NONCOMMUTATIVE SYMMETRIC FUNCTIONS AND
LAPLACE OPERATORS FOR CLASSICAL LIE ALGEBRAS

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Abstract
New systems of Laplace (Casimir) operators for the orthogonal and symplectic Lie
algebras are constructed. The operators are expressed in terms of paths in graphs
related to matrices formed by the generators of these Lie algebras with the use
of some properties of the noncommutative symmetric functions associated with a
matrix. The decomposition of the Sklyanin determinant into a product of quasi-
determinants play the main role in the construction. Analogous decomposition for
the quantum determinant provides an alternative proof of the known construction
for the Lie algebra \( \mathfrak{gl}(N) \).
A general theory of noncommutative symmetric functions is developed in the paper by Gelfand–Krob–Lascaux–Leclerc–Retakh–Thibon [GKLLRT]. Noncommutative analogues and generalizations of the classical results of the theory of symmetric functions are obtained. The background of the noncommutative theory is the Gelfand–Retakh quasi-determinants [GR1], [GR2] (see also [KL]). Various results describing the properties of the noncommutative symmetric functions and relations between them are obtained and several examples of applications of these results for different kinds of specializations are discussed. Symmetric functions associated with a matrix whose entries are elements of a noncommutative ring is one of these applications. New systems of generators of the center of the universal enveloping algebra $U(\mathfrak{gl}(N))$ are constructed by using the properties of the symmetric functions associated with the matrix $E$ formed by the generators of the Lie algebra $\mathfrak{gl}(N)$. The key role in this construction is played by the fact that the coefficients of the characteristic Capelli determinant related to the matrix $E$ are central in the algebra $U(\mathfrak{gl}(N))$ (see Howe [H], Howe–Umeda [HU]) and that this determinant is factorized by the quasi-determinants related to submatrices of $E$ (see Gelfand–Retakh [GR1], [GR2]).

The invariance of the Capelli determinant can be easily proved by using the properties of the Yangian $Y(N) = Y(\mathfrak{gl}(N))$ (see, e.g., Nazarov [N], Nazarov–Tarasov [NT]); the latter is a ‘quantum’ deformation of the universal enveloping algebra for the polynomial current Lie algebra $\mathfrak{gl}(N)[x]$ (see, e.g., Takhtajan–Faddeev [TF], Drinfeld [D]). Namely, the center of the Yangian is generated by the coefficients of a formal series called the quantum determinant, and the Capelli determinant is the image of the quantum determinant under a natural homomorphism $Y(N) \rightarrow U(\mathfrak{gl}(N))$. In this paper we give an alternative proof of the factorization of the Capelli determinant by using the decomposition of the quantum determinant in the algebra of formal power series with coefficients from $Y(N)$.

Further, we apply this approach to the case of the orthogonal and symplectic Lie algebras $\mathfrak{o}(N)$ and $\mathfrak{sp}(N)$. Here the Yangian $Y(N)$ should be replaced by the Olshanskii twisted Yangian $Y^+(N)$ or $Y^-(N)$ respectively and the quantum determinant by the Sklyanin determinant (see [O], [MNO]). In Molev [Mol] a formula for the Sklyanin determinant was found and the Capelli-type determinant for the orthogonal and symplectic Lie algebras was constructed. As in the case of $\mathfrak{gl}(N)$, this determinant is a polynomial with coefficients from the center of the universal enveloping algebra.

It turns out that an analogue of the decomposition of the quantum determinant takes place for the Sklyanin determinant as well. Moreover, the image of this decomposition under the natural homomorphism of the twisted Yangian $Y^+(N)$ to the universal enveloping algebra yields a decomposition of the Capelli-type determinant into a product of quasi-determinants related to submatrices of the matrix $F$ formed by the generators of the Lie algebra $\mathfrak{o}(N)$ or $\mathfrak{sp}(N)$.

This allows us using some results of the paper [GKLLRT] to construct families of Laplace operators and express them in terms of paths in graphs related to the
matrix $F$. While the invariant polynomials which correspond to both the Capelli
determinant and the Capelli-type determinant under the Harish-Chandra isomor-
phism are the elementary symmetric functions, the corresponding polynomials for
the Laplace operators of these families are the complete symmetric functions and
the sums of powers of variables (see [GKLLRT] for the $\mathfrak{gl}(N)$-case).

The approach based on the properties of the Yangians was also used in [Mo2]
where the quantum Liouville formulae (see [N], [MNO]) were applied to the calcu-
lation of the Perelomov–Popov invariant polynomials [PP].

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sions.
1. Construction of Laplace operators

Let us consider the Lie algebra $\mathfrak{gl}(N)$ and let $\{E_{ij}\}$, where $1 \leq i, j \leq N$ be its standard basis. The commutation relations in this basis have the form:

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}.$$ 

For $1 \leq m \leq N$ denote by $E^{(m)}$ the $m \times m$-matrix with the entries $E_{ij}$, where $i, j = 1, \ldots, m$, and set $\rho_m := -m + 1$.

Let $\mathcal{E}^{(m)}$ denote the complete oriented graph with the vertices $\{1, \ldots, m\}$, the arrow from $i$ to $j$ is labelled by the $ij$-th matrix element of the matrix $E^{(m)} + \rho_m$. Then every path in the graph defines a monomial in the matrix elements in a natural way. A path from $i$ to $j$ is called simple if it does not pass through the vertices $i$ and $j$ except for the beginning and the end of the path.

Using this graph introduce the elements $\Lambda_k^{(m)}$, $S_k^{(m)}$, $\Psi_k^{(m)}$ and $\Phi_k^{(m)}$ of the universal enveloping algebra $U(\mathfrak{gl}(N))$ as follows [GKLLRT]: for $k \geq 1$

$$( -1)^{k-1} \Lambda_k^{(m)} \text{ is the sum of all monomials labelling simple paths in } \mathcal{E}^{(m)} \text{ of length } k \text{ going from } m \text{ to } m;$$

$S_k^{(m)} \text{ is the sum of all monomials labelling paths in } \mathcal{E}^{(m)} \text{ of length } k \text{ going from } m \text{ to } m;$$

$\Psi_k^{(m)} \text{ is the sum of all monomials labelling paths in } \mathcal{E}^{(m)} \text{ of length } k \text{ going from } m \text{ to } m, \text{ the coefficient of each monomial being the length of the first return to } m;$

$\Phi_k^{(m)} \text{ is the sum of all monomials labelling paths in } \mathcal{E}^{(m)} \text{ of length } k \text{ going from } m \text{ to } m, \text{ the coefficient of each monomial being the ratio of } k \text{ to the number of returns to } m.$

Denote by $L(\lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_N)$, the highest weight representation of the Lie algebra $\mathfrak{gl}(N)$. That is, $L(\lambda)$ is generated by a nonzero vector $v$ such that

$$E_{ii}v = \lambda_iv, \quad 1 \leq i \leq N, \quad \text{and} \quad E_{ij}v = 0, \quad 1 \leq i < j \leq N.$$ 

For $i = 1, \ldots, N$ set $l_i := \lambda_i + \rho_i$.

