An $SO(2)$ Gauge Description of the Kepler Problem*

Michael Trunk

Universität Freiburg
Fakultät für Physik
Hermann-Herder-Str. 3
D-79104 Freiburg
Germany

Abstract

Starting from the fact that not all the unitary irreducible representations of the symmetry algebra which accounts for the “accidental” degeneracy of the hydrogen energy levels do occur as energy eigenspaces, an $SO(2)$ gauge description of the Kepler problem is given. This extended system is used to test an algebraic procedure for the quantization and reduction of gauge systems, which is entirely based upon observable quantities.

*This paper is based upon the diploma thesis [1].

1 Introduction

As is well known, the energy levels of a particle in a $1/r$ -potential are degenerate not only in the magnetic quantum number m but also in the angular momentum quantum number l . The degeneracy in m is a consequence of the obvious $SO(3)$ -symmetry of the Kepler Hamiltonian H , whereas the degeneracy in l reflects the existence of a larger symmetry algebra for the Kepler problem (KP) which is generated by the angular momentum vector $\mathbf{L}$, the Lenz-Runge vector $\tilde{\mathbf{M}}$, and H . In the eigenspaces of H with energy $E < 0$, $E = 0$ or $E > 0$ the symmetry algebra can be reduced to the Lie algebras $so(4)$, $e(3)$ and $so(1,3)$ respectively. The corresponding eigenspaces carry irreducible representations of the groups $SO(4)$, $E(3)$ and $SO(1,3)$ [2, 3, 4]. However, not all the unitary irreducible representations of the three groups do occur as energy eigenspaces. The reason for this is that there exist relations between the generators of the symmetry algebra, which select the physically relevant representations by imposing constraints on the allowed eigenvalues of the Casimir operators. The fact that not all the representations of the symmetry algebra are physically realized is a typical feature of gauge systems if we look upon them as follows.

A gauge system is characterized by a constraint algebra and an algebra of observables, where the observables are defined by their commutation relations with the constraints. The main objective of the quantization of the system consists in constructing the algebra of observables and its unitary irreducible representations on a Hilbert space $\mathcal{H}$. However, the presence of the constraints also implies that not all the representations can be physically relevant. The selection of the physically relevant representations is effected by means of the constraints through algebraic dependences. As this selection has observable consequences, it should be achieved by the observable content of the constraint algebra. Defining the observables to strongly commute with the constraints, the observable content of the constraint algebra is expressed by its Casimir operators. Again by the definition of the observables, the Casimirs of the constraint algebra are also Casimirs of the algebra of observables. Now, the “vanishing” of the constraints implies the projection onto the zero eigenspaces of the Casimirs of the constraint algebra and of the corresponding Casimirs of the algebra of observables, thereby selecting the physically relevant representations of the algebra of observables. By this procedure one can construct the physical Hilbert space of the system, which is spanned by the physically relevant representations, in a purely algebraic manner that is entirely based on observable quantities.

The above considerations apply as well to the observable part of the symmetry algebra of the system, because the Casimirs of the constraint algebra are also Casimirs of the observable part of the symmetry algebra. The symmetry alge-

bra and its representations, on the other hand, are proven tools for constructing and structuring the physical Hilbert space of the system. Therefore, the physical Hilbert space can also be obtained by determining the physically relevant representations of the (full) observable symmetry algebra, which are often well studied and readily available in the mathematical literature.

As to the KP, this point of view suggests to describe it as the reduced form of a higher-dimensional gauge system. The observable part of the symmetry algebra of this extended system should be isomorphic to the symmetry algebra of the KP, but the relations that fix the eigenvalues of its Casimir operators should be modified by the Casimirs of the constraint algebra. Due to these modified relations the Casimir operators of the symmetry algebra can take more general values in the extended system than in the original one and all the unitary irreducible representations of the symmetry algebra within a certain class (see below) will occur as energy eigenspaces. The reduction can, then, be performed by selecting the physically relevant representations.

In the following we will carry out this program by application of the so-called KS-transformation, giving rise to an $SO(2)$ gauge description of the KP with the desired features. It will be shown that the algebraic reduction of the extended quantum system yields the well known quantum theory of the hydrogen atom.

