**The Internal Motion of Superparticle and the Equation of Motion
of Supersymmetric Partners**

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The internal motion of the superparticle and the equation of motion of its supersymmetric partners are discussed based upon the conservation laws resulting from the invariances possessed by the Lagrangian of the superparticle, one of which is the Lorentz invariance and the other is a new one pointed out here. It is shown that this leads to the existence of a new quantum number which is related to the spin but gives information independent of it.

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The superparticle has been discussed by many authors as the simplest model of supersymmetry [1], [2]. It is natural to try to interpret it as a particle with internal structure identifying its bosonic and fermionic coordinates with variables describing respectively the external and the internal motion, and several authors have discussed the relation between the superparticle and the spinning particle [3]. In this paper we would like to pursue the same problem furthermore based upon the conservation laws resulting from the invariances possessed by the Lagrangian of the superparticle.

As usual, we shall take the Lagrangian of the superparticle as

$$L = p_\mu \{ \dot{x}^\mu - i(\xi^* \sigma^\mu \dot{\xi} - \dot{\xi}^* \sigma^\mu \xi) \} - \frac{1}{2} e (p_\mu p^\mu + m^2), \quad (1)$$

where ξ is a two-component Grassmann spinor and e is a Lagrangian multiplier. In this paper we assume $m \neq 0$.

From this Lagrangian momenta P_μ , π^α and $\pi^{*\dot{\alpha}}$ canonically conjugate to x^μ , ξ_α and ξ_α^* respectively are given by^{*)}

$$\left. \begin{aligned} P_\mu &\equiv \partial L / \partial \dot{x}^\mu = p_\mu, \\ \pi^\alpha &\equiv \partial L / \partial \dot{\xi}_\alpha = -i p_\mu (\xi^* \sigma^\mu)^\alpha, \\ \pi^{*\dot{\alpha}} &\equiv \partial L / \partial \dot{\xi}_\alpha^* = -i p_\mu (\sigma^\mu \xi)^{\dot{\alpha}}, \end{aligned} \right\} \quad (2)$$

and the Euler equations are written as

$$\dot{p}_\mu = 0, \quad (3)$$

$$p_\mu p^\mu + m^2 = 0, \quad (4)$$

$$(p_\mu \sigma^\mu) \dot{\xi} = 0. \quad (5)$$

The Lagrangian (1) is clearly invariant under the Lorentz transformation, and the resulting conservation law is that of the four-dimensional angular momentum $M_{\mu\nu}$ given by [1]

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad (6)$$

where $S_{\mu\nu}$ is the “spin” angular momentum coming from the internal motion described by ξ which is given by

$$S_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} p^\rho (\xi^* \sigma^\sigma \xi), \quad (7)$$

where $\varepsilon_{\mu\nu\rho\sigma} = -1$ or $+1$ according to whether μ, ν, ρ and σ are even or odd permutation of 1, 2, 3 and 0.

In addition to this, it should be noted that the Lagrangian (1) is invariant under the transformation

$$\xi \rightarrow a\xi - b\Sigma' i\sigma_2 \xi^* \quad (8)$$

^{*)} Unless otherwise noted, we use always right derivative for Grassmann variables.

as can be shown easily noticing that ξ is a Grassmann spinor,^{*)} where a and b are arbitrary constants satisfying $|a|^2 + |b|^2 = 1$, and Σ' is given by $\sigma_2 \Sigma^T \sigma_2$ with $\Sigma = p_\mu \sigma^\mu / ip$ ($p \equiv \sqrt{p_\mu p^\mu}$), satisfying $\Sigma \Sigma' = \Sigma' \Sigma = 1$. For infinitesimal transformation it is more convenient to parametrize a and b as $a = 1 - (i/2)d\alpha_3$ and $b = -(i/2)d\alpha_1 + (1/2)d\alpha_2$, and this invariance leads to the conservation law of T_r ($r = 1, 2, 3$) given by

$$\left. \begin{aligned} T_1 &= m(\xi i \sigma_2 \xi + c.c.)/2, \\ T_2 &= -im(\xi i \sigma_2 \xi - c.c.)/2, \\ T_3 &= m(\xi^* \Sigma \xi - \xi \Sigma^T \xi^*)/2. \end{aligned} \right\} \quad (9)$$