**Theorem 1.1** [GKLLRT, Section 7.5]. The center of the algebra $U(\mathfrak{gl}(N))$ is generated by the scalars and each of the following families of elements.

$$\Lambda_k = \sum_{i_1 + \ldots + i_N = k} \Lambda_{i_1}^{(1)} \cdots \Lambda_{i_N}^{(N)}, \quad \text{(1.1)}$$
$$S_k = \sum_{i_1 + \ldots + i_N = k} S_{i_1}^{(1)} \cdots S_{i_N}^{(N)}, \quad \text{(1.2)}$$
$$\Psi_k = \sum_{m=1}^{N} \Psi_k^{(m)}, \quad \text{(1.3)}$$
$$\Phi_k = \sum_{m=1}^{N} \Phi_k^{(m)}, \quad \text{(1.4)}$$
Let $k = 1, 2, \ldots, N$. Moreover, $\Psi_k = \Phi_k$ for any $k$, and the eigenvalues of $\Lambda_k, S_k$ and $\Psi_k$ in the representation $L(\lambda)$ are, respectively, the elementary, complete and power sums symmetric functions of degree $k$ in the variables $l_1, \ldots, l_N$.

Now we formulate an analogue of this theorem for the orthogonal and symplectic Lie algebras. Set $\mathfrak{g}(n) := \mathfrak{o}(2n), \mathfrak{o}(2n+1)$ or $\mathfrak{sp}(2n)$. We shall consider all the three cases simultaneously. It will be convenient to parametrize the basis elements $E_{ij}$ of the Lie algebra $\mathfrak{gl}(N)$ by the indices $i, j = -n, -n+1, \ldots, n-1, n$, where $n := \lfloor N/2 \rfloor$ and the index 0 is skipped when $N$ is even. Then the Lie algebra $\mathfrak{g}(n)$ can be realized as a subalgebra in $\mathfrak{gl}(N)$ spanned by the elements

$$F_{ij} := \begin{cases} E_{ij} - E_{-j,-i}, & \text{in the orthogonal case;} \\ E_{ij} - \text{sgn}(i)\text{sgn}(j)E_{-j,-i}, & \text{in the symplectic case.} \end{cases}$$

For $i = 1, \ldots, n$ set

$$\rho_{-i} = -\rho_i := \begin{cases} i - 1 & \text{for } \mathfrak{g}(n) = \mathfrak{o}(2n), \\ i - \frac{1}{2} & \text{for } \mathfrak{g}(n) = \mathfrak{o}(2n+1), \\ i & \text{for } \mathfrak{g}(n) = \mathfrak{sp}(2n). \end{cases}$$

We also set $\rho_0 := 1/2$ in the case of $\mathfrak{g}(n) = \mathfrak{o}(2n+1)$. Let $1 \leq m \leq n$. Denote by $F^{(m)}$ the matrix with the entries $F_{ij}$, where $i, j = -m, -m+1, \ldots, m$ (the index 0 is skipped if $N = 2n$). Let us consider the complete oriented graph $F_m$ with the vertices $\{-m, -m+1, \ldots, m\}$, the arrow from $i$ to $j$ is labelled by the $ij$-th matrix element of the matrix $F^{(m)} + \rho_m$.

Introduce now the elements $\Lambda_k^{(m)}, \bar{\Lambda}_k^{(m)}, S_k^{(m)}, \bar{S}_k^{(m)}, \Phi_k^{(m)}, \bar{\Phi}_k^{(m)}$ of the universal enveloping algebra $U(\mathfrak{g}(n))$ as follows: for $k \geq 1$

$$(-1)^{k-1} \Lambda_k^{(m)} \text{ (resp. } -\bar{\Lambda}_k^{(m)}) \text{ is the sum of all monomials labelling simple paths in } F^{(m)} \text{ (resp. simple paths that do not pass through } -m) \text{ of length } k \text{ going from } m \text{ to } m;$$

$$S_k^{(m)} \text{ (resp. } (-1)^k \bar{S}_k^{(m)}) \text{ is the sum of all monomials labelling paths in } F^{(m)} \text{ (resp. paths that do not pass through } -m) \text{ of length } k \text{ going from } m \text{ to } m;$$

$$\Phi_k^{(m)} \text{ (resp. } (-1)^k \bar{\Phi}_k^{(m)}) \text{ is the sum of all monomials labelling paths in } F^{(m)} \text{ (resp. paths that do not pass through } -m) \text{ of length } k \text{ going from } m \text{ to } m, \text{ the coefficient of each monomial being the ratio of } k \text{ to the number of returns to } m.$$ 

Denote by $L(\lambda), \lambda = (\lambda_1, \ldots, \lambda_n)$, the highest weight representation of the Lie algebra $\mathfrak{g}(n)$. That is, $L(\lambda)$ is generated by a nonzero vector $v$ such that

$$F_{ii}v = \lambda_i v, \quad 1 \leq i \leq n; \quad \text{and} \quad F_{ij}v = 0, \quad -n \leq i < j \leq n.$$

For $i = 1, \ldots, n$ set $l_i := \lambda_i + \rho_i$. 
Theorem 1.2. Each of the following families of elements is contained in the center of the algebra $U(\mathfrak{g}(n))$:

$$
\Lambda_{2k} = \sum_{i_1 + \cdots + i_{2n} = 2k} \Lambda_{i_1}^{(1)} \Lambda_{i_2}^{(1)} \cdots \Lambda_{i_{2n-1}}^{(n)} \Lambda_{i_{2n}}^{(n)},
$$

$$
S_{2k} = \sum_{i_1 + \cdots + i_{2n} = 2k} S_{i_1}^{(1)} S_{i_2}^{(1)} \cdots S_{i_{2n-1}}^{(n)} S_{i_{2n}}^{(n)},
$$

$$
\Phi_{2k} = \sum_{m=1}^{n} (\tilde{\Phi}_{2k}^{(m)} + \Phi_{2k}^{(m)}),
$$

$k = 1, 2, \ldots$.

Moreover, the eigenvalues of $(-1)^k \Lambda_{2k}$, $S_{2k}$ and $\Phi_{2k}/2$ in the representation $L(\lambda)$ are, respectively, the elementary, complete and power sums symmetric functions of degree $k$ in the variables $l_1^2, \ldots, l_n^2$.