2 The symmetry algebra of the Kepler problem

The Hamiltonian for the KP

$$H = \frac{1}{2} \mathbf{p}^2 - \frac{k}{r} \quad , \quad k > 0 \quad (1)$$

($r = \sqrt{\mathbf{x}^2}$) admits six constants of motion, the components of the angular momentum vector $\mathbf{L}$ and the Lenz-Runge vector $\tilde{\mathbf{M}}$

$$\mathbf{L} = \mathbf{x} \times \mathbf{p} \quad \tilde{\mathbf{M}} = \frac{1}{2} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - k \frac{\mathbf{x}}{r}. \quad (2)$$

Together with H they generate an algebra $\mathcal{A}$ which is characterized by the commutation relations

$$[H, L_i] = 0 = [H, \tilde{M}_i] \quad [L_i, L_j] = i \varepsilon_{ijk} L_k \quad (3)$$

$$[L_i, \tilde{M}_j] = i \varepsilon_{ijk} \tilde{M}_k \quad [\tilde{M}_i, \tilde{M}_j] = -2H i \varepsilon_{ijk} L_k. \quad (4)$$

Up to powers of H , this algebra possesses two Casimir elements

$$\tilde{C}_1 = \tilde{\mathbf{M}}^2 - 2H \mathbf{L}^2 \quad \tilde{C}_2 = \mathbf{L} \cdot \tilde{\mathbf{M}} = \tilde{\mathbf{M}} \cdot \mathbf{L}. \quad (5)$$

In the realization of $\mathcal{A}$ by the dynamical quantities (1) and (2) of the KP the generators are not independent. There exist two relations between them which, together with the energy, completely fix the eigenvalues of the Casimirs*

$$\tilde{C}_1 = 2H + k^2 \quad \tilde{C}_2 = 0. \quad (6)$$

As the algebra $\mathcal{A}$ represents the full symmetry of the KP, the energy eigenspaces carry irreducible representations of $\mathcal{A}$, but only those which are determined by these relations.

3 The KS–Kepler problem

An $SO(2)$ gauge description of the KP can be obtained by application of the KS-transformation (KS) [5, 6]. The KS is a “canonical” transformation

$$\kappa : T^*(\mathbf{R}^4 \setminus \{0\}) \longrightarrow T^*(\mathbf{R}^3 \setminus \{0\}) \quad (u, v) \longmapsto (\mathbf{x}, \mathbf{p}). \quad (7)$$

It is explicitly given by

$$\mathbf{x} = \Lambda(u) u \quad \mathbf{p} = \frac{1}{2u^2} \Lambda(u) v \quad (8)$$

$$\Lambda(u) = \begin{pmatrix} u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_1 & -u_2 & -u_3 & u_4 \end{pmatrix}. \quad (9)$$

After introducing Euler angles in u -space

$$\begin{aligned} u_1 &= |u| \cos \frac{\varphi + \psi}{2} \cos \frac{\vartheta}{2} & u_2 &= |u| \cos \frac{\varphi - \psi}{2} \sin \frac{\vartheta}{2} \\ u_3 &= |u| \sin \frac{\varphi - \psi}{2} \sin \frac{\vartheta}{2} & u_4 &= |u| \sin \frac{\varphi + \psi}{2} \cos \frac{\vartheta}{2} \end{aligned} \quad (10)$$

$$0 \leq \varphi < 2\pi \quad 0 \leq \vartheta < \pi \quad 0 \leq \psi < 4\pi$$

and using the identity $r = \sqrt{\mathbf{x}^2} = u^2$, the transformation $\mathbf{x} = \mathbf{x}(u)$ becomes

$$x_1 = r \sin \vartheta \cos \varphi \quad x_2 = r \sin \vartheta \sin \varphi \quad x_3 = r \cos \vartheta \quad (11)$$

showing that φ and ϑ are polar angles for $\mathbf{R}^3$.

