What is the lightest excited state of the strongly selfcoupled Higgs field?

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Abstract

We argue for the existence of an upper bound $m_*$ on the Higgs mass at which the lowest excited state of the Higgs field ceases to be the conventional plane wave. An explicit construction of an alternative nonperturbative state is discussed. This excitation is spatially localized. The field fluctuations inside the localization region are large. The energy of the excitation is smaller than the mass of the plane wave state $m$ at $m > m_*$. An approximate value of $m_*$ is found to be $m_* \approx 4.75$ times the vacuum expectation value. This is an upper bound at the tree level.

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Let us consider the one-component Higgs field specified by the action

\[ S = \int d^4x \left[ \frac{1}{2} (\partial_\mu \varphi)^2 - U(\varphi) \right] \]

where the potential term, \( U(\varphi) \), is defined as

\[ U(\varphi) = \lambda (\varphi^2 - v^2)^2. \]

The canonical quantization suggests that the minimal-energy excitation above the vacuum is the one-particle state with the mass squared \( m^2 = 8\lambda v^2 \). This conclusion is based on the small fluctuation analysis on the fixed classical background \( \varphi = v \). Any feedback effect is neglected. This is valid for small values of the coupling constant because the energy associated with deviations of the classical \( \varphi \)-field from its vacuum value is typically \( m/\lambda \gg m \). But there is no reason to expect the Higgs particle with the mass about \( \sqrt{8\lambda} v \) at large values of \( \lambda \). Moreover, some indications of nontrivial excitations in the Higgs sector have been collected [1].

In this paper we discuss a scenario of the departure from the canonical picture at large \( \lambda \). We present some steps to the explicit construction of a nonperturbative excited state. As it shown by the estimates presented below this state is lighter than the mass of the conventional plane wave at \( m > m_\approx \approx 4.75v \). Unlike the plane wave, it is localized in a region of the order of \((0.1 - 0.3) v^{-1}\). The mean value of the Higgs field inside this volume is smaller than \( v \). The field fluctuations there are of the order of \( v \).

A very naive approach to treat such a state is to take some space dependent field configuration \( \varphi = \Phi(x) < v \) with \( \Phi(\infty) = v \) and consider the small-fluctuation spectrum around it. There is, however, a well-known fact that any classical configuration collapses to the origin in the usual Higgs system [2]. To overcome this, we shall consider the lowest eigenvalue, \( \Omega_0^2 \), of this spectrum. Then one can expect to find the energy of the lightest excited state \( \mathcal{E} \) as the minimum of \( E[\Phi(x)] + \Omega_0[\Phi(x)] \). Such a minimum can only be reached when \( \Omega_0 < m \) corresponds to a bound state in the potential created by \( \Phi(x) \). While the order of magnitude of \( \Omega_0 \) is \( m \), the value of \( E \) is typically \( m/\lambda \). These two values can balance each other at large \( \lambda \) only.

A crucial question that one has to address to this plan is how to make distinguish between the background field and the fluctuations around it. Moreover, it is clear that one has to treat both of them simultaneously performing the quantization if the coupling constant is large. An approach presented here allows us to proceed toward this aim.

Let us start with an illustrative model. Consider a system with two degrees of freedom described by the Schrödinger equation

\[ \left( -\frac{1}{2\mu} \frac{\partial^2}{\partial x^2} - \frac{1}{2} \frac{\partial^2}{\partial y^2} + V(x) + \frac{1}{2} \omega^2(x)y^2 \right) \Psi(x, y) = \mathcal{E} \Psi(x, y). \]

Suppose \( V(x), x \geq 0 \), is a monotonously increasing function, \( V(0) = 0 \). Unlikely, the function \( \omega^2(x) \) decreases. So, the full potential in eq. (3) looks like a valley that leads
upward, but becomes wider as \( x \) increases. What are the stationary states of this system? Do the wave functions concentrate near the origin?