Remarks. (i) One can define the elements $\Psi_{2k}$ by analogy with the $\mathfrak{gl}(N)$-case. However, the assertion of Theorem 1.2 for them is wrong.

(ii) Replacing $2k$ with $2k - 1$ in the formulae (1.5)–(1.7) one could define the central elements $\Lambda_{2k-1}$, $S_{2k-1}$ and $\Phi_{2k-1}$, but they turn out to be equal to 0.

(iii) The scalars and each of the families (1.5)–(1.7) with $k = 1, \ldots, n$ generate the center of $U(\mathfrak{g}(n))$, except for the case of $\mathfrak{g}(n) = \mathfrak{o}(2n)$. To get generators in the latter case, the $2n$-th elements in each family should be replaced with $\Lambda_{2n}^{1/2}$ which coincides, up to a constant, with the central Pfaffian-type element; see [Mo1].
2. Quasi-determinants and noncommutative symmetric functions

The main point in the proof of Theorems 1.1 and 1.2 is the use of the decomposition of the Capelli-type determinants into a product of quasi-determinants. In the case of the Lie algebra $\mathfrak{gl}(N)$ this decomposition was obtained in [GR1] and used in [GKLLRT, Section 7.4] to prove Theorem 1.1. We outline that proof here. Then we formulate an analogue of this decomposition for the orthogonal and symplectic Lie algebras (for the proof see Section 4) and using some ideas of the paper [GKLLRT] derive Theorem 1.2.

The algebra of noncommutative symmetric functions [GKLLRT] is defined as the free associative algebra with countably many generators $\Lambda_1, \Lambda_2, \ldots$. These generators are called the elementary symmetric functions. The complete symmetric functions $S_k$, the power sums symmetric functions of the first kind $\Psi_k$ and the power sums symmetric functions of the second kind $\Phi_k$ are defined as follows. Set

$$\lambda(t) = 1 + \sum_{k=1}^{\infty} \Lambda_k t^k,$$

where $t$ is an intermediate commuting with all the $\Lambda_k$. Then

$$1 + \sum_{k=1}^{\infty} S_k t^k := \lambda(-t)^{-1},$$

$$\sum_{k=1}^{\infty} \Psi_k t^{k-1} := \lambda(-t) \cdot \frac{d}{dt} (\lambda(-t)^{-1}),$$

$$\sum_{k=1}^{\infty} \Phi_k t^{k-1} := -\frac{d}{dt} \log \lambda(-t).$$

These relations generalize the corresponding relations between classical symmetric functions; see, e.g. Macdonald [M]. In particular, in the commutative case formulae (2.3) and (2.4) give the same families of power sums functions, that is, $\Psi_k = \Phi_k$ for all $k = 1, 2, \ldots$. Various applications of both the commutative and noncommutative theory can be obtained by means of specializations. That is, the series $\lambda(t)$ is replaced with an arbitrary series whose coefficients are still regarded as (formal) elementary symmetric functions. Then the other symmetric functions are constructed by using formulae (2.2)–(2.4) (see [M], [GKLLRT]).

Let $X$ be an $n \times n$-matrix over a ring with the unity and suppose that there exists the matrix $X^{-1}$, and its $ji$-th entry $(X^{-1})_{ji}$ is invertible. Then the $ij$-th quasi-determinant of $X$ is defined by the formula

$$|X|_{ij} := ((X^{-1})_{ji})^{-1}.$$  

Note that the definition of $|X|_{ij}$ can be made much less restrictive. In particular, the quasi-determinant can be defined for some non-invertible matrices (see [GR1],...
Applying Theorem 1.1 to the Lie algebras $\mathfrak{gl}/n5bGR/2/n5d/n29$. If the ring is commutative, then $[X]_{ij} = (-1)^{i+j} \det X / \det X^{ij}$ where $X^{ij}$ is the submatrix of $X$ obtained by removing the $i$-th row and $j$-th column.

Let $E$ denote the $N \times N$-matrix with the entries $E_{ij}$ and $\rho$ denote the diagonal matrix with the diagonal entries $\rho_1, \ldots, \rho_N$. Consider the Capelli determinant $\det (1 + t(E + \rho))$, where ‘det’ denotes the usual alternating sum of the products of the entries of the matrix, provided that the first element in each product is taken from the first column, the second one from the second column etc. This determinant is a polynomial in $t$ of degree $N$ and it was proved in Howe-Umeda [HU] that all its coefficients belong to the center of the algebra $U(\mathfrak{gl}(N))$. On the other hand, one has the following decomposition of the Capelli determinant into the product of quasi-determinants.

**Theorem 2.1 [GR1].** In the algebra of formal series in $t$ with coefficients from $U(\mathfrak{gl}(N))$ one has

$$\det (1 + t(E + \rho)) = (1 + tE_{11})[1 + t(E^{(2)} + \rho_2)]22 \cdots [1 + t(E^{(N)} + \rho_N)]_{NN}. \quad (2.5)$$

Now we apply this decomposition to prove Theorem 1.1. It was proved in [GKLLRT, Proposition 7.20] that the generating series of the elements $\Lambda_k^{(m)}$, $S_k^{(m)}$, $\Psi_k^{(m)}$ and $\Phi_k^{(m)}$ introduced in Section 1 can be written in terms of quasi-determinants in the following way:

$$1 + \sum_{k=1}^{\infty} \Lambda_k^{(m)} t^k = [1 + t(E^{(m)} + \rho_m)]_{mm}, \quad (2.6)$$

$$1 + \sum_{k=1}^{\infty} S_k^{(m)} t^k = [1 - t(E^{(m)} + \rho_m)]_{mm}^{-1}, \quad (2.7)$$

$$\sum_{k=1}^{\infty} \Psi_k^{(m)} t^{k-1} = [1 - t(E^{(m)} + \rho_m)]_{mm} \frac{d}{dt} [1 - t(E^{(m)} + \rho_m)]_{mm}^{-1}, \quad (2.8)$$

$$\sum_{k=1}^{\infty} \Phi_k^{(m)} t^{k-1} = -\frac{d}{dt} \log([1 - t(E^{(m)} + \rho_m)]_{mm}). \quad (2.9)$$

Applying Theorem 2.1 to the Lie algebras $\mathfrak{gl}(m)$ where $m = 1, \ldots, N$ we see that the quasi-determinant $[1 + t(E^{(m)} + \rho_m)]_{mm}$ is represented as the ratio of two Capelli determinants corresponding to the Lie algebras $\mathfrak{gl}(m)$ and $\mathfrak{gl}(m - 1)$. This implies that the elements $\Lambda_k^{(m)}$ with $k \geq 1$ and $1 \leq m \leq N$ generate a commutative subalgebra in $U(\mathfrak{gl}(N))$ [GKLLRT, Theorem 7.26].

