[\phi, \tau^1, \tau^2] + W[\phi, \tau^1, \tau^2] \), where \( S_g[\phi, \tau^1, \tau^2] \) and \( W[\phi, \tau^1, \tau^2] \) are given by eq.(38 - b) and (41 - b). (More precisely, the pseudo-Riemannian version of eq.(38 - b).) The mathematical expression is exactly the same as the Riemannian expression, eq.(38 - b).) It should be noted that the first variations of \( S[\phi, \tau^1, \tau^2] \) w.r.t. \( \phi \) and \( (\tau^1, \tau^2) \) reproduce exactly eqs.(30 - 32), provided that the lapse \( N(t) \) is restored and set \( N(t) = e^\phi \), in accordance with the metric \( \hat{g}_{\alpha \beta} = e^\phi \hat{g}_{\alpha \beta}, \hat{g}_{\alpha \beta} = (1, \hat{h}_{ab}) \), used to define \( S_g \). Here, \( \Omega \) in eq.(41 - b) should be interpreted as \( \Omega = \int d^3x \sqrt{\hat{g}} = \int dt e^{\phi(t)}V \). This result shows the following two points.

First, our approximation for the estimation of \( Det^{1/2} \left( \frac{1}{\hbar}(-\hat{g}^2 + \frac{1}{8} \hat{R}) \right) \), treating \( \tau^1, \tau^2 \) as if they were constant so that \( \hat{R} = 0 \), corresponds to the approximation
used to solve the semiclassical Einstein equation, eq.(1). Namely, \( \langle T_{\alpha\beta} \rangle \), calculated on a conformally flat background, is used in eq.(1) to estimate the deviation from the original background geometry. As is discussed at the beginning of §§2 - b and in §§4 - a, the latter approximation has been implemented for the tractability of the problem, at the expense of the self-consistency of eq.(1). Such an approximation is what is usually meant by the term "back-reaction", and this may be the best we can do in practice.

Second, we reproduced eq.(1) (or equivalently, eqs.(30-32)) from the phase part \( S_g + W \) in the partition function \( Z \) with matter part integrated, and without taking care of the contributions from the measure for \( \phi \) and \( (\tau^1, \tau^2) \) (cf. eq.(39)). However, we now know explicitly the non-trivial path-integral measure for \( \phi \) and \( (\tau^1, \tau^2) \) as is shown in eq.(39). This fact shows the limitation of the discussions of semiclassical gravity using eq.(1) as a starting point. There should be \( O(\hbar) \) correction to eq.(1) coming from the path-integral measure for \( g_{\alpha\beta} \) and this correction causes a non-trivial correction to the dynamics of \( g_{\alpha\beta} \).

Let us consider this effect explicitly in our case. From eq.(39), one finds that the correction to the (pseudo-Riemannian) action coming from the measure, \( S_{\text{measure}}(\phi, \tau^1, \tau^2) \), can be given as

\[
S_{\text{measure}}(\phi, \tau^1, \tau^2) = \int dt \left( \frac{9}{2} \phi - \frac{\pi}{3} \tau^2 - \ln \tau^2 + 2 \sum_{n=1}^{\infty} \ln |1 - e^{i2\pi n \tau}|^2 \right) 
\]

\[
= \int dt \left( \frac{9}{2} \phi - \ln \tau^2 + 2 \sum_{n=1}^{\infty} \ln(\sin \pi n \tau \sin \pi n \bar{\tau}) \right),
\]

where we have used the identity \( \ln |1 - e^{i2\pi n \tau}|^2 = \ln 4 - 2\pi^2 n + \ln(\sin \pi n \tau \sin \pi n \bar{\tau}) \) and have discarded \( \sum_n \ln 4 \), absorbing it in the cosmological constant, and have made a regularization \( \sum_{n=1}^{\infty} = \zeta(1) = -\frac{1}{12} \) by analytic continuation. Note that the correction of the Hamiltonian eq.(29) due to \( S_{\text{measure}} = \int dt L_{\text{measure}} \) can be
given by

