Abstract

Nagoya University, Tokyo 101, Japan

Department of Physics, Faculty of Science and Technology

Takahisa Fujita and Teruaki Segrada

SU(N) Massive Gross-Neveu Model

Bound State Spectrum

1. Introduction
denoted by $M$ and the fermion mass by $m$. On the other hand, Dashen, Haaslacher and Neveu [17] calculated the bound state spectrum employing path integral method with semiclassical approximations. There, they found that the binding energy behaves as $O(N^{-2})$ in contradiction with Schonfeld's result.

Therefore, it should be worthwhile to check in which way the binding energy behaves, $O(N^{-4})$ or $O(N^{-2})$. In this paper, we find that the mass of scalar boson (charge conjugation $C = 1$ state) in the Gross-Neveu model behaves similarly to that obtained by Dashen et al. Namely, the quantity $\frac{M}{m} - 1$ is $O(N^{-2})$. In addition, there exists a vector boson (charge conjugation $C = -1$ state). This vector boson can only be obtained when one carefully calculates the effect of $\frac{1}{N}$ since it appears like higher orders in $\frac{1}{N}$. However, one can see that this is not in the higher order of $\frac{1}{N}$, and in fact the mass of the vector boson does not depend on $N$.

Further, we discuss a connection of our calculation to the Ising model with random bonds [36]. It has recently been argued that the Ising model with random bonds is equivalent to the Gross-Neveu model with $N = 0$ [36]-[39]. Here, we do not want to examine the validity of this equivalence [40]. Instead, we make an analytic continuation of our result to the $N = 0$ case. It is found that the mass of the boson for $N = 0$ case shows a phase transition at some critical value of the coupling constant. However, its relation to the Ising model with random bonds is still not very clear here.

This paper is organized in the following way. In the next section, we briefly explain the Gross-Neveu model and review some of the results obtained so far. In section 3, we introduce the $\frac{1}{N}$ expansion method and apply it to the Gross-Neveu model. In section 4, we present our results of the calculated spectrum of scalar as well as vector bosons. In section 5, we discuss the connection of our calculated results to the Ising model with random bonds. Section 6 summarizes what we have learned and clarified from the present study.

2. SU(N) massive Gross-Neveu model

The SU(N) massive Gross-Neveu model is a two dimensional field theory which is described by the following lagrangian density,

$$\mathcal{L} = \bar{\psi}(i\gamma_{\mu}\partial^{\mu} - m_{0})\psi - g_{0}(\bar{\psi}\psi)^{2}$$

(2.1)

where we have suppressed the label which runs over the $N$ distinct fermion species, i.e.

$$\bar{\psi}\psi = \sum_{i=1}^{N} \bar{\psi}_{i}\psi_{i}.$$  

(2.2)

Here, $: :$ denotes a normal ordering.

In the SU(N) massive Gross-Neveu model, mass and coupling constant must be renormalized. For example, the self-energy diagrams as shown in Fig. 1 diverge, and thus have to be renormalized. Here, we add the two counter terms $\delta E_{a}$ and $\delta E_{b}$ corresponding to Figs. 1 (a) and (b).

Therefore, the lagrangian density can be written as

$$\mathcal{L} = \bar{\psi}(i\gamma_{\mu}\partial^{\mu} - m_{0})\psi - g_{0}(\bar{\psi}\psi)^{2} + \delta E_{a} + \delta E_{b},$$

(2.3)

where $m$ is a renormalized fermion mass. Now, the hamiltonian $H$ can be written as

$$H = \int dx : \bar{\psi}(m - i\gamma^{\lambda}\partial_{\lambda})\psi : + g_{0}\int dx (\bar{\psi}\psi)^{2} - \int dx (\delta E_{a} + \delta E_{b}).$$

(2.4)

We quantize the fermion field $\psi_{i}$ in a box $L$,

$$\psi_{i} = \frac{1}{\sqrt{L}} \sum_{n} |\frac{E_{n}}{L} u_{n} e^{(2\pi i nx/L)} + \delta_{0}^{(i)} u_{n} e^{(-2\pi i nx/L)}|.$$  

(2.5)

The Dirac spinors are defined as

$$u_{n} = \frac{1}{\sqrt{2E_{n}(E_{n} + m)}} \begin{pmatrix} E_{n} + m \\ p_{n} \end{pmatrix},$$

(2.6a)

$$v_{n} = \frac{1}{\sqrt{2E_{n}(E_{n} + m)}} \begin{pmatrix} p_{n} \\ -E_{n} \end{pmatrix}$$

(2.6b)

with

$$p_{n} = \frac{2\pi n}{L}, \quad E_{n} = \sqrt{p_{n}^{2} + m^{2}}.$$  

(2.7)
are derived from the leading order term of \( \mathcal{O} \). To get the leading order term of \( \partial \), the leading order term of \( \partial \) is also extracted from the leading order term of \( \partial \). The leading order term of \( \partial \) is given by (3.2):

\[
\partial \psi = \frac{N \psi}{\sqrt{V}} + \mathcal{O}
\]

Thus, we arrive at the leading order form of (3.2), which is valid for the leading order term of \( \partial \).