It is very easy to answer these questions in the adiabatic approximation that is valid if the motion in the \( x \) direction is much slower than that in \( y \). In this case the wave function can be found in the form

\[
\Psi(x, y) = f_n(y; \omega(x))\psi(x)
\]

(4)

where \( f_n(y; \omega) \) is the \( n \)-th normalized eigenfunction of the harmonic oscillator with the frequency \( \omega \). One substitutes (4) to the equation (3), then multiplies it by \( f_n \) and integrates over \( y \) taking into account that \( \int dy f_n(y)\partial f_n(y)/\partial \omega = 0 \). It yields an effective Schrödinger equation for \( \psi(x) \):

\[
\left( -\frac{1}{2\mu} \frac{\partial^2}{\partial x^2} + V(x) + (n + \frac{1}{2})\omega(x) \right) \psi(x) = E\psi(x).
\]

(5)

One observes that an effective potential \( V_{\text{eff}} = V + (n + 1/2)\omega \) enters this equation. The minimum of \( V_{\text{eff}} \) may be far from the point \( x = 0 \). In other words, the wave function is repulsed from the region where the potential is too narrow.

What is the first correction to the adiabatic approximation? In order to find it out one substitutes \( \Psi(x, y) = \psi(x)f_n(y; \omega(x)) \) into the equation (3) and finds the first order correction \( V^{(1)} \) to \( V \). It reads

\[
V^{(1)} = -\frac{1}{2\mu} \left( \frac{d\omega}{dx} \right)^2 \int dy f_n(y; \omega(x)) \frac{\partial^2 f_n(y; \omega(x))}{\partial \omega^2}.
\]

(6)

If one takes into account that \( E \sim n\omega \) then the correction (6) divided by \( E \) turns into \( V^{(1)}/E \sim n/(\omega \mu \Delta x^2) \). Here \( \Delta x \) denotes the characteristic length at which \( \omega \) changes essentially. \( \mu \Delta x^2 \) is the typical time of the motion over this interval.

Let us discuss the relation of this model to our problem. We would like to construct a valley in the field configuration space similar to that formed by the potential in eq. (3). A coordinate \( X \), an analog of \( x \), parametrizes the bottom of this valley. The point \( X = 0 \) corresponds to \( \Phi = v \). The larger \( X \) the farther \( \Phi(x) \) deviates from the vacuum expectation value. The curvature of the valley cross section is to be smaller as \( X \) increases. As we see soon it is related to \( \Omega_0 \), the lowest normal frequency of the fluctuations around \( \Phi(x) \). This suggests a choice of the coordinate \( X \). We take just

\[
X = m - \Omega_0.
\]

(7)

With this choice we can specify the bottom of the valley we are going to consider. It is the configuration \( \Phi(x) \) at which the classical field energy

\[
E[\varphi(x)] = \int d^3 x \left[ \frac{1}{2}(\nabla \varphi)^2 + U(\varphi) \right]
\]

(8)

is minimal at fixed value of \( \Omega_0 \). Let us write this condition in the form

\[
E[\Phi(x)] = \min_{\text{fixed } \Omega_0} E[\varphi(x)].
\]

(9)
Here $\Omega_0$ is the lowest eigenvalue of the problem
\[
\left(-\Delta + m^2 + U''(\varphi)\right)\psi(x) = \Omega^2\psi(x).
\] (10)

The corresponding eigenfunction $\psi_0(x)$ is spherically symmetric and nodeless.

Let us specify an analog of the coordinate $y$ of our quantum mechanical example (3). It is convenient to end to expand the field near the bottom of the valley
\[
\varphi(x) = \Phi(x) + \chi(x)
\] (11)
and consider the first and the second variations of the classical energy, $\delta E$ and $\delta^2 E$ up to the terms of the order $\chi^2$.

All the functions $\chi(x)$ constitute a real linear space, $\mathcal{H}$, with the inner product
\[
(\chi_1, \chi_2) = \int d^3x \chi_1(x)\chi_2(x).
\] (12)

There are two specific coordinate systems in it. The introduction of the first one relates to $\delta^2 E$. This value reads
\[
\delta^2 E = \int d^3x A(\Phi(x))\chi(x).
\] (13)

Here is $A(\Phi(x))$ the operator in the LHS of eq. (10). The eigenfunctions of $A$ form a complete set. Let us refer to these functions, the corresponding eigenvalue and coordinates as, respectively, $\psi_\alpha(x)$, $\Omega_\alpha^2$, and $\xi_\alpha$, $\alpha = 0, 1, \ldots$.