On the other hand, formulae (2.5) and (2.6) imply that the coefficients of the Capelli determinant coincide with the elements $\Lambda_k$ given by (1.1); that is,

$$1 + \Lambda_1 t + \cdots + \Lambda_N t^N = \det (1 + t(E + \rho)). \quad (2.10)$$

Let us regard now these elements as specializations of (commutative) elementary symmetric functions. Then, comparing formulae (2.2)–(2.4) with (2.5)–(2.9) we
conclude that the corresponding specializations of the complete symmetric functions $S_k$ and the power sums symmetric functions $\Psi_k (= \Phi_k)$ are given by formulae (1.2)-(1.4). This proves the first part of Theorem 1.1.

It is clear that to prove the second part it suffices to find the eigenvalues of the elements of one of the families (1.1)-(1.4) in the representation $L(\lambda)$. It can be easily done for the coefficients of the Capelli determinant as well as for the power sums functions $\Psi_k$ (or $\Phi_k$). The proof is complete.

**Remark.** It can be easily seen from formulae (2.1)-(2.4) and (2.6)-(2.9) that for any fixed $m$ the elements $\Lambda_k^{(m)}$, $s_k^{(m)}$, $\Psi_k^{(m)}$ and $\Phi_k^{(m)}$ can be regarded as specializations of the corresponding noncommutative symmetric functions. These elements were introduced in [GKLLRT] as a special case of the noncommutative symmetric functions associated with a matrix over an arbitrary ring. However, as we have seen in the above proof, these functions (associated with the matrix $E^{(m)} + \rho_m$) form a commutative subalgebra in $U(\mathfrak{gl}(N))$.

Let us turn now to the case of the orthogonal and symplectic Lie algebras. We shall keep using the notation introduced in Section 1.

Let $F$ denote the $N \times N$-matrix with the entries $F_{ij}$, where $-n \leq i, j \leq n$ and let $\rho$ denote the diagonal matrix with the diagonal entries $\rho_{-n}, \ldots, \rho_n$. We introduce an analogue of the Capelli determinant for the orthogonal and symplectic Lie algebras in the following way. First, we define a map

$$\mathfrak{S}_N \rightarrow \mathfrak{S}_{N-1}, \quad p \mapsto p'$$

(2.11)

of the symmetric group $\mathfrak{S}_N$ to its natural subgroup $\mathfrak{S}_{N-1}$ as follows (see [Mo1, Section 3]). Let us consider the graph $\Gamma_N$ whose vertices are identified with the elements $\mathfrak{S}_N$, and two permutations $p = (p(1), \ldots, p(N))$ and $q = (q(1), \ldots, q(N))$ are connected by an edge if $q$ can be obtained from $p$ by interchanging two indices $p(k)$ and $p(l)$, provided that either

(i) $p(k)$ and $p(l)$ are the two maximal elements of the set $\{p(i), p(i+1), \ldots, p(N-i+1)\}$ for some $1 \leq i \leq n$; or

(ii) $k < l < N-k+1$ and $p(k), p(l), p(N-k+1)$ are the three maximal elements of the set $\{p(k), p(k+1), \ldots, p(N-k+1)\}$.

The graph $\Gamma_N$ admits the following properties. It has $(N-1)!$ connected components and each of them is isomorphic (as a graph) to the $k$-dimensional cube for some $1 \leq k \leq N-1$. Moreover, given $k$ the number of components, isomorphic to the $k$-dimensional cube, equals the signless Stirling number of the first kind $c(N-1, k)$ (see, e.g., Stanley [S]). Each connected component of the graph $\Gamma_N$ contains a unique vertex $p_0 = (p_0(1), \ldots, p_0(N))$ such that $p_0(n+1) = N$. Then, by definition, all the vertices of this component have the same image $p' = (p_0(N), \ldots, p_0(n+2), p_0(n), \ldots, p_0(1))$ under the map (2.11).

Now the Capelli-type determinant $\widetilde{\det}(1+t(F+\rho))$ is defined by the formula

$$\widetilde{\det}(1+t(F+\rho)) =$$

$$(-1)^n \sum_{p \in \mathfrak{S}_N} \text{sgn}(pp')(1+t(F+\rho_{-n}))^{-i_{\nu(1)}, i_{\nu'(1)}}, (1+t(F+\rho_n))^{-i_{\nu(N)}, i_{\nu'(N)}},$$

(2.12)
where \((i_1, \ldots, i_N)\) is an arbitrary permutation of the indices \((-n, \ldots, n)\) (the right hand side does not depend on this permutation).

It was proved in [Mo1] that this determinant can be represented in the form

\[
\widetilde{\det} (1 + t(F + \rho)) = \Lambda(t^2), \quad \text{if} \quad N = 2n \quad \text{and}
\]

\[
\widetilde{\det} (1 + t(F + \rho)) = (1 + t/2) \Lambda(t^2), \quad \text{if} \quad N = 2n + 1,
\]

where \(\Lambda(s)\) is a polynomial in \(s\) of degree \(n\) whose coefficients belong to the center of the algebra \(U(\mathfrak{g}(n))\).

For \(1 \leq m \leq n\) denote by \(\widetilde{F}^{(m)}\) the submatrix of the matrix \(F^{(m)}\) (see Section 1) obtained by removing the row and column enumerated by \(-m\). The following analogue of Theorem 2.1 will be proved in Section 4.

**Theorem 2.2.** In the algebra of formal series in \(t\) with coefficients from \(U(\mathfrak{g}(n))\) one has the decomposition:

\[
\Lambda(t^2) = [1 - t(\widetilde{F}^{(1)} + \rho_1)]_{11} [1 + t(F^{(1)} + \rho_1)]_{11} \cdots [1 - t(\widetilde{F}^{(n)} + \rho_n)]_{nn} [1 + t(F^{(n)} + \rho_n)]_{nn}.
\]

(2.13)

Moreover, all the factors on the right hand side of (2.13) are permutable.