(3.3)

\[
\partial \psi + \frac{N \psi}{\sqrt{V}} = \mathcal{O}
\]

This equation is valid for all states in the massless limit. The massless limit is valid for the leading order term of \( \partial \), and the leading order term of \( \partial \) is given by (3.3).

3. 1/\( k \) expansion method

The leading order terms of \( \psi \) and \( \phi \) are expanded in terms of \( k \). The leading order terms of \( \psi \) and \( \phi \) are given by (3.4):

\[
(\bar{k} - \hat{d} \cdot \vec{k}) \psi = k \lambda \kappa \phi = \hat{d} \cdot \vec{k} \phi
\]

\[
\bar{k} \phi = \hat{d} \cdot \vec{k} \phi
\]

where \( k \) and \( \hat{d} \) are defined as (3.4).

\[
(\bar{k} - \hat{d} \cdot \vec{k}) \psi = k \lambda \kappa \phi = \hat{d} \cdot \vec{k} \phi
\]

\[
\bar{k} \phi = \hat{d} \cdot \vec{k} \phi
\]

where \( k \) is defined as the momentum of \( \psi \) and \( \phi \) and \( \hat{d} \) is the momentum of \( \phi \) and \( \psi \). The leading order terms of \( \psi \) and \( \phi \) are given by (3.4).

\[
\bar{k} \phi = \hat{d} \cdot \vec{k} \phi
\]

where \( k \) is defined as the momentum of \( \phi \) and \( \psi \) and \( \hat{d} \) is the momentum of \( \phi \) and \( \psi \). The leading order terms of \( \psi \) and \( \phi \) are given by (3.4).

\[
H \psi + \hat{d} \phi + \psi \hat{d} \phi = \frac{T}{\sqrt{V}} + \left( k \lambda \kappa \phi + \hat{d} \cdot \vec{k} \phi \right)
\]

\[
H \phi + \hat{d} \psi + \phi \hat{d} \psi = \frac{T}{\sqrt{V}} + \left( k \lambda \kappa \phi + \hat{d} \cdot \vec{k} \phi \right)
\]

where \( H \) is the Hamiltonian of the system and \( \hat{d} \) is the momentum of \( \psi \) and \( \phi \). The leading order terms of \( \psi \) and \( \phi \) are given by (3.4).

\[
H \psi + \hat{d} \phi + \psi \hat{d} \phi = \frac{T}{\sqrt{V}} + \left( k \lambda \kappa \phi + \hat{d} \cdot \vec{k} \phi \right)
\]

\[
H \phi + \hat{d} \psi + \phi \hat{d} \psi = \frac{T}{\sqrt{V}} + \left( k \lambda \kappa \phi + \hat{d} \cdot \vec{k} \phi \right)
\]

where \( H \) is the Hamiltonian of the system and \( \hat{d} \) is the momentum of \( \psi \) and \( \phi \). The leading order terms of \( \psi \) and \( \phi \) are given by (3.4).

The creation and annihilation operators obey the anti-commutation relations.
in the SU(N) massive Gross-Neveu model cancels out. The term proportional to \(1/K\) appears as \((m/K)^2\) terms of \(F\) and \(G\) in the interaction hamiltonian.

Therefore, the equation for the invariant mass squared \(M^2\) becomes

\[
M^2 = \int_0^1 dx \left( \frac{1}{x} + \frac{1}{1-x} \right) |f(x)|^2 - \frac{g_0}{2\pi} \int_0^1 dy f(y) \left[ \left( \frac{1}{y} + \frac{1}{1-y} \right) \left( \frac{1}{1-x} + \frac{1}{x} \right) + \mathcal{N} \left( \frac{1}{y} - \frac{1}{1-y} \right) \left( \frac{1}{x} - \frac{1}{1-x} \right) \right]
\]

where \(M_R\) is defined as \(M_R = \frac{M}{m}\).

Note that \(f(x)\) satisfies the normalization condition \(\int_0^1 dx |f(x)|^2 = 1\).

Equivalently, we can write an integral equation by making variations with respect to \(f(x)\),

\[
M_R^2 f(x) = \left( \frac{1}{x} + \frac{1}{1-x} \right) f(x) - \frac{g_0}{2\pi} \int_0^1 dy f(y) \left[ \left( \frac{1}{y} + \frac{1}{1-y} \right) \left( \frac{1}{1-x} + \frac{1}{x} \right) + \mathcal{N} \left( \frac{1}{y} - \frac{1}{1-y} \right) \left( \frac{1}{x} - \frac{1}{1-x} \right) \right]
\]

Here, if we make a large \(N\) limit and neglect the first term of the interaction part, then this equation becomes identical to that obtained by Thies and Ohta. However, as we will see later, the first part of the interaction term plays an important role for the negative charge conjugation state.