In order to arrive at the desired gauge description of the KP we start from the Kepler Lagrangian

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} \dot{\mathbf{x}}^2 + \frac{k}{r} \quad (12)$$

*The term $2H$ in the first relation appears only quantum mechanically as an order $\hbar^2$ correction term of the corresponding classical relation.

and express $\mathbf{x}$ and $\dot{\mathbf{x}}$ via the KS by u and $\dot{u}$. The Lagrangian for the KS-KP reads

$$L(u, \dot{u}) = 2 u^2 \dot{u}^2 - 2 \dot{u}^T (u^* \cdot u^{*T}) \dot{u} + \frac{k}{u^2} \quad (13)$$

$$u^{*T} = (u_4, -u_3, u_2, -u_1) \quad (14)$$

The canonical momenta are $v = \partial L / \partial \dot{u}$, and the Jacobian $\left| \frac{\partial v}{\partial \dot{u}} \right|$ vanishes. Performing the constraint analysis according to Dirac [7] we obtain the constraint $C = 0$ with

$$C = 2I = u_4 v_1 - u_3 v_2 + u_2 v_3 - u_1 v_4. \quad (15)$$

With respect to the canonical symplectic structure $\omega = dv \wedge du$ on $T^*(\mathbf{R}^4 \setminus \{0\})$ I generates an $SO(2)$ action described by $(v^{*T} = (v_4, -v_3, v_2, -v_1))$

$$(u, v) \mapsto (\bar{u}, \bar{v}) = (u, v) \cos \phi + (u^*, v^*) \sin \phi \quad \kappa(\bar{u}, \bar{v}) = \kappa(u, v). \quad (16)$$

The fibres are circles in phase space, after projection into u -space they are parametrized by the angle ψ .

Making use of the arbitrariness of the Lagrange multiplier λ , the extended Hamiltonian can be brought into the form

$$H_E = \frac{1}{8 u^2} v^2 - \frac{k}{u^2} + \lambda I = H + \lambda I. \quad (17)$$

The observable part of the symmetry algebra of the canonical Hamiltonian H (the Poisson brackets of observables with I have to vanish strongly) is isomorphic to the symmetry algebra $\mathcal{A}$ of the KP, extended by the central element I . It is generated by H , I and the KS-transforms (mod I) of the vectors $\mathbf{L}$ and $\tilde{\mathbf{M}}^*$

$$\mathbf{L} = \frac{1}{2} \Lambda(u) v^* \quad \tilde{\mathbf{M}} = -\frac{1}{8} \Lambda(v) v + \Lambda(u) u H. \quad (18)$$

The relations (6) between the generators of $\mathcal{A}$ are modified by functions of the constraint I

$$\tilde{C}_1 = \tilde{\mathbf{M}}^2 - 2 H \mathbf{L}^2 = -2 H I^2 + k^2 \quad \tilde{C}_2 = \mathbf{L} \cdot \tilde{\mathbf{M}} = k I. \quad (19)$$

Observe that the modification is by observable elements of the constraint algebra, because I itself must be addressed as an observable (it commutes with itself). The observable status of I is also confirmed by its proportionality with $\tilde{C}_2$.

*We shall denote the KS-equivalents of three-dimensional quantities by the same symbols.

4 Quantization of the KS-Kepler problem

In this section we will perform the quantization of the KS-KP and its reduction to the hydrogen atom. This will be done according to the algebraic scheme outlined in the introduction, supplemented by some physical requirements to determine the scalar product.

The extended Hilbert space

As the fundamental observables to be quantized we shall take the quantities H , $\mathbf{L}$, $\tilde{\mathbf{M}}$, $\mathbf{x}$, $\mathbf{p}$ and I as given in (8,15,17,18). Classically these observables generate a transitive action on the constraint surface $I^{-1}(0)$, quantum mechanically they generate the algebra of observables and allow to connect all the states within the physical subspace.