Using this result we prove now Theorem 1.2. We have the following generating series for the elements \(\Lambda_k^{(m)}, S_k^{(m)}\) and \(\Phi_k^{(m)}\) introduced in Section 1 (see [GKLLRT, Proposition 7.26]):

\[
1 + \sum_{k=1}^{\infty} \Lambda_k^{(m)} t^k = [1 + t(F^{(m)} + \rho_m)]_{mm},
\]

(2.14)

\[
1 + \sum_{k=1}^{\infty} S_k^{(m)} t^k = [1 - t(F^{(m)} + \rho_m)]_{mm}^{-1},
\]

(2.15)

\[
\sum_{k=1}^{\infty} \Phi_k^{(m)} t^{k-1} = -\frac{d}{dt} \log ([1 - t(F^{(m)} + \rho_m)]_{mm}).
\]

(2.16)

Similarly, for the generating series of the elements \(\widehat{\Lambda}_k^{(m)}, \widehat{S}_k^{(m)}, \widehat{\Phi}_k^{(m)}\) we have

\[
1 + \sum_{k=1}^{\infty} \widehat{\Lambda}_k^{(m)} t^k = [1 - t(\widehat{F}^{(m)} + \rho_m)]_{mm},
\]

(2.17)

\[
1 + \sum_{k=1}^{\infty} \widehat{S}_k^{(m)} t^k = [1 + t(\widehat{F}^{(m)} + \rho_m)]_{mm}^{-1},
\]

(2.18)

\[
\sum_{k=1}^{\infty} \widehat{\Phi}_k^{(m)} t^{k-1} = -\frac{d}{dt} \log ([1 + t(\widehat{F}^{(m)} + \rho_m)]_{mm}).
\]

(2.19)

We obtain from formulae (2.13), (2.14) and (2.17) that the coefficients of the polynomial \(\Lambda(s)\) coincide with the elements \(\Lambda_{2k}\) given by (1.5); that is,

\[
1 + \Lambda_2 t^2 + \cdots + \Lambda_{2n} t^{2n} = \Lambda(t^2).
\]
As in the $\mathfrak{gl}(N)$-case, let us regard this polynomial (in the variable $t$) as a specialization of the generating series $\lambda(t)$ for the elementary symmetric functions (see (2.1)). Then using Theorem 2.2 and formulae (2.14)–(2.19) we obtain that the corresponding specializations of the complete symmetric functions $S_{2k}$ and the power sums symmetric functions $\Phi_{2k}$ are given by (1.6) and (1.7). This proves the first part of Theorem 1.2.

To prove the second part we can either find the eigenvalues of the elements $\Phi_{2k}$ in the representation $L(\lambda)$ directly, or use the fact (see [Mol]) that the image of the polynomial $\Lambda(s)$ in $L(\lambda)$ is $(1 - s l_1^2) \cdots (1 - s l_n^2)$, which completes the proof of Theorem 1.2.
3. Yangian for $\mathfrak{gl}(N)$ and the decomposition
of the quantum determinant

Here we use the approach based on the properties of the Yangians to obtain an
alternative proof of the decomposition of the Capelli determinant into the product
of quasi-determinants (Theorem 2.1). Various results describing the algebraic
structure of the Yangians are collected in the paper [MNO]. We reproduce those
of them we shall use below.

The Yangian $Y(N) = Y(\mathfrak{gl}(N))$ is the complex associative algebra with the
generators $t^{(1)}_{ij}, t^{(2)}_{ij}, \ldots$ where $1 \leq i, j \leq N$. The quadratic defining relations are
written in the following way. First, for any $i, j = \ldots, N$ introduce the formal
power series

$$t_{ij}(u) = \delta_{ij} + t^{(1)}_{ij} u^{-1} + t^{(2)}_{ij} u^{-2} + \ldots \in Y(N)[[u^{-1}]] \quad (3.1)$$

and combine these series into a single $T$-matrix:

$$T(u) := \sum_{i,j=1}^{N} t_{ij}(u) \otimes E_{ij} \in Y(N)[[u^{-1}]] \otimes \text{End}(\mathbb{C}^{N}), \quad (3.2)$$

where $E_{ij}$ are the standard matrix units. We need to consider the multiple tensor
products of the form $\mathbb{C}^{N} \otimes \ldots \otimes \mathbb{C}^{N}$ and operators therein. For an operator $X \in \text{End}(\mathbb{C}^{N})$ and a number $m = 1, 2, \ldots$ we set

$$X_k := 1 \otimes^{(k-1)} \otimes X \otimes 1 \otimes^{(m-k)} \in \text{End}(\mathbb{C}^{N})^{\otimes m}, \quad 1 \leq k \leq m. \quad (3.3)$$

If $X \in \text{End}(\mathbb{C}^{N})^{\otimes 2}$ then for any $k, l$ such that $1 \leq k, l \leq m$ and $k \neq l$, we denote by $X_{kl}$ the operator in $(\mathbb{C}^{N})^{\otimes m}$ which acts as $X$ in the product of $k$-th and $l$-th copies and as $1$ in all other copies. That is,

$$X = \sum_{r,s,t,u} a_{rstu} E_{rs} \otimes E_{tu}, \quad a_{rstu} \in \mathbb{C} \quad \Rightarrow$$

$$\Rightarrow X_{kl} = \sum_{r,s,t,u} a_{rstu}(E_{rs})_k (E_{tu})_l. \quad (3.4)$$

Further, given formal variables $u_1, \ldots, u_m$ we set for $k = 1, \ldots, m$

$$T_k(u_k) := \sum_{i,j=1}^{N} t_{ij}(u_k) \otimes (E_{ij})_k \in Y(N)[[u_1^{-1}, \ldots, u_m^{-1}]] \otimes \text{End}(\mathbb{C}^{N})^{\otimes m}. \quad (3.5)$$

We let $P$ denote the permutation operator in $\mathbb{C}^{N} \otimes \mathbb{C}^{N}$:

$$P := \sum_{i,j} E_{ij} \otimes E_{ji}.$$
Now the defining relations for $t_{ij}^{(k)}$ can be written as the ternary relation on the $T$-matrix:

$$R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v),$$ \quad (3.6)

where

$$R(u) = R_{12}(u) := 1 - u^{-1}P_{12}$$

is the Yang $R$-matrix.

Let

$$a_N = (N!)^{-1} \sum_{p \in S_N} \text{sgn}(p) \cdot p \in \mathbb{C}[S_N]$$

denote the normalized antisymmetrizer in the group ring. Consider the natural action of $\mathfrak{S}_N$ in the tensor space $(\mathbb{C}^N)^{\otimes N}$ and denote by $A_N$ the image of $a_N$.