Let us now analyze the internal motion based upon these conservation laws. For this purpose it becomes necessary to quantize the internal motion, and we must write down the Hamiltonian formalism. The basic variables satisfy the Poisson bracket relations $(\xi_\alpha, \xi_\beta)_{PB} = (\xi_\alpha^*, \xi_\beta^*)_{PB} = 0$, $(\pi^\alpha, \pi^\beta)_{PB} = (\pi^{\alpha\dot{}}, \pi^{\beta\dot{}})_{PB} = 0$, $(\xi_\alpha, \pi^\beta)_{PB} = \delta_\alpha^\beta$ and $(\xi_\alpha^*, \pi^{\beta\dot{}})_{PB} = \delta_\alpha^{\beta\dot{}}$, where in case q_i and p_i are Grassmann numbers the Poisson bracket is defined by [1]

$$(F, G)_{PB} = (\partial F / \partial q_i)(\partial / \partial p_i)G + (\partial F / \partial p_i)(\partial / \partial q_i)G, \quad (10)$$

with $(\partial F / \partial q_i)$ and $(\partial / \partial p_i)G$ denoting respectively the right and the left derivative. These variables are, however, not dynamically independent. In fact they satisfy

$$\left. \begin{aligned} \eta^\alpha &\equiv \pi^\alpha - im \xi_\rho^* \Sigma^{\rho\alpha} = 0 \\ \eta^{\dot{\alpha}} &\equiv \pi^{\dot{\alpha}} - im \Sigma^{\dot{\alpha}\rho} \xi_\rho = 0 \end{aligned} \right\} \quad (11)$$

as seen from Eqs. (2), where it is understood that p_μ in Σ is constant because of Eq. (3) and satisfies the mass shell condition (4) and further that the time ordering parameter has already been fixed. Therefore it is necessary to treat the system as a constrained one.

^{*)} Substituting $\xi \rightarrow a\xi - b\Sigma' i \sigma_2 \xi^*$ and $\dot{\xi} \rightarrow a\dot{\xi} - b(\dot{\Sigma}' i \sigma_2 \xi^* + \Sigma' i \sigma_2 \dot{\xi}^*)$ together with their complex conjugates, L is transformed as

$$L \rightarrow L + p(ab^* \xi i \sigma_2 \Sigma' \dot{\xi} - c.c.).$$

Substituting explicit forms for Σ and Σ' , we find

$$\xi i \sigma_2 \Sigma' \dot{\xi} = \xi i \sigma_2 \{(\mathbf{p}\boldsymbol{\sigma} + p^0)(\dot{\mathbf{p}}\boldsymbol{\sigma} - \dot{p}^0) - p_\mu \dot{p}^\mu\} \xi / p^2.$$

Here terms proportional to σ_i in the curled bracket vanish since for Grassmann spinor $\xi i \sigma_2 \sigma_i \xi = 0$. Therefore the first term reduces to $p_\mu \dot{p}^\mu$, which cancels with the second term. So $\xi i \sigma_2 \Sigma' \dot{\xi} = 0$, and L is invariant under the transformation (8).