In the Schrödinger representation ($\hbar = 1$)

$$u \longmapsto u \qquad v \longmapsto \frac{1}{i} \frac{\partial}{\partial u} \qquad \mathcal{H} = L^2(\mathbf{R}^4, d\mu) \quad (20)$$

a consistent factor ordering for the fundamental observables is given by ($\partial_\mu = \partial/\partial u_\mu$, $\partial^T = (\partial_1, \partial_2, \partial_3, \partial_4)$, $\partial^{*T} = (\partial_4, -\partial_3, \partial_2, -\partial_1)$)

$$H = -\frac{1}{8u^2} \Delta^{(4)} - \frac{k}{u^2} \qquad \Delta^{(4)} = \partial \cdot \partial = \sum_{\mu=1}^4 \frac{\partial^2}{\partial u_\mu^2} \quad (21)$$

$$\mathbf{L} = \frac{1}{2i} \Lambda(u) \partial^* \qquad \mathbf{x} = \Lambda(u) u \quad (22)$$

$$\tilde{\mathbf{M}} = \frac{1}{8} \Lambda(\partial) \partial + \Lambda(u) u H \qquad \mathbf{p} = -\frac{i}{2u^2} \Lambda(u) \partial \quad (23)$$

$$I = \frac{1}{2i} (u_4 \partial_1 - u_3 \partial_2 + u_2 \partial_3 - u_1 \partial_4). \quad (24)$$

The scalar product, induced by a measure $d\mu$ on $\mathbf{R}^4$, should meet the following requirements: i) invariance under the maximal part of the symmetry algebra that operates in u -space (generated by $\mathbf{L}$ and I), and ii) symmetry of the fundamental observables with respect to it.

The first condition can be satisfied by putting

$$d\mu(u) = c |u|^n d^4 u \qquad c = \text{const.} \quad (25)$$

the second can be used to fix the integer n . Because the operators H , $\tilde{\mathbf{M}}$ and $\mathbf{p}$ are symmetric only for $n = 2$, whereas $\mathbf{L}$, $\mathbf{x}$ and I are symmetric for arbitrary n , we choose $\mathcal{H} = L^2(\mathbf{R}^4, c u^2 d^4 u)$ as the Hilbert space for the KS-KP (cf. [8]).

The physical Hilbert space

The commutation relations between the observables H , $\mathbf{L}$, $\tilde{\mathbf{M}}$ and I generate the algebra $\mathcal{A} \oplus I$ (i.e. $\mathcal{A}$ extended by the central element I). In the realization (21)-(24) of this algebra there are two relations between its generators*

$$\tilde{C}_1 = \tilde{\mathbf{M}}^2 - 2H\mathbf{L}^2 = -2H(I^2 - 1) + k^2 \quad \tilde{C}_2 = \mathbf{L} \cdot \tilde{\mathbf{M}} = kI. \quad (26)$$

As a consequence, the Hilbert space $\mathcal{H}$ will be made up of those irreducible representations of $\mathcal{A} \oplus I$ in which these identities hold. These representations can be described as follows.

As H and I are central elements, they must be constant in any irreducible representation of $\mathcal{A} \oplus I$. For constant H , i.e. in the eigenspaces of H with energy $E < 0$, $E = 0$ or $E > 0$, the nonlinearity in the commutation relations between the $\tilde{M}_i$ can be removed by replacing $\tilde{\mathbf{M}}$ with

$$\mathbf{M} = \begin{cases} \tilde{\mathbf{M}}/\sqrt{-2H} & E < 0 \\ \tilde{\mathbf{M}} & E = 0 \\ \tilde{\mathbf{M}}/\sqrt{2H} & E > 0. \end{cases} \quad (27)$$

The commutation relations of $\mathbf{L}$ and $\mathbf{M}$ form the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{e}(3)$ and $\mathfrak{so}(1,3)$ with the Casimir operators

$$C_1 = \begin{cases} \mathbf{L}^2 + \mathbf{M}^2 \\ \mathbf{M}^2 \\ \mathbf{L}^2 - \mathbf{M}^2 \end{cases} \quad C_2 = \mathbf{L} \cdot \mathbf{M} \quad (28)$$