There exists a formal series

$$q\det T(u) := 1 + d_1u^{-1} + d_2u^{-2} + \cdots \in Y(N)[[u^{-1}]]$$ \quad (3.7)

such that the following identity holds:

$$A_N T_1(u) \cdots T_N(u - N + 1) = q\det T(u) A_N.$$ \quad (3.8)

The series $q\det T(u)$ is called the quantum determinant of the matrix $T(u)$. Explicit formulae for the quantum determinant can be easily derived from the identity (3.8). We shall use the following one below:

$$q\det T(u) = \sum_{p \in S_N} \text{sgn}(p)t_{p(1),1}(u) \cdots t_{p(N),N}(u - N + 1).$$ \quad (3.9)

The coefficients $d_1, d_2, \ldots$ of the quantum determinant $q\det T(u)$ are algebraically independent generators of the center of the algebra $Y(N)$.

For $1 \leq m \leq N$ denote by $T^{(m)}(u)$ the submatrix of $T(u)$ corresponding the first $m$ rows and columns.

**Theorem 3.1.** One has the following decomposition of $q\det T(u)$ in the algebra $Y(N)[[u^{-1}]]$:

$$q\det T(u) = t_{11}(u) |T^{(2)}(u - 1)|_{22} \cdots |T^{(N)}(u - N + 1)|_{NN}.$$ \quad (3.10)

**Proof.** In fact, this theorem is a special case of Theorem 7.3 from [MNO] (see also [KS], [NT]), and easily follows from formula (3.8). Indeed, let us define the matrix $\hat{T}(u) = (\hat{t}_{ij}(u))$ by the formula

$$\hat{T}(u) = q\det T(u) T^{-1}(u - N + 1).$$ \quad (3.11)

Then multiplying both sides of (3.8) by $T_{N-1}^{-1}(u - N + 1)$ from the right we obtain the relation

$$A_N T_1(u) \cdots T_{N-1}(u - N + 2) = A_N \hat{T}_N(u).$$ \quad (3.12)
Taking the $NN$-th entry of the matrices on the left and right hand sides of (3.11) we get
\[ q\det T(u)(T^{-1}(u - N + 1))_{NN} = T_{NN}(u). \]
It follows from (3.12) that $T_{NN}(u) = q\det T^{(N-1)}(u)$, and so,
\[ q\det T(u) = q\det T^{(N-1)}(u) |T(u - N + 1)|_{NN}. \]
An easy induction proves the theorem.

Let us show now that the decomposition (2.5) is a consequence of (3.10). Indeed, it is not difficult to verify [D] (see also [MNO]) that the mapping
\[ \xi : t_{ij}(u) \mapsto \delta_{ij} + E_{ij}u^{-1} \]
defines the algebra homomorphism
\[ \xi : Y(N) \to U(gl(N)). \]
Formula (3.9) implies that
\[ \prod_{m=1}^{N} (1 + \rho_m u^{-1}) \xi(q\det T(u)) = \det (1 + (E + \rho)u^{-1}), \]
while
\[ (1 + \rho_m u^{-1}) \xi(|T^{(m)}(u - m + 1)|_{mm}) = |1 + (E^{(m)} + \rho_m)u^{-1}|_{mm}. \]
So, applying the homomorphism $\xi$ to both sides of (3.10) and replacing $u^{-1}$ with $t$, we obtain the decomposition (2.5) of the Capelli determinant.
4. Twisted Yangian and the decomposition

of the Sklyanin determinant

In this section following Olshanskii [O], [MNO] we define the algebra \( Y^\pm(N) \) which is an analogue of the Yangian for the orthogonal and symplectic Lie algebras and introduce a formal series — the Sklyanin determinant whose coefficients generate the center of \( Y^\pm(N) \). Then we obtain an analogue of Theorem 3.1 for the Sklyanin determinant and use it to prove Theorem 2.2.

Let \( \{e_i\} \) be the canonical basis of the vector space \( \mathbb{C}^N \). From now on we shall parametrize the basis vectors by the numbers \( i = -n, -n + 1, \ldots, n-1, n \), where \( n := [N/2] \) and \( i = 0 \) is skipped when \( N \) is even.

It will be convenient to use the symbol \( \theta_{ij} \) which is defined as follows:

\[
\theta_{ij} := \begin{cases} 
1, & \text{in the orthogonal case;} \\
\text{sgn}(i)\text{sgn}(j), & \text{in the symplectic case.}
\end{cases}
\]

Whenever the double sign \( \pm \) or \( \mp \) occurs, the upper sign corresponds to the orthogonal case and the lower sign to the symplectic one.

By \( X \mapsto X^t \) we will denote the matrix transposition which is defined on the matrix units as follows:

\[
(E_{ij})^t = \theta_{ij} E_{-j,-i}.
\]

The twisted Yangian \( Y^\pm(N) \) is the complex associative algebra with the generators \( s_{ij}^{(1)}, s_{ij}^{(2)}, \ldots \), where \( -n \leq i, j \leq n \). The defining relations are written in the following way. First, for \( -n \leq i, j \leq n \) introduce the formal power series

\[
s_{ij}(u) = \delta_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \cdots
\]

and set

\[
S(u) = \sum_{i,j} s_{ij}(u) \otimes E_{ij} \in Y^\pm(N)[[u^{-1}]] \otimes \text{End}(\mathbb{C}^N).
\]

Further, introduce the following element of the algebra \( \text{End}(\mathbb{C}^N)^{\otimes 2} \):

\[
Q := \sum_{i,j} \theta_{ij} E_{-j,-i} \otimes E_{ji},
\]

and set

\[
R^0(u) = 1 - u^{-1} Q.
\]

As in the case of the Yangian \( Y(N) \) we adopt here the notation of type (3.3)–(3.5) for the twisted Yangian. Now the defining relations for \( s_{ij}^{(k)} \) can be written as the quaternary relation and the symmetry relation for the matrix \( S(u) \):

\[
R(u - v) S_1(u) R^0(-u - v) S_2(v) = S_2(v) R^0(-u - v) S_1(u) R(u - v),
\]

(4.3)
Note that the twisted Yangian $Y^\pm(N)$ can be identified with a subalgebra in $Y(N)$ by setting $S(u) = T(u)T^\dagger(-u)$.

There exists a formal series

$$s\det S(u) \in Y^\pm(N)[[u^{-1}]]$$

such that the following identity holds:

$$A_NS_1(u)R_{12}^1 \cdots R_{1N}^1 S_2(u-1)R_{23}^2 \cdots R_{2N}^2 S_3(u-2) \cdots S_{N-1}(u-N+2)R_{N-1,N}^N S_N(u-N+1) = s\det S(u)A_N,$$

where $R_{ij}^l := R_{ij}(-2u + i + j - 2)$. The series $s\det S(u)$ is called the Sklyanin determinant of the matrix $S(u)$. A formula for $s\det S(u)$ was obtained in [Mo1]. To write it down we use the map (2.11). For any permutation $(i_1, \ldots, i_N)$ of the indices $(-n, \ldots, n)$ we have

$$s\det S(u) = (-1)^n \gamma_N(u) \sum_{p \in \Theta_N} \text{sgn}(pp^t) s_{i_1,i_2}^t(u) \cdots s_{i_{N-1},i_N}^t(u-n+1) \cdot s_{i_N+1,i_{N+1}}(u-n) \cdots s_{i_{N+1},i_1}(u-N+1),$$

where $s_{ij}^t(u)$ are the matrix elements of the matrix $S^t(u)$ and $\gamma_N(u) = (2u+1)/(2u-N+1)$ in the symplectic case.