^{**) In our case which is massive with $D = 4$ the Lagrangian (1) is invariant under the gauge transformation given by [4]}

$$\begin{aligned} \delta x^\mu &= ap^\mu + \delta \bar{\theta} \gamma^\mu \theta, & \delta \theta &= \gamma^\mu p_\mu \kappa, \\ \delta a &= 4\bar{\theta} \kappa, & \delta p^\mu &= \delta e = 0, \end{aligned}$$

where θ is a Majorana spinor given by $\begin{pmatrix} \xi \\ i \sigma_2 \xi^* \end{pmatrix}$ and κ is an arbitrary Majorana spinor. At first sight it seems that this variation of θ contains a part of (8) as its special case and so (8) does not necessarily represent a new invariance. But what is described by this transformation is an arbitrariness in the fermionic coordinate *linked* with the arbitrariness in the choice of the time ordering parameter, which has nothing to do with (8). The significance of the transformation (8) is that it represents a symmetry of the system which remains *after* the gauge fixing, which is done by fixing the time ordering parameter.

η^α and $\eta^{\dot{\alpha}}$ satisfy

$$\left. \begin{aligned} (\eta^\alpha, \eta^\beta)_{PB} &= (\eta^{\dot{\alpha}}, \eta^{\dot{\beta}})_{PB} = 0, \\ (\eta^\alpha, \eta^{\dot{\beta}})_{PB} &= -2im\Sigma^{\dot{\beta}\alpha}. \end{aligned} \right\} \quad (12)$$

Therefore C defined by $C^{ab} = (\eta^a, \eta^b)_{PB}$, where $a = (\alpha, \dot{\alpha})$ and $b = (\beta, \dot{\beta})$, has non-vanishing determinant, and C^{-1} is given by

$$C^{-1} = (i/2m) \begin{pmatrix} 0 & \Sigma'_{\alpha\dot{\beta}} \\ \Sigma'^T_{\dot{\alpha}\beta} & 0 \end{pmatrix}. \quad (13)$$

This means that Eqs. (11) are the second class constraints, and we must use the Dirac bracket defined by

$$(F, G)_{DB} = (F, G)_{PB} - (F, \eta^a)_{PB} C_{ab}^{-1} (\eta^b, G)_{PB} \quad (14)$$

instead of the Poisson bracket [5]. From Eqs. (11) we have $(\xi_\alpha, \eta^\beta)_{PB} = \delta_\alpha^\beta$, $(\xi_\alpha, \eta^{\dot{\beta}})_{PB} = 0$, $(\xi_\alpha^*, \eta^\beta)_{PB} = 0$ and $(\xi_\alpha^*, \eta^{\dot{\beta}})_{PB} = \delta_\alpha^{\dot{\beta}}$. Thus relevant Dirac brackets are given by

$$\left. \begin{aligned} (\xi_\alpha, \xi_\beta)_{DB} &= 0, \quad (\xi_\alpha^*, \xi_\beta^*)_{DB} = 0, \\ (\xi_\alpha \xi_\beta^*)_{DB} &= -\frac{i}{2m} \Sigma'_{\alpha\dot{\beta}}. \end{aligned} \right\} \quad (15)$$

We shall assume, according to the general procedure given by Dirac, that in dynamical systems described by Grassmann numbers with the second class constraint the quantization should be done as a Fermi system and the quantum mechanical anti-commutation relation between two observables A and B is given by $\{A, B\} = i(A, B)_{DB}$, where $\{A, B\} \equiv AB + BA$. Then we have from Eqs. (15)

$$\left. \begin{aligned} \{\xi_\alpha, \xi_\beta\} &= 0, \quad \{\xi_\alpha^*, \xi_\beta^*\} = 0, \\ \{\xi_\alpha, \xi_\beta^*\} &= \frac{1}{2m} \Sigma'_{\alpha\dot{\beta}}. \end{aligned} \right\} \quad (16)$$

Therefore a_α and $\bar{a}^\alpha$ defined by

$$a_\alpha = \sqrt{2m} \xi_\alpha, \quad \bar{a}^\alpha = \sqrt{2m} \Sigma^{T\alpha\dot{\rho}} \xi_{\dot{\rho}}^* \quad (17)$$

satisfy

$$\left. \begin{aligned} \{a_\alpha, a_\beta\} &= \{\bar{a}^\alpha, \bar{a}^\beta\} = 0, \\ \{a_\alpha, \bar{a}^\beta\} &= \delta_\alpha^\beta. \end{aligned} \right\} \quad (18)$$