The other set of coordinates includes the direction of the bottom of the valley defined by (9) as one of the basis vectors. This set can be chosen as the eigenfunctions of an operator $\hat{A} = PAP$ where $P$ is the projector on $\Sigma$, the hypersurface of fixed $\Omega_0$ at $\varphi(x) = \Phi(x)$. The vector that is tangential to the bottom of the valley in the configuration space is just the gradient of the energy $\psi_E(x)$. It is orthogonal to $\Sigma$ due to the condition (9) and, therefore, it is an eigenfunction of $\hat{A}$ with zero eigenvalue. Its explicit form is
\[
\psi_E(x) = -\Delta \Phi(x) + U'(\varphi).
\] (14)

In terms of this function the first differential of the energy for the direction $\chi(x)$ in $\mathcal{H}$ reads $\delta E$,
\[
\delta E = \int d^3x \psi_E(x)\chi(x).
\] (15)

All other eigenfunctions of $\hat{A}$ denoted by $\chi_\beta$, $\beta = 0, 1, \ldots$, belong to $\Sigma$. Let the corresponding eigenvalues and the coordinates be $\omega_\beta$ and $\eta_\beta$. Note that for each $\chi(x)$ in $\Sigma$ the mean value $\langle \chi, \hat{A}\chi \rangle = \langle \chi, A\chi \rangle$. The minimal $\langle \chi, A\chi \rangle$ among all normalized $\chi$ in $\Sigma$ is just $\omega_0^2$, the minimal nonzero eigenvalue of $\hat{A}$.

Let us go on constructing the valley similar to one in (3). To this end we make use of a trial wave functional
\[
\Psi[\eta, \omega] = \psi(X)f_1(\eta_0; \omega_0)\prod_{\beta=1}^{\infty} f_0(\eta_\beta; \omega_\beta).
\] (16)
The average energy of the quantum motion in the hypersurface $\Sigma$ given by (16) is

$$\frac{3}{2} \omega_0 + \sum_{\beta=1}^{\infty} \frac{1}{2} \omega_\beta$$

(17)

minus the vacuum zero energy. If we add to this sum one more frequency related to the motion along the valley then the expression (17) turns to $\omega_0$ plus the one-loop correction to $E[\Phi(x)]$. A preliminary result for the one-loop corrections is discussed later and we neglect them now. To this accuracy, the energy of the excited Higgs field takes the form

$$E_{eff}[\Phi(x)] = E[\Phi(x)] + \omega_0(\Omega).$$

(18)

In the following we refer to this value as the bottom of the effective valley.

What is the value of $\omega_0$? If we are lucky the gradient $\psi_E$ is orthogonal to $\psi_0$. In this case we reach the minimal value $E_{eff} = E + \Omega_0$. In the opposite limiting case of the bad luck $\psi_E \sim \psi_0$. Then the minimal value of the effective energy is shifted to $E_{eff} = E + \Omega_1$. This implies the nontrivial behaviour of the Higgs configuration does not appear till larger values of the Higgs mass where our possible approximations are not good.

The intermediate case can be illustrated in a simplified way if we consider in addition to $\xi_0$ only one coordinate $\xi_1$ corresponding to the eigenvalue $\Omega_1^2$. Let the projections of $\psi_E$ on $\psi_0$ and $\psi_1$ be proportional to $\cos \theta_{0E}$ and $\sin \theta_{0E}$. Then

$$\omega_0^2 = \Omega_0^2 \sin^2 \theta_{0E} + \Omega_1^2 \cos^2 \theta_{0E}.$$

(19)

One sees that a small value of $\cos \theta_{0E}$ yields a quadratic correction to $\omega_0$:

$$\omega_0 \approx \Omega_0 + \frac{\Omega_1^2 - \Omega_0^2}{2 \Omega_0} \cos^2 \theta_{0E}.$$

(20)

In reality we have to consider the complete set of coordinates and eigenvalues together with $\Omega_1$. Those eigenvalues are typically of the order $m$. The expression (20) is, of course, only an order-of-magnitude estimate.

Let us present our results. In order to probe the structure of $E$ in the configuration space we use the following spherically symmetric ansatz for the $\varphi$-field (fig.1):

$$\varphi(r) = v \left[ 1 - b \left( 1 + \frac{r^2}{a^2} \right)^3 \exp \left( -\frac{r^2}{a^2} \right) \right].$$

(21)

This trial function has two parameters $a$ and $b$ related to the width and the amplitude of the configuration.