Let us set

$$c(u) := \frac{1}{\gamma_N(u+N/2-1/2)} s\det S(u+N/2-1/2).$$

Then $c(u)$ is an even formal series in $u^{-1}$, $c(u) = 1 + c_2u^{-2} + c_4u^{-4} + \cdots$, and the elements $c_2, c_4, \ldots$ are algebraically independent generators of the center of the algebra $Y^\pm(N)$.

Let $1 \leq m \leq n$. Denote by $S^{(m)}(u)$ the submatrix of $S(u)$ corresponding the rows and columns enumerated by $-m, -m + 1, \ldots, m$, and by $\tilde{S}^{(m)}(u)$ the submatrix of $S^{(m)}(u)$ obtained by removing the row and column enumerated by $-m$. We have the following analogue of Theorem 3.1.

**Theorem 4.1.** If $N = 2n$ then

$$c(u) = |S^{(1)}(-u - 1/2)|_{11} \cdots |S^{(1)}(u - 1/2)|_{11} \cdots |S^{(n)}(-u - n + 1/2)|_{nn} \cdot |S^{(n)}(u - n + 1/2)|_{nn},$$

if $N = 2n + 1$ then

$$c(u) = s_{00}(u) \cdot |S^{(1)}(-u - 1)|_{11} \cdots |S^{(1)}(u - 1)|_{11} \cdots |\tilde{S}^{(n)}(-u - n)|_{nn} \cdot |S^{(n)}(u - n)|_{nn}.$$
Moreover, all the factors on the right sides of (4.7) and (4.8) are permutable.

**Proof.** Let us define the matrix \( \hat{S}(u) = (\hat{s}_{ij}(u)) \) by the formula

\[
\hat{S}(u) = \text{sdet} S(u) S^{-1}(u - N + 1).
\]

Then, multiplying both sides of (4.5) by \( S^{-1}_N(u - N + 1) \) from the right we obtain the relation

\[
A_N S_1(u) R'_{12} \cdots R'_{1N} S_2(u - 1) R'_{23} \cdots R'_{2N} S_3(u - 2) \cdots S_{N-1}(u - N + 2) R'_{N-1,N} = A_N \hat{S}_N(u).
\]

Taking the \( nn \)-th entry of the matrices on the left and right hand sides of (4.9) we get

\[
\text{sdet} S(u) (S^{-1}(u - N + 1))_{nn} = \hat{s}_n(u).
\]

Hence,

\[
\text{sdet} S(u) = \hat{s}_n(u) |S(u - N + 1)|_{nn}.
\]

Now we use formula (4.10). It can be easily verified by using the symmetry relation (4.4) (see [Mo1]) that

\[
A_N S_1(u) R'_{12} \cdots R'_{1N} = \frac{2u + 1}{2u \pm 1} A_N S_1'(u).
\]

Denote by \( A_N^{(2)} \) the normalized antisymmetrizer corresponding to the subgroup of \( \Theta_N \) consisting of the permutations which preserve the first index. Clearly, \( A_N = A_N A_N^{(2)} \). Note that \( A_N^{(2)} \) is permutable with \( S'_1(-u) \), and \( R'_{kl} \) is permutable with \( R'_{kl} \) and \( S_k(u) \), provided that the indices \( i, j, k, l \) are distinct. So, we can rewrite formula (4.10) in the form:

\[
\frac{2u + 1}{2u \pm 1} A_N S_1'(-u) A_N^{(2)} S_2(u - 1) R'_{23} \cdots R'_{2,N-1} S_3(u - 2) \cdots S_{N-1}(u - N + 2) R'_{2,N-1,N} = A_N \hat{S}_N(u).
\]

Let us apply the operators in both sides of this formula to the vector \( v_i = \epsilon_{-i} \otimes \epsilon_{-n+1} \otimes \epsilon_{-n+2} \otimes \cdots \otimes \epsilon_{-n-1} \otimes \epsilon_n \), where \( i \in \{-n + 1, \ldots, n\} \). For the right hand side we clearly obtain

\[
A_N \hat{S}_N(u) v_i = \delta_{in} \hat{s}_n(u) \zeta,
\]

where \( \zeta := A_N (\epsilon_{-n} \otimes \epsilon_{-n+1} \otimes \cdots \otimes \epsilon_n) \). To calculate the left hand side we note first that

\[
R'_2 N \cdots R'_{N-1,N} v_i = v_i.
\]

Further, let us introduce the formal series

\[
C_{a_2, \ldots, a_{N-1}}(u - 1) \in Y^{\pm}(N)[[u^{-1}]], \quad -n \leq a_i \leq n,
\]
as follows:

\[ A_N^{(2)} S_2(u - 1) R_{23}^t \cdots R_{2,N-1}^t S_3(u - 2) \cdots S_{N-1}(u - N + 2)(e_{a_{N+1}} \otimes \cdots \otimes e_{a_{N-1}}) \]

\[ = \sum_{a_2, \ldots, a_{N-1}} C_{a_2, \ldots, a_{N-1}}(u - 1)(e_{a_2} \otimes \cdots \otimes e_{a_{N-1}}). \]

In particular,

\[(N - 2)! C_{-n+1, \ldots, -n-1}(u - 1) = \det S^{(n-1)}(u - 1), \quad (4.15)\]

and the series \( C_{a_2, \ldots, a_{N-1}}(u - 1) \) is skew-symmetric with respect to permutations of the indices \( a_2, \ldots, a_{N-1} \). This allows us to write the left hand side of (4.13) applied to \( v_i \) in the form:

\[ \frac{2u + 1}{2u + 1} (N - 2)! \sum_{k=1}^{N-1} (-1)^{k-1} s_{b_k, -i}(-u) C_{b_1, \ldots, b_{k+1}, \ldots, b_{N-1}}(u - 1) \zeta \]

\[ = \frac{2u + 1}{2u + 1} (N - 2)! \prod_{k=1}^{N-1} \theta_{i_n} \sum_{k=1}^{N-1} (-1)^{k-1} \prod_{\theta_{b_k} = n} C_{b_1, \ldots, b_{k+1}, \ldots, b_{N-1}}(u - 1) \zeta, \]

where \( (b_1, \ldots, b_{N-1}) = (-n, -n + 1, \ldots, n - 1) \). Put

\[ C_{-b_k}(u - 1) := (N - 2)! (-1)^{k-1} \prod_{\theta_{b_k} = n} C_{b_1, \ldots, b_{k+1}, \ldots, b_{N-1}}(u - 1). \]