Thus a_α and $\bar{a}^\alpha$ are annihilation and creation operators, and the vacuum state $|0\rangle$ and the occupation number operator n_α having eigenvalue 0 and 1 are defined by

$$a_\alpha |0\rangle = 0, \quad (\alpha = 1, 2), \quad (19)$$

$$n_\alpha = \bar{a}^\alpha a_\alpha, \quad (\alpha : \text{not summed}). \quad (20)$$

In constructing the internal wave functions we shall work in the rest frame of the center of mass defined by $p^\mu = (0, 0, 0, m)$ (C.M. frame). In this frame $S_{\mu\nu}$ reduces to

$S_i(\equiv S_{jk}) = m(\xi^* \sigma_i \xi)(i, j, k = 1, 2 \text{ or } 3 \text{ and are cyclic})$ and $S_{i0} = 0$, and substituting Eqs. (17) S_i can be written as

$$S_i = \frac{1}{2}(\bar{a} \sigma_i a). \quad (21)$$

On the other hand we have

$$\left. \begin{aligned} T_1 &= \frac{1}{4}(ai\sigma_2 a + h.c.), \\ T_2 &= -\frac{i}{4}(ai\sigma_2 a - h.c.), \\ T_3 &= \frac{1}{2}(\bar{a}a - 1). \end{aligned} \right\} \quad (22)$$

Then using the anti-commutation relations (18) it can be checked easily that they satisfy

$$\left. \begin{aligned} [S_i, S_j] &= iS_k, \\ [T_r, T_s] &= -iT_t, \\ [S_i, T_r] &= 0, \end{aligned} \right\} \quad (23)$$

where $[]$ means the quantum mechanical commutator: $[A, B] \equiv AB - BA$. It is of interest to note that Eqs.(23) correspond to just the Poisson bracket relations to be satisfied by the laboratory and the body frame component of the angular momentum of a non-relativistic rigid sphere [6]. In the following we shall call $\mathbf{T}$ superspin.

From Eqs. (23) we find $[S_i, \mathbf{S}^2] = [T_r, \mathbf{T}^2] = 0$, where $\mathbf{S}^2$ and $\mathbf{T}^2$ are given respectively by ΣS_i^2 and ΣT_r^2 . Therefore $\mathbf{S}^2$, $\mathbf{T}^2$, S_3 and T_3 form a set of mutually commuting observables. They can be expressed in terms of number operators as follows:

$$\mathbf{S}^2 = \frac{1}{2}(n_1 + n_2) \left\{ \frac{1}{2}(n_1 + n_2) + 1 \right\} - 2n_1 n_2, \quad (24)$$

$$\mathbf{T}^2 = \frac{1}{2}(n_1 + n_2) \left\{ \frac{1}{2}(n_1 + n_2) - 2 \right\} + n_1 n_2 + \frac{3}{4}, \quad (25)$$

$$S_3 = \frac{1}{2}(n_1 - n_2), \quad (26)$$

$$T_3 = \frac{1}{2}(n_1 + n_2 - 1). \quad (27)$$

Thus the eigenfunction of the internal motion can be constructed by operating the creation operator $\bar{a}^\alpha$ onto the vacuum state $|0\rangle$, and writing the eigenvalue of $\mathbf{S}^2$ and $\mathbf{T}^2$ as $S(S+1)$ and $T(T+1)$ respectively we have result as shown on Table I. Here an important question is whether $\mathbf{S}$ and $\mathbf{T}$ are dynamically independent or not. First we note that under the constraints (11) the degree of freedom of the internal motion is two. Therefore the eigenfunctions are specified by giving the eigenvalue of, for example, n_1 and n_2 , or equivalently of $\mathbf{S}^2$ and S_3 . The eigenfunctions specified by the eigenvalue of n_1 and n_2 are non-degenerate. It should be noted, however, that, since $\mathbf{S}^2$ is quadratic in n_1 and n_2 , the correspondence between (n_1, n_2) and $(\mathbf{S}^2, S_3)$ is not unique, and the eigenvalue of $\mathbf{S}^2$ and S_3 are not enough to specify the eigenfunctions uniquely, as seen in the sector with $S = 0$ in Table I. Therefore additional information is necessary. This is given by