Eq. (3.6) can be solved analytically since it is a separable type interaction. Here, we define the following two quantities \(A\) and \(B\),

\[
A = \int_0^1 dx f(x), \quad (3.7a)
\]

\[
B = \int_1^0 dx \frac{f(x)}{x}, \quad (3.7b)
\]

where we have introduced the cutoff \(\epsilon\).

Using \(A\) and \(B\), we can solve Eq. (3.6) for \(f(x)\). The solutions depend on the choice of the charge conjugation since it depends on the symmetry of the wave function \(f(x)\). In what follows, we treat the positive and negative charge conjugation states separately.

(i) Charge conjugation \(C = 1\) state

For positive charge conjugation state of \(C = 1\), which we may call a scalar boson, \(f(x)\) satisfies \(f(x) = -f(1-x)\). In this case, using \(A\) and \(B\), we can solve Eq. (3.6) for \(f(x)\) and obtain

\[
f(x) = \frac{\frac{1}{1-x}}{\left( \frac{g_0}{2\pi} \right) \left( \frac{1}{x} + \frac{1}{1-x} \right)} \left( \frac{1}{x} + \frac{1}{1-x} \right) [B(1-x) - A] - B(1-x)\]

(3.8)

We now put this \(f(x)\) into Eqs. (3.7a) and (3.7b). In this case, however, the constant \(A\) becomes zero due to the symmetry. Thus, we obtain

\[
B = \frac{g_0}{2\pi} \left( 2N - 1 \right) \left( \frac{1}{x} + \frac{1}{1-x} \right) \left( \frac{1}{x} + \frac{1}{1-x} \right) - B(1-x)
\]

(3.9)

This equation can be rewritten since \(B\) is nonzero,

\[
\frac{g_0}{2\pi} \left( 2N - 1 \right) \ln \epsilon = \left( \frac{1}{x} + \frac{1}{1-x} \right) \left( \frac{1}{x} + \frac{1}{1-x} \right)
\]

(3.10)

Here, we define a renormalized coupling constant \(g\) by

\[
\frac{\phi}{\pi} = \frac{1}{g_0} + \frac{1}{2\pi} \left( 2N - 1 \right) \ln \epsilon.
\]

(3.11)

Now, we introduce \(\alpha\) as \(M_R = 2 \cos \alpha\).

In this case, the eigenvalue equation (3.10) becomes

\[
\frac{g_0}{\pi} = \frac{1}{2N-1} \cot \alpha.
\]

(3.12)

For large \(N\) limit, \(\alpha\) becomes

\[
\alpha \approx \frac{1}{gN^2}.
\]

(3.13)

Putting this into \(\Delta M_R = 2(1 - \cos \alpha)\), we obtain

\[
\Delta M_R \approx \frac{1}{gN^2}.
\]

(3.14)

Therefore, the behavior of the binding energy at large \(N\) agrees with Dashen's result [17].

We also want to comment on the running coupling constant \(g\) defined by (3.11). This can be written as

\[
g = \frac{g_0}{1 - (2N - 1) \frac{g_0}{\pi} \ln \frac{1}{\epsilon}}.
\]

(3.15)
The expression is not clear due to the lack of visible characters or symbols. It appears to be a mathematical or scientific equation, possibly related to a physics or engineering context. Without more context or a clearer image, it is challenging to determine the exact nature of the expression.
5. Summary

We have presented the bound state spectrum of the SU(N) massive Gross-Neveu model. The spectrum is obtained analytically. This is done with the help of the 1/K expansion method. We find that the binding energy of the charge conjugation of C = 1 state behaves like \( O(N^{-2}) \) as suggested by the semiclassical calculation, and not like \( O(N^{-4}) \) as suggested by Schonfeld.

For charge conjugation of \( C = -1 \) state, the eigenvalue equation is found to be equivalent to the U(1) massive Thirring model. This is independent of the number of flavour \( N \).

Further, we have discussed the correspondence between the \( N = 0 \) Gross-Neveu model and the Ising model with random bonds. There, we found that the \( N = 0 \) Gross-Neveu model predicts a phase transition at some critical value of the coupling constant.

However, we do not know yet how we can evaluate a critical exponent from our bound state energy result.

Acknowledgements

We would like to thank C. Itoi, H. Mukaida and A. Ogura for useful discussions.
References
Fig. 1. The Feynman diagrams corresponding to the mass renormalization.

Fig. 2. The Feynman diagrams corresponding to the coupling constant renormalization.

Fig. 3. The mass spectrum of the charge zero sector for the SU(N) massive Gross-Neveu model as the function of the coupling constant $\frac{g}{\Lambda}$. The dashed lines are calculated by Eq.(3.12). The solid line is calculated by Eq.(3.22).