Then, taking into account (4.14), we get the following matrix relation:

\[ \frac{2u + 1}{2u + 1} S^{(n)}(-u) \begin{pmatrix} C_{-n+1}(u - 1) \\ \vdots \\ C_n(u - 1) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \tilde{S}_{n,n}(u) \end{pmatrix} \]

Multiplying its both sides by the matrix \( (\tilde{S}^{(n)}(-u))^{-1} \) from the left and comparing the \( n \)-th coordinates of the vectors, we obtain using (4.15) that

\[ \frac{2u + 1}{2u + 1} \det S^{(n-1)}(u - 1) = (\tilde{S}^{(n)}(-u))^{-1}_{n,n} \tilde{S}_{n,n}(u), \]

and hence,

\[ \tilde{S}_{n,n}(u) = \frac{2u + 1}{2u + 1} |\tilde{S}^{(n)}(-u)|_{n,n} \det S^{(n-1)}(u - 1). \]

Together with (4.11) this gives

\[ \det S(u) = \frac{2u + 1}{2u + 1} |\tilde{S}^{(n)}(-u)|_{n,n} \cdot \det S^{(n-1)}(u - 1) \cdot |S(u - N + 1)|_{n,n}. \quad (4.16) \]

Note that the series \( |S(u - N + 1)|_{n,n} \) commutes with the matrix elements of the matrix \( S^{(n-1)}(v) \) (see [MNO, Proposition 7.5]) and hence with the series \( \det S^{(n-1)}(u - 1) \).
1). Since the coefficients of \( \text{sdet} S(u) \) belong to the center of the algebra \( Y^\pm(N) \) all the factors in the right hand side of (4.16) are mutually permutable. So, (4.16) can be rewritten as

\[
\text{sdet} S(u) = \frac{2u + 1}{2u} \text{sdet} S^{(n-1)}(u - 1) \cdot |\tilde{S}^{(n)}(-u)|_{nn} \cdot |S(u - N + 1)|_{nn}.
\] (4.17)

Performing an easy calculation and using the induction argument we complete the proof of formulae (4.7) and (4.8).

Further, using again the fact that the coefficients of the Sklyanin determinant belong to the center of the Yangian, we obtain from formula (4.17) that the product of the quasi-determinants

\[
|\tilde{S}^{(n)}(-u - N/2 + 1/2)|_{nn} \cdot |S^{(n)}(u - N/2 + 1/2)|_{nn}
\]

is permutable with each of the factors in formula (4.7) or (4.8). On the other hand, it was noticed above that the series \( |S^{(n)}(u - N/2 + 1/2)|_{nn} \) is also permutable with each of the factors, which proves the theorem.

Theorem 2.2 can be derived now as follows. It is not difficult to verify (see [O], [MNO]) that the mapping

\[
\xi : s_{ij}(u) \mapsto \delta_{ij} + F_{ij}(u \pm \frac{1}{2})^{-1}
\]

defines the algebra homomorphism

\[
\xi : Y^\pm(N) \to U(\mathfrak{g}(n)).
\]

Formula (4.6) implies that

\[
\prod_{i=-n}^{n} (1 + \rho_i u^{-1}) \xi(c(u)) = \text{det} (1 + (F + \rho)u^{-1}).
\]

On the other hand, if \( 1 \leq m \leq n \) then in the case of \( N = 2n \) we have

\[
(1 + \rho_m u^{-1}) \xi(|S^{(m)}(u - m + 1/2)|_{mm}) = |1 + (F^{(m)} + \rho_m)u^{-1}|_{mm}
\]

and

\[
(1 + \rho_{-m} u^{-1}) \xi(|\tilde{S}^{(m)}(-u - m + 1/2)|_{mm}) = |1 - (F^{(m)} + \rho_m)u^{-1}|_{mm},
\]

while in the case of \( N = 2n + 1 \)

\[
(1 + \rho_m u^{-1}) \xi(|S^{(m)}(u - m)|_{mm}) = |1 + (F^{(m)} + \rho_m)u^{-1}|_{mm}
\]

and

\[
(1 + \rho_{-m} u^{-1}) \xi(|\tilde{S}^{(m)}(-u - m)|_{mm}) = |1 - (F^{(m)} + \rho_m)u^{-1}|_{mm}.
\]
So, applying the homomorphism $\xi$ to both sides of the decomposition (4.7) or (4.8) and replacing $u^{-1}$ with $t$ we complete the proof of Theorem 2.2.

**Remarks.** (i) An analogue of the Cayley–Hamilton theorem for a generic matrix $A$ with noncommutative entries was obtained in [GKLLRT, Theorem 8.17] and the *pseudo-determinants* $\det_i A$ of $A$ were introduced. In particular, for $A = E$ or $F$ the Cayley–Hamilton identity coincides with the characteristic identity satisfied by $E$ or $F$ (see resp. [NT] or [Mol]) and the images of these identities in highest weight representations give the Bracken–Green identities [BG], [G]. The pseudo-determinants of the matrices $E + \rho_N$ or $F + \rho_n$ coincide with the leading coefficients of the Capelli determinant $\det (1 - t(E + \rho))$ or the Capelli-type determinant $\det (1 - t(F + \rho))$ respectively. In particular, for any $i = -n, \ldots, n$

$$\det_i (F + \rho_n) = (-1)^{N+n} \sum_{\rho \in S_N} \text{sgn}(pp')(F + \rho_n)_{-i_{\rho}(1), i'_{\rho}(1)} \cdots (F + \rho_n)_{-i_{\rho(N), i'_{\rho(N)}},}.$$

(ii) The approach used in Section 1 for the construction of Laplace operators for the Lie algebras $\mathfrak{gl}(N)$, $\mathfrak{o}(N)$, $\mathfrak{sp}(N)$ can be applied to the algebras $Y(N)$ and $Y^\pm(N)$ themselves. One can consider the specializations of the symmetric functions taking the coefficients $d_1, d_2, \ldots$ of the quantum determinant or the coefficients $c_2, c_4, \ldots$ of the Sklyanin determinant as elementary symmetric functions. Then a description of other families of central elements of $Y(N)$ and $Y^\pm(N)$ as complete or power sums symmetric functions can be obtained by using Theorems 3.1 and 4.1.
References