The identities (26) permit to express the spectra of H and I by the eigenvalues of C_1 and C_2 , so that the problem of finding the irreducible representations of $\mathcal{A} \oplus I$ in which (26) hold is reduced to the problem of finding the irreducible representations of $\mathfrak{so}(4)$, $\mathfrak{e}(3)$ and $\mathfrak{so}(1,3)$, where we can restrict ourselves to those representations which correspond to unitary representations of the groups $\text{SO}(4)$, $\text{E}(3)$ and $\text{SO}(1,3)$. The latter, including the spinor representations, can be characterized by two numbers [9, 10]**

$$\begin{array}{llll} \text{SO}(4) & : & (j_1, j_2) & 2j_1 \in \mathbf{N}_0 \quad 2j_2 \in \mathbf{N}_0 \\ \text{E}(3) & : & (k_o, \tau) & 2k_o \in \mathbf{Z} \quad \tau \geq 0 \\ \text{SO}(1,3) & : & (k_o, c) & 2k_o \in \mathbf{Z} \quad c = i\rho, \rho \geq 0 \\ & & \text{or} & k_o = 0 \quad 0 < c \leq 1 \end{array} \quad (29)$$

*The term $2H$ in the first relation again appears as a quantum correction of order $\hbar^2$ to the corresponding classical relation(19).

**The representations of $\text{SO}(1,3)$ with $c = i\rho$ constitute the principal series, the others the supplementary series of unitary irreducible representations.

which are related to the eigenvalues of C_1 and C_2 by*

$$C_1 = \begin{cases} 2(j_1(j_1 + 1) + j_2(j_2 + 1)) \\ \tau^2 \\ k_o^2 + c^2 - 1 \end{cases} \quad C_2 = \begin{cases} j_1(j_1 + 1) - j_2(j_2 + 1) \\ k_o \tau \\ -ik_o c \end{cases} \quad (30)$$

The identities (26) are not compatible with all these representations. In the case of $SO(1,3)$, the first identity, via the positivity of H , excludes the representations of the supplementary series, for $E(3)$ it restricts τ to k . The second identity, then, leaves us with the representations

$$\begin{array}{lll} SO(4) & : & (j_1, j_2) \\ E(3) & : & (k_o, \tau) \quad \tau = k \\ SO(1,3) & : & (k_o, c) \quad c = i\rho. \end{array} \quad (31)$$

These representations are in one-to-one correspondence with the common eigenspaces of H and I belonging to the eigenvalues

$$E = \begin{cases} -\frac{k^2}{2(j_1 + j_2 + 1)^2} & =: -\frac{k^2}{2n^2} \\ 0 \\ -\frac{k^2}{2c^2} & =: +\frac{k^2}{2\nu^2} \end{cases} \quad I = \begin{cases} j_1 - j_2 \\ k_o \\ k_o \end{cases} \quad (32)$$

The eigenspaces $\mathcal{H}_E$ of H alone are not irreducible with respect to the three groups, they decompose according to

$$\begin{aligned} \mathcal{H}_n &= \bigoplus_{j_1 - j_2 = -(n-1), \dots, (n-1)}^{j_1 + j_2 = n-1} (j_1, j_2) \\ \mathcal{H}_0 &= \bigoplus_{2k_o \in \mathbb{Z}} (k_o, k) \\ \mathcal{H}_\nu &= \bigoplus_{2k_o \in \mathbb{Z}} (k_o, i\nu). \end{aligned} \quad (33)$$

The representations can be explicitly constructed (see [11]). The corresponding basis states, labelled by the eigenvalues of the complete set of compatible observables H , $\mathbf{L}^2$, L_3 and I , are given by ($r = u^2$, φ, ϑ, ψ are the Euler coordinates (10))

$$\Phi_{nlm}^{l_o} = C_{nl} \left(\frac{2kr}{n} \right)^l e^{-kr/n} {}_1F_1(l - n + 1, 2l + 2, \frac{2kr}{n}) Y_{ml_o}^l(\varphi, \vartheta, \psi) \quad (34)$$

$$\Phi_{0lm}^{l_o} = C_{0l} \frac{1}{2\sqrt{r}} J_{2l+1}(\sqrt{8kr}) Y_{ml_o}^l(\varphi, \vartheta, \psi) \quad (35)$$

$$\Phi_{\nu lm}^{l_o} = C_{\nu l} \left(\frac{2kr}{\nu} \right)^l e^{-ikr/\nu} {}_1F_1(l + i\nu + 1, 2l + 2, \frac{2ikr}{\nu}) Y_{ml_o}^l(\varphi, \vartheta, \psi) \quad (36)$$