T_3 . In this sense $\mathbf{S}$ and $\mathbf{T}$ may be said to be dynamically independent, although they are related on the other side by equation like $\mathbf{S}^2 - \mathbf{T}^2 = -3(n_1 - \frac{1}{2})(n_2 - \frac{1}{2}) = \pm 3/4$. The eigenfunctions in an arbitrary frame can be obtained from the listed ones through Lorentz transformation assuming that $|0\rangle$ is invariant under the Lorentz transformation.

state	configuration	n_1	n_2	S	S_3	T	T_3
w_1	$ 0\rangle$	0	0	0	0	1/2	-1/2
v^1	$\bar{a}^1 0\rangle$	1	0	1/2	1/2	0	0
v^2	$\bar{a}^2 0\rangle$	0	1	1/2	-1/2	0	0
w_2	$\frac{1}{2}(\bar{a}i\sigma_2\bar{a}) 0\rangle$	1	1	0	0	1/2	1/2

Table I

An unusual feature of this result is the fact that the vacuum state carries quantum number $T = 1/2$ and $T_3 = -1/2$, which is the consequence of an additional constant term -1 in T_3 in Eqs. (22), which is essential for the closure of the algebra.

Among the members listed in Table I, v^α and w_a (α and $a=1,2$) are supersymmetric partners. They have spin 1/2 and 0 respectively. Contrary to this, however, their superspin is 0 and 1/2. Therefore if we identify v^α and w_a with the ordinary particle and its supersymmetric partner (s -particle) respectively, the s -particle is characterized by non-vanishing superspin.

Now let us consider their equation of motion. First we shall discuss the spin 1/2 multiplet. The key point in this discussion is the inversion. We assume that under the inversion ξ transforms as

$$\xi \rightarrow i\sigma_2\xi^*, \quad (28)$$

which leaves the Lagrangian (1) invariant. Therefore the left-hand side of Eq. (19) is transformed as $a_\alpha|0\rangle = \sqrt{2m}\xi_\alpha|0\rangle \rightarrow \sqrt{2m}(i\sigma_2\xi^*)^\dot{\alpha}|0\rangle' = \Sigma^{\dot{\alpha}\rho}(i\sigma_2\bar{a})_\rho|0\rangle' = 0$, which gives

$$\bar{a}^\alpha|0\rangle' = 0, \quad (29)$$

with the explicit solution^{*)}

$$|0\rangle' = -\frac{1}{2}(\bar{a}i\sigma_2\bar{a})|0\rangle. \quad (30)$$

Therefore under the inversion v^α transforms as

$$\begin{aligned} v^\alpha &\equiv \bar{a}^\alpha|0\rangle = \sqrt{2m}\Sigma^{T\alpha\dot{\rho}}\xi_\dot{\rho}^*|0\rangle \\ &\rightarrow \sqrt{2m}\Sigma_{\dot{\alpha}\rho}^{'T}(i\sigma_2\xi)^\rho|0\rangle' = \Sigma_{\dot{\alpha}\rho}^{'T}(i\sigma_2a)^\rho|0\rangle' \equiv u_{\dot{\alpha}}^*. \end{aligned} \quad (31)$$

^{*)} The transformation (8) can be expressed as a three-dimensional rotation generated by T_r given by Eqs. (22). In terms of this, the inversion law (28) is given, in C. M. frame, by a rotation through $(-\pi)$ around the second axis. The phase factor in Eq. (30) has been fixed as the result of this rotation.