\[
\Delta(NH) = -L_{\text{measure}} = -\frac{9}{2}\phi + \ln \tau^2 - 2 \sum_{n=1}^{\infty} \ln(\sin \pi n \tau \sin \pi n \bar{\tau})
\]

\[
= e^\phi \left( -\frac{9}{4} \frac{1}{\sqrt{V}} \ln V + \frac{1}{\sqrt{V}} \ln \tau^2 - \frac{2}{\sqrt{V}} \sum_{n=1}^{\infty} \ln(\sin \pi n \tau \sin \pi n \bar{\tau}) \right)
\]

noting that \( N = e^\phi, V = e^{2\phi} \) in this case. Thus, after setting \( N = 1 \), the modified Hamiltonian becomes, instead of eq.(29),

\[
\alpha H = (\tau^2)^2(p_1^2 + p_2^2) - \frac{1}{2} \alpha^2 \sigma^2 V - (\tau^2)^{3/2} f(\tau)V^{1/2}/\alpha
\]

\[
- \frac{9}{4} \frac{\hbar \alpha}{\sqrt{V}} \ln V/\alpha^2 + \frac{\hbar \alpha}{\sqrt{V}} \ln \tau^2 - \frac{2\hbar \alpha}{\sqrt{V}} \sum_{n=1}^{\infty} \ln(\sin \pi n \tau \sin \pi n \bar{\tau}) \quad (29')
\]

where \( \hbar \) has been inserted to indicate the \( O(\hbar) \) correction terms. Thus eqs.(30-32) are modified as

\[
\dot{V} = -\alpha \sigma V \quad (30 - a')
\]

\[
\dot{\sigma} = \frac{\alpha}{2} \frac{\sigma^2}{\alpha^2} + \frac{1}{2\alpha^2} (\tau^2)^{3/2} f(\tau)V^{-1/2} + \frac{9}{8} \hbar V^{-3/2} (2 - \ln V/\alpha^2) + \frac{\hbar}{2} V^{-3/2} \ln \tau^2
\]

\[- \hbar V^{-3/2} \sum_{n=1}^{\infty} \ln(\sin \pi n \tau \sin \pi n \bar{\tau}) \quad (30 - b')
\]

\[
\dot{\tau}^1 = \frac{2}{\alpha} (\tau^2)^2 p_1 \quad , \quad (31 - a')
\]

\[
\dot{p}_1 = \frac{1}{\alpha^2} (\tau^2)^{3/2} \frac{\partial f(\tau)}{\partial \tau^1} V^{1/2}
\]

\[
+ 4\pi \hbar V^{-1/2} \sum_{n=1}^{\infty} \frac{n \sin \pi n \tau \cos \pi n \bar{\tau}}{\cos^2 \pi n \bar{\tau} \sin^2 \pi n \tau + \sin^2 \pi n \tau \cos^2 \pi n \bar{\tau}} \quad , \quad (31 - b')
\]

\[
\dot{\tau}^2 = \frac{2}{\alpha} (\tau^2)^2 p_2 \quad , \quad (32 - a')
\]

\[
\dot{p}_2 = -\frac{2}{\alpha} \tau^2 (p_1^2 + p_2^2) + \frac{3}{2\alpha^2} (\tau^2)^{1/2} f(\tau)V^{1/2} + \frac{1}{\alpha^2} (\tau^2)^{3/2} \frac{\partial f(\tau)}{\partial \tau^2} V^{1/2}
\]

\[- \frac{\hbar}{\tau^2} V^{-1/2}
\]