*The numbers $j_1(j_1 + 1)$ and $j_2(j_2 + 1)$ are the eigenvalues of the squares of the two commuting angular momenta $\mathbf{J}_1 = \frac{1}{2}(\mathbf{L} + \mathbf{M})$, $\mathbf{J}_2 = \frac{1}{2}(\mathbf{L} - \mathbf{M})$; $so(4)$ is isomorphic to $su(2) \oplus su(2)$.

$$\begin{aligned}
Y_{ml_o}^l(\varphi, \vartheta, \psi) &= \frac{\sqrt{2l+1}}{4\pi} e^{il_o\psi} e^{im\varphi} P_{ml_o}^l(\cos \vartheta) \\
P_{ml_o}^l(z) &= N_{ml_o}^l (1+z)^{-\frac{m+l_o}{2}} (1-z)^{-\frac{m-l_o}{2}} \frac{d^{l-m}}{dz^{l-m}} \left((1+z)^{l+l_o} (1-z)^{l-l_o} \right)
\end{aligned} \tag{37}$$

where C_{nl} , C_{0l} , $C_{\nu l}$ and $N_{ml_o}^l$ are normalization constants and l_o is the eigenvalue of $-I$ given in (32). l and m are as usual with the range $-l \leq m \leq l$ and

$$\begin{aligned}
l &= |l_o|, |l_o| + 1, \dots, n-1 = |j_1 - j_2|, \dots, j_1 + j_2 \\
l &= |l_o|, |l_o| + 1, \dots = |k_o|, |k_o| + 1, \dots \\
l &= |l_o|, |l_o| + 1, \dots = |k_o|, |k_o| + 1, \dots
\end{aligned} \tag{38}$$

The physical subspace $\mathcal{H}_{phys}$ is determined by the relations between the Casimir of the constraint algebra, here the constraint I itself, and the Casimirs of the symmetry algebra as expressed by the identities (26). As I is a Casimir of the algebra of observables, it is consistent to restrict it to a value in its spectrum. Restricting it to zero fixes the eigenvalues of the Casimirs of the symmetry algebra which correspond to the physical representations. $\mathcal{H}_{phys}$ is the direct sum of these representations.

For $I = 0$ the identities (26) single out the representations

$$\begin{array}{lll}
\text{SO}(4) & : & j_1 = j_2 = j \\
\text{E}(3) & : & k_o = 0 \quad \tau = k \\
\text{SO}(1,3) & : & k_o = 0 \quad c = i\rho
\end{array} \tag{39}$$

The corresponding wave functions Φ_{Nlm}^o ($l_o = 0$, $N = n, 0, \nu$) are just the 0-eigenstates of I . They span the subspace of functions in $\mathcal{H}$ which are independent of ψ and 2π -periodic in φ . This can also be seen by expressing I in Euler coordinates, where it takes the form

$$I = i \frac{\partial}{\partial \psi}. \tag{40}$$

The 2π -periodicity in φ is equivalent to the invariance under the inversion $u \rightarrow -u$, which corresponds to the finite gauge transformation $\psi \rightarrow \psi + 2\pi$. By equations (11) the physical states can be put into one-to-one correspondence with the states in $L^2(\mathbf{R}^3, d^3x)$

$$\Phi(\mathbf{x}) = \Psi_{phys}([u]) \tag{41}$$

where $[u]$ is the equivalence class of points that are mapped onto $\mathbf{x}$ under the KS.

Now, writing the measure $d\mu$ ((25), $n = 2$) in Euler coordinates

$$d\mu(u) = c \frac{1}{8} |u|^5 du \sin \vartheta d\vartheta d\varphi d\psi \tag{42}$$

we see that its restriction to the subspace $\mathcal{H}_{phys}$ of physical states can be done by simply performing the integration over ψ (the integration over ψ is the projector

onto the 0-eigenspace of I). Putting $c = 4/\pi$ and $r = u^2$, the measure on the physical subspace reduces to

$$d\mu_{phys} = r^2 dr \sin \vartheta d\vartheta d\varphi \quad (43)$$

showing that the correspondence (41) is a Hilbert space isomorphism of $\mathcal{H}_{phys}$ with $L^2(\mathbf{R}^3, d^3x)$.