It is straightforward to find the bottom of the valley (9). We do it by the variation of $a$ and $b$ and searching for the minima in small $\Omega_0$ intervals. Typically the total number of data point is a few ten-thousands. A few tens from them are chosen. Some results of this procedure are presented in figs. 2 – 5. The valley shapes in the $(a,b)$-plane are shown in fig. 6.
The values of the cosine of the angle between $\psi_0$ and $\psi_E$

$$\cos \theta_{0E} = \frac{(\psi_0, \psi_E)}{\sqrt{(\psi_E, \psi_E)}}$$  \hspace{1cm} (22)

are also shown in figs. 2–5. It turns out that the ansatz (21) is sufficiently good in the sense that it yields small values of $\cos \theta_{0E}$ except for the region of $\Omega_0$ close to zero. In order to estimate the correction to $E_{\text{eff}}$ due to nonorthogonality of $\psi_E$ and $\psi_0$ we make use of the equation (19) with $\Omega_1 = m$. The resulted increase of $E_{\text{eff}}$ near $\Omega_0 \approx 0$ is seen in figs. 2–5. The value of $\cos \theta_{0E}$ up to 0.3 can be neglected with, approximately, one per cent accuracy in $\omega_0$ (20).

The region $\Omega_0 \approx m$ can not be described at all with our definition of the valley (9). The reason is that the value of $\Omega_0$ remains $\Omega_0 = m$ as long as the potential in eq. (10) is too weak to give rise to a bound state. In particular it means that $\Omega_0$ is a nonanalytic function of $a$ and $b$ as they are small. Note that the potential in (10) can be treated perturbatively in this case.

Let us discuss the results. If the Higgs particle is light, then $E_{\text{eff}}$ monotonously increases as $\Omega_0$ gets smaller. This confirms that the lightest excited state is the conventional plane wave. A nontrivial local minimum of $E_{\text{eff}}$ first appears at $m \approx 4.38v$. It is illustrated with fig. 2.

The bottom of the effective valley (18) becomes degenerate with the plane wave energy $m$ at $m = 4.75v$ (fig. 3). One can expect large fluctuations in the $\Omega_0$ direction (we discuss the width of the appropriate wave function below). The lowest excited state ceases to be the plane wave.

A minimum of $E_{\text{eff}}$ appears and becomes deeper as $m$ increases (figs. 4, 5). In $m > m_*$ we use the parameter $m$ just instead of less manifest value of $\lambda = m^2/8v^2$. The actual energy of the lowest excited state $E$ is less than $m$. Its value without any corrections due to the quantization of $\Omega_0$ is just $E_{\text{eff}}$ at the minimum. It is plotted in fig. 7 as a function of $m$.

The value of $E_{\text{eff}}$ at the left edge of the effective valley, i.e. at $\Omega_0 \approx 0$, becomes smaller than at the local minimum $\Omega_0^{\text{min}}$ at $m \approx 6.1v$ (fig. 7). At $m \approx 6.78v$ the minimum of $E_{\text{eff}}$ at $\Omega_0^{\text{min}}$ merges with a local maximum and disappears. The bottom of the effective valley at larger $m$ decreases monotonously as $\Omega_0$ gets smaller. This could be a hint on a the more complicated configuration with even lower energy. Unfortunately, our data do not allow to make such a conclusion. The correction to $E_{\text{eff}}$ due to the nonorthogonality of $\psi_0$ to $\psi_E$ shown in figs. 2–5 increases $E_{\text{eff}}$ near $\Omega_0 \approx 0$. This can be either a meaningful behaviour or an artifact of our ansatz.

The quantization of the motion along the valley is not straightforward. Doing it properly we have to consider a nonlocal action that arises after all the degrees of freedom across the valley are integrated out. It is the adiabatic approach that allows us to neglect this complication. Thus, a naive way is just to make use of the Schrödinger equation

$$\left(-\frac{1}{2\mu_0} \frac{\partial^2}{\partial \Omega_0^2} + E_{\text{eff}}[\Phi(x)]\right) \psi(\Omega_0) = \mathcal{E}\psi(\Omega_0)$$  \hspace{1cm} (23)
with

$$
\mu_\Omega(\Omega_0) = \int d^3 x \left( \frac{\partial \Phi}{\partial a} \frac{d a}{d \Omega_0} + \frac{\partial \Phi}{\partial b} \frac{d b}{d \Omega_0} \right)^2.
$$

(24)

We face, however, the problem of ordering because this mass parameter \( \mu_\Omega \) depends on the coordinate, \( \Omega_0 \). In our case, we first apply the differential operator to the wave function then multiply it by \( 1/\mu_\Omega \). This naive choice gives the qualitative character to our discussion.