\[
+ 4\pi \hbar V^{-1/2} \sum_{n=1}^{\infty} \frac{n \sin \pi n \tau \cos \pi n \bar{\tau}}{\cos^2 \pi n \bar{\tau} \sin^2 \pi n \tau + \sin^2 \pi n \tau \cos^2 \pi n \bar{\tau}} \quad . \quad (32 - b')
\]
Let us pay attention to the last term in eq.\((32 - b')\). Strictly speaking, \((\tau^1, \tau^2)\) should be regarded as \((\tau^1, \tau^2) \in D(H_+)/\sim (\simeq \mathcal{M}_{g=1})\), where \(D(H_+) = \{z \in H_+||\text{Re}z| \leq 1/2, |z| \geq 1\}\) and \(\sim\) indicates the identification \((\tau^1, \tau^2) = -(\tau^1, \tau^2)\) on the boundary of \(D(H_+)\). Then, as long as \(\tau^1 \neq 0\), this summation converges. However, when \(\tau^1 = 0\), this summation becomes divergent, like \(\sim \sum_{n=1}^{\infty} n = \zeta(-1)\). Thus, some appropriate regularization should be understood. In practice, the regularization is implemented automatically, coming from the truncation of the infinite summation in eq.\((32 - b')\).

Figures 4 - a, b, c show the modified dynamics of \((\tau^1, \tau^2)\), \((p^1, p^2)\) and \((V, \sigma)\) with the same initial conditions as those in Figures 3 - a, b, c. Eq.\((29')\) has been used as a constraint function, and the initial condition for \(p^2\) has been decided using it. The correction terms change the initial condition for \(p^2\) only slightly, since they are \(O(h)\), which is the general feature irrespective of the choice of the set of initial conditions. However, these terms change the dynamics of the system prominently. In general, the correction terms make the system more unstable, and the evolution to the catastrophe proceeds rapidly, compared to the case without the correction terms (let us call the latter case as “case A”, for brevity). It can be seen most obviously in the behavior of \((p^1, p^2)\) (Figure 4 - b, comparing with Figure 3 - b). Generally, the correction terms cause \(p^2\) to become large, faster than the case A. Accordingly, \(\tau^2\) approaches zero before \(\tau^1\) evolves significantly, compared to the case A (Figure 4 - a, comparing with Figure 3 - a). The evolution of \((V, \sigma)\), namely \(V \to 0, \sigma \to \infty\), proceeds much more rapidly compared to the case A (Figure 4 - c, comparing with Figure 3 - c). These are general features irrespective of the initial conditions.

Among the correction terms, the term \(\frac{\lambda a}{\sqrt{V}} \ln \tau^2\) in eq.\((29')\) (or the term \(-\frac{1}{2}V^{-1/2}\) in eq.\((32 - b')\)) is responsible for the acceleration of \(\tau^2\) to 0. On the other hand, the term \(-\frac{9}{8} \frac{\lambda a}{\sqrt{V}} \ln V/\alpha^2\) in eq.\((29')\) (or the term \(\frac{9}{8}hV^{-3/2}(2-\ln V/\alpha^2)\), \(>0\) for \(V < \alpha^2\) in eq.\((30 - b')\)) is responsible for the acceleration of \(V\) to 0. The term \(-\frac{2a}{\sqrt{V}} \sum_{n=1}^{\infty} \ln(\sin \pi n \tau \sin \pi n \bar{\tau})\) in eq.\((29')\) (or the terms including an infinite summation in eqs.\((30 - b')\), \((31 - b')\) and \((32 - b')\)) causes a fine oscillatory behavior.
in the trajectories (see Figures 4 – d, e, f). This term, however, does not change drastically the dynamics as a whole, as long as the regularization is implemented. These points can be confirmed by dropping the term in question artificially in the evolution equations, and comparing the resultant evolution with the original one.

In summary, the path-integral measure influences the semiclassical dynamics of the system in a non-trivial manner.

5. Conclusion

In this paper, we have investigated the semiclassical dynamics of the topological degrees of freedom, $(\tau^1, \tau^2)$, which has been seldom discussed so far. By reducing the spacetime dimension to 3, we could concentrate on the study of a finite number of topological modes and we could describe the back-reaction effect from matter to topological modes explicitly. We observed a non-trivial dynamics caused by the back-reaction. This observation implies the importance of the investigation of topological aspects for the deeper understanding of quantum gravity. Moreover, we could fix the path-integral measure for $(\tau^1, \tau^2)$ and observed its non-trivial influence on the semiclassical dynamics of $(\tau^1, \tau^2)$. This gives an explicit example indicating the significance of fixing the measure in semiclassical gravity.