This means that to include the inversion it is necessary to include both of v^α and u_α^* on the same footing. Substituting Eq. (30) for $|0\rangle'$ and using the anti-commutation relations (18), we find that u_α^* and v^α are related by

$$u_\alpha^* = \Sigma_{\dot{\alpha}\rho}^{\prime T} v^\rho. \quad (32)$$

Here let us consider the total wave function of the superparticle $\Phi(x^\mu, \xi)$. Confining our attention to its single Fourier component $\Phi(p_\mu, \xi)$ for simplicity, this can be expanded into the eigenfunctions of the internal motion $U_a(p_\mu, \xi)$ as $\Sigma \Phi_a(p_\mu) U_a(p_\mu, \xi)$, where a is an abbreviation for (S, S_3, T, T_3) . Then Φ_a describes a particle with definite spin, $\dots$ etc. specified by a . U_a is expected to satisfy the ortho-normality condition of the form $\int U_a^* U_{a'} d\xi = \delta_{aa'}$. Therefore Φ_a should be given by expression like $\int U_a^* \Phi d\xi$. This means that the transformation property of Φ_a is governed by U_a^* and so if we anticipate the application to particle physics it is more convenient, in discussing the transformation property of the internal wave functions, to consider their complex conjugate rather than considering the original ones. In view of this we shall consider u_α and $v^{*\dot{\alpha}}$ instead of u_α^* and v^α , and combine them into the form

$$\psi = \begin{pmatrix} u_\alpha \\ v^{*\dot{\alpha}} \end{pmatrix}. \quad (33)$$

Then from Eq. (32) we see immediately that u_α and $v^{*\dot{\alpha}}$ satisfy the equation $v^{*\dot{\alpha}} = \Sigma^{\dot{\alpha}\rho} u_\rho$, which, under the mass shell condition (4), is equivalent to

$$(i\gamma^\mu p_\mu + m)\psi = 0. \quad (34)$$

The four-component spinor ψ defined by Eq. (33) transforms as the Dirac spinor under the Lorentz transformation and the space inversion, and Eq. (34) is nothing but the Dirac equation. Therefore the spin 1/2 multiplet behaves exactly as the Dirac particle.

Next we shall discuss w_a . This has spin 0 and transforms as scalar under the Lorentz transformation. At the same time it has superspin 1/2 and transforms as spinor under the rotation generated by T_r :

$$\varphi \rightarrow (1 + \frac{i}{2} d\alpha_r \tau_r) \varphi, \quad (35)$$

where φ is given by

$$\varphi = \begin{pmatrix} w_1^* \\ w_2^* \end{pmatrix}. \quad (36)$$

and $\tau_r (r = 1, 2, 3)$ is the Pauli matrix in a two-dimensional space spanned by w_1^* and w_2^* . Therefore we can ascribe a new quantum number $\boldsymbol{\tau}$ to φ , which is related to $\boldsymbol{T}$ by $\boldsymbol{T} = -\boldsymbol{\tau}/2$.

The transformation (35) is a global transformation, and it becomes necessary to introduce a gauge field $A_\mu^r (r = 1, 2, 3)$ to extend this to a local one. Therefore, including

this gauge field, the equation of motion for φ should be of the form

$$\{\eta^{\mu\nu}(\partial_\mu - \frac{i}{2}\boldsymbol{\tau} \cdot \mathbf{A}_\mu)(\partial_\nu - \frac{i}{2}\boldsymbol{\tau} \cdot \mathbf{A}_\nu) - m^2\}\varphi = 0, \quad (37)$$

where it is understood that φ here is the Fourier transform of that given by Eq. (36). Therefore the superspin $\mathbf{T}$ can be observed through A_μ^r and is conserved if the total Lagrangian including the interaction term is invariant under the transformation (8). Thus, for example, the production of the s -particle must be in pairs so that the total $\mathbf{T}$ vanishes, or the decay of a single s -particle into ordinary particles is forbidden, and it is expected that the superspin should play an important role in the physics of s -particles, just as the ordinary spin does in the physics of ordinary particles.

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