In spite of the poor accuracy of the equation (23) we use it in order to make some estimates. First of all it is worth to check the validity of the adiabatic approximation. Its parameter can be chosen as \( \delta_1 = \nu/\Omega_0 \) where \( \nu = (E_{eff}/\mu_\Omega)^{1/2} \) is the frequency of the small oscillations around the minimum of \( E_{eff} \). Some values of \( \delta_1 \) are collected in table 1. We also present there \( \delta_2 \), the maximum - by absolute value - of the ratio of \( V^{(1)} \) eq. (6) to \( E_{eff} \), with \( x = m - \Omega_0, \omega = \Omega_0 \) and \( n = 1 \). As it is discussed below the minimum of \( E_{eff} \) can hardly be considered as a harmonic oscillator. This suggests that the parameter \( \delta_2 \) is more relevant than \( \delta_1 \). The smallness of \( \delta_2 \) and not too large value of \( \delta_1 \) let us hope that our results are qualitatively valid. We have to remind, however, that the values of both parameters depend on the form of the kinetic term in the equation (23).

The further use of this equation allows us to estimate the characteristic distance and time of the motion in the effective valley. Let \( \Delta \Omega_0 \) be the fluctuation of \( \Omega_0 \). One sees from fig. 3 that the bottom of the effective valley (18) is flat at \( m = 4.75 \nu \). So, one can expect the large fluctuations \( \Delta \Omega_0 \approx m \). As the Higgs mass is larger than 4.75\( \nu \), \( E_{eff} \) looks like an oscillator potential rather than a flat one (figs. 4, 5). Is it a harmonic oscillator? If it is the case then the wave function of the motion along the valley is \( \psi(\Omega_0) \approx f_0(\Omega_0; \nu) \) and \( \Delta \Omega_0 \approx (2/(\mu_\Omega \nu))^{1/2} \). The numerical results collected in table 1 show that it is not the case, especially at \( m \approx m_* \). The boundary condition for \( \psi(\Omega_0) \) at the edges of the effective valley should be also essential.

A characteristic time \( \tau_{free} \) for the flat \( E_{eff} \) can be estimated by the use of the Green function of the one-dimensional free motion. If one creates a well localized state near \( \Omega_0 = m \) then the time of the wave function propagation on the characteristic distance \( \Delta \Omega_0 \) is \( \tau_{free} \approx \mu_\Omega \Delta \Omega_0^2/2 \). Its numerical values are given in table 1 for \( \Delta \Omega_0 = \Omega_0^{\text{min}} \). One observes that this time is shorter than the inverse of the excitation energy only at \( m = 4.38 \nu \). At larger mass a possible propagation from the plane wave state to \( \Omega_0^{\text{min}} \) takes the time which is longer than \( 1/E \). Another estimate of \( \tau \) can be done for \( m > m_* \) in the form \( \tau_{osc} \approx 2\pi/\nu \). This is comparable with the values of \( \tau_{free} \) at \( m = 5.5 \nu \) and \( m = 6.78 \nu \) (table 1).

To sum up, we have shown by an explicit construction at the tree level that there is an upper bound \( m_* \) on the Higgs mass at which the lightest excited state ceases to be the conventional plane wave. This state becomes to be spatially localized and has the mass smaller than the naively expected Higgs mass \( m \) (fig. 7). We have found an approximate upper bound of \( m_* \approx 4.75 \nu \). In the minimal standard model it would correspond to \( m \approx 1.17 \text{ TeV} \).