ACKNOWLEDGEMENTS

The author thanks R. Balsubramanyan, T. Ghosh, S.D. Mohanty, T. Padmanabhan, V. Sahni and S. Sinha for helpful discussions. This work has been financially supported in part by the Honda Fellowship of the Japan Association for Mathematical Sciences.
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Figure Captions.

Figure 1-a: The fundamental region for a torus. A torus can be constructed either by identifying the boundary of this region, \((\xi^1, 0) \sim (\xi^1, 1)\) and \((0, \xi^2) \sim (1, \xi^2)\), or equivalently, by the identification of the whole \(\mathbb{R}^2\), \((\xi^1 + n, \xi^2 + m) \sim (\xi^1, \xi^2)\) \((n, m \in \mathbb{Z})\).

Figure 1-b: Generators of the torus, \(c_1 = (1/\sqrt{\tau^2}, 0)\) and \(c_2 = (\tau^1/\sqrt{\tau^2}, \sqrt{\tau^2})\). The 2-volume of the torus is unity.

Figure 2-a: The plot of the function \(f(\tau^1, \tau^2)\) for the range \(\tau^1 : 0 - 1\) and \(\tau^2 : 0.5 - 0.8\). The infinite summation has been truncated at -200 and 200.

Figure 2-b: The contour plot of \(f(\tau^1, \tau^2)\), with the same range and the truncation points as in Figure 2 - a. The lines indicate the values (from bottom to top) 30, 28, 26, 24, 22, 20, 15 and 5.

Figure 3-a: The trajectory of \((\tau^1, \tau^2)\) determined by eqs.(29)-(32). The infinite summation in the definition of \(f(\tau)\) has been truncated at -100 and 100. \(\alpha\) has been set to unity. The initial conditions are \(\tau^1 = 0.000, \tau^2 = 0.500, p^1 = 1.000, p^2 = 1.620, V = 1.000\) and \(\sigma = 0.000\). Points \(A, B, C, D\) and \(Z\) indicate typical points on the trajectory. \(A : (0.000, 0.500)\) at \(t = 0.000, B : (0.728, 0.888)\) at \(t = 0.676, C : (1.163, 0.765)\) at \(t = 0.996, D : (1.551, 0.165)\) at \(t = 2.040\) and \(Z : (1.557, 0.030)\) at \(t = 3.000\).

Figure 3-b: The trajectory of \((p^1, p^2)\) determined by eqs.(29)-(32). Initial conditions are the same as in Figure 3 - a. \(A : (1.000, 1.620)\) at \(t = 0.000, B : (0.949, -0.005)\) at \(t = 0.676, C : (0.949, -0.588)\) at \(t = 0.996, D : (0.832, -5.114)\) at \(t = 2.040\) and \(Z : (-2.011, -30.08)\) at \(t = 3.000\).

Figure 3-c: The trajectory of \((V, \sigma)\) determined by eqs.(29)-(32). Initial conditions are the same as in Figure 3 - a. \(A : (1.000, 0.000)\) at \(t = 0.000, B : (0.913, 0.265)\) at \(t = 0.676, C : (0.821, 0.407)\) at \(t = 0.996, D : (0.379, 1.206)\) at \(t = 2.040\) and \(Z : (0.026, 7.171)\) at \(t = 3.000\).

Figure 4-a: The trajectory of \((\tau^1, \tau^2)\) determined by eqs.(29')-(32'). The initial condi-
tions are \( \tau^1 = 0.000, \tau^2 = 0.500, p^1 = 1.000, p^2 = 1.625, V = 1.000 \) and \( \sigma = 0.000 \). The infinite summation in the definition of \( f(\tau) \) has been truncated at -100 and 100. \( \alpha \) has been set to unity. \( \lambda \) has been set to 0.001. The initial conditions are \( \tau^1 = 0.000, \tau^2 = 0.500, p^1 = 1.000, p^2 = 1.625, V = 1.000 \) and \( \sigma = 0.000 \). Points \( A', B', C' \) and \( Z' \) indicate typical points on the trajectory. \( A' : (0.000, 0.500) \) at \( t = 0.000 \), \( B' : (0.296, 0.686) \) at \( t = 0.394 \), \( C' : (0.652, 0.340) \) at \( t = 0.997 \) and \( Z' : (0.673, 0.238) \) at \( t = 1.124 \).