Reduction of operators

In order to complete the reduction we still have to find the representation of the fundamental observables (21)-(23) on the physical Hilbert space $\mathcal{H}_{phys}$. This can easiest be done by working in Euler coordinates.

First of all we observe that the physical states are the common eigenstates of $I = i\partial/\partial\psi$ and the operator P of spatial inversion belonging to the eigenvalues 0 and 1 respectively. Furthermore, the observables (21)-(23) commute with both I and P . So, if we express the operators representing the observables on $\mathcal{H}$ in Euler coordinates, we can obtain the reduced operators representing them on $\mathcal{H}_{phys}$ by dropping all terms containing a derivative with respect to ψ .

We will show how this works for the Hamiltonian H (21). In Euler coordinates (with $r = u^2$) H reads

$$H = -\frac{1}{2} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\varphi\vartheta\psi} \right) - \frac{k}{r} \quad (44)$$

$$\Delta_{\varphi\vartheta\psi} = \frac{\partial^2}{\partial \vartheta^2} + \cot \vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \left(\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \psi^2} - 2 \cos \vartheta \frac{\partial^2}{\partial \varphi \partial \psi} \right). \quad (45)$$

Dropping the last two terms in $\Delta_{\varphi\vartheta\psi}$ yields

$$H_{phys} = -\frac{1}{2} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial^2}{\partial \vartheta^2} + \cot \vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \right) \right) - \frac{k}{r}. \quad (46)$$

By equations (11) H_{phys} can be identified with the Hamiltonian for the hydrogen atom. In the same way the reduced forms of $\mathbf{L}$, $\tilde{\mathbf{M}}$, $\mathbf{x}$ and $\mathbf{p}$ can be shown to coincide with the corresponding three-dimensional expressions (cf. [12]).

Finally, the wave functions Φ_{Nlm}^o can directly be seen to coincide with the energy eigenstates of the hydrogen atom as given in [13].

5 Conclusions

As we have seen, the algebraic method provides a powerful and elegant tool for the reduction of the quantized KS-KP. This is to be compared with other methods

for the quantization of gauge systems, for example the BRST method. The latter can also be used to perform the quantization of the KS-KP and its reduction to the quantum theory of the hydrogen atom, but its success depends crucially on the choice of the scalar product on the ghost sector (cf. [1]). Unfortunately, there is no physically motivated unique (a priori) choice for this scalar product.

The use of the (nonlinear) symmetry algebra $\mathcal{A}$ has proved to be advantageous for the algebraic quantization of the KS-KP. Moreover, it allows for a largely unified treatment of the three sections of the spectrum. In the former approaches to the problem, the cases $E < 0$, $E = 0$ and $E > 0$ had to be treated separately (e.g. [4]).

References

- [1] Trunk, M.: Die Eichung des Kepler Problems. Diploma thesis, Universität Freiburg 1993
- [2] Pauli, W.: Z. Phys. **36**, 336–363 (1926)
- [3] Fock, V.: Z. Phys. **98**, 145–154 (1935)
- [4] Bander, M., Itzykson, C.: Rev. Mod. Phys. **38**, 330–345, 346–358 (1966)
- [5] Kustaanheimo, P., Stiefel, E.L.: J. Reine Angew. Math. **218**, 204–219 (1965)
- [6] Kummer, M.: Commun. Math. Phys. **84**, 133–152 (1982)
- [7] Dirac, P.A.M.: Lectures on quantum mechanics. Yeshiva University, New York 1964
- [8] Iwai, T.: J. Math. Phys. **23**, 1093–1099 (1982)
- [9] Wybourne, B.G.: Classical groups for physicists. Wiley 1974
- [10] Naimark, M.A.: Linear representations of the Lorentz group. Pergamon 1964
- [11] Barut, A.O., Schneider, C.K.E., Wilson, R.: J. Math. Phys. **20**, 2244–2256 (1979)
- [12] Ikeda, M., Miyachi, Y.: Math. Japon. **15**, 127–142 (1970)
- [13] Landau, L.D., Lifshitz, E.M.: Quantum mechanics. Pergamon 1962