Let us discuss briefly some corrections to our results. The effective valley (18) appears in balancing the energy \( E \) of the classical field with the lowest excitation \( \Omega_0 \) on its background.
This, obviously, violates the basis of the usual loop expansion. Normally such corrections should be small. Another approximation we have used is the adiabatic approach. It does not require the effect to be small, but needs the motion along the valley to be much slower than that across. In the table 2 we present our preliminary estimates of the one-loop corrections to (8) obtained by the method described in [3]. These data indicate relatively good quality of the tree approximation. Note that both the one-loop correction to $E$ and the correction (6) are negative. So, they make the valley deeper\footnote{We neglect here the one-loop contributions from other particle species. The Higgs particle in our consideration is much heavier than the top-quark, the heaviest particle of the standard model [4]. Its contribution to $E[\Phi(x)]$ can be found by the method of ref. [5].}. We have neglected also the anharmonicity of the fluctuations around $\Phi(x)$. This obviously corresponds to the omission of the loop contributions to $\Omega_0$.

Let us note that the introduction of the valley in the functional space is a step to distinguish between the classical background and the fluctuations around it. Indeed, it allows us to perform simultaneously the quantization of the coordinate related to the background and the fluctuations if one leaves behind the adiabatic approach.

Another source of correction is our use of the ansatz. Normally, when one proceeds in this way, the resulted value of the energy is an upper bound on the true minimum. An ansatz-free calculation of the bottom-of-the-valley energy will be published elsewhere.

Are our results applicable to the Standard Model of the electroweak interaction? The answer based on the data presented here is rather "no" because of too large threshold value of the Higgs mass $m_\ast \approx 4.75v \approx 1170$ GeV. This is near the number at which the Higgs decay width becomes equal to its mass [6]. However, the sources of the corrections discussed above give the negative contributions to the excitation energy. If they result in essential decreasing of $m_\ast$ then the consideration of the nontrivial Higgs field excitation in the Standard Model can become meaningful.

It would be interesting to relate the phenomenon we have discussed with the so-called triviality of the Higgs field [7]. This question is, however, outside the scope of this paper.

We are grateful to Alexander von Humboldt Foundation for the support of this work. V. G. K. thanks K. Kang for a discussion.
References

[7] See ref. [6]. A more recent list of the publications can be found, for example, in B. A. Knie1, Phys. Rep. 240 (1994) 211. We apologize to the authors of many interesting papers for such an indirect citation.
### Tables

<table>
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<th>$\mathcal{E}/v$</th>
<th>$\Omega_0^{min}/v$</th>
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Table 1: Some parameters of the effective valley defined in the text. $\Omega_0^{min}$ means the value of $\Omega_0$ at the minimum of $E_{eff}$.

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Table 2: The values of $a$, $b$, and the one-loop correction to $\mathcal{E}$ at the minimum of $E_{eff}$ at different $m$. The values of $\mathcal{E}$ are presented again for the reader’s convenience.
Figure Captions

Fig. 1 The shape of $\phi(x)$ given by the ansatz (21) at different values of $b$.

Fig. 2 The bottom of the valley $E[\Phi(x)]$ defined in eq. (9) (the lower solid line), the bottom of the effective valley (18) with $\omega_0 = \Omega_0$ (the upper solid line), the cosine of the angle between $\psi_E$ and $\psi_0$ (22) (short-dashed line), and the value of $E_{\text{eff}}$ corrected accordingly to eq. (19) (long-dashed line) at $m = 4.38v$. The data points are shown in the short-dashed line. The thick horizontal line indicates the energy equal $m$.

Fig. 3 The same as in fig. 2 at $m = 4.75v$. The irregular behaviour at small $\Omega_0$ is due to the poor statistics collected in this region.

Fig. 4 The same as in fig. 2 at $m = 5.5v$.

Fig. 5 The same as in fig. 2 at $m = 6.78v$. See also the caption to fig. 3.

Fig. 6 The valley shapes in the $(a, b)$-plane. Four curves are labelled with the value of $m/v$. Note that the relatively large error in $a$ and $b$ yields an essentially smaller variation of $E_{\text{eff}}$ computed near its minimum.

Fig. 7 The minimal value of $E_{\text{eff}}$ (18) with $\omega_0 = \Omega_0$ as a function of $m$ (solid line). This curve consists of a few hundreds points obtained by searching for $\Omega_0^{\text{min}}$ only. A few points found as the minima of $E_{\text{eff}}$ after generation of the effective valley are shown with the circles. The plane wave energy $m$ is indicated with the short-dashed line. The crosses connected with the long-dashed line show the value of $E_{\text{eff}}$ at the left edge of the effective valley. The numerical error seen in this value is due to the poor statistics of the Monte Carlo procedure at small $\Omega_0$. 

11