**Figure 4-b:** The trajectory of \( (p^1, p^2) \) determined by eqs. (29')-(32'). Initial conditions are the same as in Figure 4-a. \( A' : (1.000, 1.625) \) at \( t = 0.000 \), \( B' : (0.937, -0.002) \) at \( t = 0.394 \), \( C' : (0.980, -3.510) \) at \( t = 0.997 \) and \( Z' : (1.015, -7.524) \) at \( t = 1.124 \).

**Figure 4-c:** The trajectory of \( (V, \sigma) \) determined by eqs. (29')-(32'). Initial conditions are the same as in Figure 4-a. \( A' : (1.000, 0.000) \) at \( t = 0.000 \), \( B' : (0.871, 0.758) \) at \( t = 0.394 \), \( C' : (0.187, 8.090) \) at \( t = 0.997 \) and \( Z' : (0.001, 918.4) \) at \( t = 1.124 \).

**Figure 4-d:** Figure 4-a magnified. The portion of the trajectory from \( t = 0.000 \) (\( A' \)) to \( t = 0.162 \) has been magnified.

**Figure 4-e:** Figure 4-b magnified. The same portion as in Figure 4-d.

**Figure 4-f:** Figure 4-c magnified. The same portion as in Figure 4-d.
Figure 1 – \( a \)
tions are \( \tau^1 = 0.000, \tau^2 = 0.500, p^1 = 1.000, p^2 = 1.625, V = 1.000 \) and \( \sigma = 0.000 \). The infinite summation in the definition of \( f(\tau) \) has been truncated at \(-100\) and \(100\). \( \alpha \) has been set to unity. \( \h \) has been set to 0.001. The initial conditions are \( \tau^1 = 0.000, \tau^2 = 0.500, p^1 = 1.000, p^2 = 1.625, V = 1.000 \) and \( \sigma = 0.000 \). Points \( A', B', C' \) and \( Z' \) indicate typical points on the trajectory. \( A' : (0.000, 0.500) \) at \( t = 0.000 \), \( B' : (0.296, 0.686) \) at \( t = 0.394 \), \( C' : (0.652, 0.340) \) at \( t = 0.997 \) and \( Z' : (0.673, 0.238) \) at \( t = 1.124 \).

**Figure 4-b:** The trajectory of \((p^1, p^2)\) determined by eqs.(29')-(32'). Initial conditions are the same as in Figure 4-a. \( A' : (1.000, 1.625) \) at \( t = 0.000 \), \( B' : (0.937, -0.002) \) at \( t = 0.394 \), \( C' : (0.980, -3.510) \) at \( t = 0.997 \) and \( Z' : (1.015, -7.524) \) at \( t = 1.124 \).

**Figure 4-c:** The trajectory of \((V, \sigma)\) determined by eqs.(29')-(32'). Initial conditions are the same as in Figure 4-a. \( A' : (1.000, 0.000) \) at \( t = 0.000 \), \( B' : (0.871, 0.758) \) at \( t = 0.394 \), \( C' : (0.187, 8.090) \) at \( t = 0.997 \) and \( Z' : (0.001, 918.4) \) at \( t = 1.124 \).

**Figure 4-d:** Figure 4-a magnified. The portion of the trajectory from \( t = 0.000 \) \((A')\) to \( t = 0.162 \) has been magnified.

**Figure 4-e:** Figure 4-b magnified. The same portion as in Figure 4-d.

**Figure 4-f:** Figure 4-c magnified. The same portion as in Figure 4-d.
Figure 1 – $a$
Figure 1 - b
Figure 2 - a
Figure 3 – a
Figure 3 – c
Figure 4 – a
Figure 4 - c
Figure 4 - d