VARIATIONAL ESTIMATES FOR EXCITED STATES

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Abstract. Approximate energies for excited states of two and three body systems (with confining power law potentials) are obtained by a naïve application of the variational method. The error in the excited state energies is similar to the error for ground state energies, less than 1%. The asymptotic form of the energy is obtained directly by semiclassical arguments: the form is correct but the leading coefficient has a small error. A classical variational principle, for the expectation value of the Hamiltonian, for periodic motion with constant action is also discussed. Variational estimates are used to confirm and extend a negative result on nucleon resonances due to Høgaasen and Richard.

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1. Introduction

Three body systems are of great interest for the physics of quarks in baryons. The spectroscopy of baryons has been very successfully treated within constraints suggested by QCD which restricts choices of interactions [1]. A parallel activity in analysing the data takes an "inverse" point of view, whereby one is trying to "derive" QCD like features from the data directly. This point of view has been rather successful in dealing with quarkonia [2,3] but less so with baryons [11]. We are taking this "inverse" approach here, and use approximate estimates to analyze three body energies. A review of many accurate methods for three body systems was recently given by Richard [4]. Our interest here is in approximate estimates which have an accuracy of say better than 1%, which at 2000 MeV is about 20 MeV, an error which is quite acceptable. Our level of mathematical rigor is closer to [2], than to reference [3].

The method which we elaborate and use, is a naïve extension to excited states of the usual variational procedure for ground states. We spend the next section discussing this method for a well known one dimensional example, where we can get into a lot of detail. We follow this with a discussion of the underlying mathematical principles and an application to three dimensions. Next we go to three body systems where this method can be tested against some accurate numerical results available. Only then do we attempt an application to physical data on baryon resonances. We

give references in the appropriate sections later, when it is clearer what we are doing rather than in this brief Introduction.

We outline here the main idea of this Note. According to the Variational Principle [5] the extrema of an integral $\langle \psi | H | \psi \rangle$ (the expectation value of an operator H) under variations of the normalized wavefunction ψ are eigenvalues of the operator H . The lowest eigenvalue of H is the absolute minimum of the integral, but the other eigenvalues are also local extrema (in Hilbert space) of the integral. So we can estimate eigenvalues by evaluating integrals if we are in the right places in Hilbert space. So how to find these regions? We take an orthonormal set of functions (eg Harmonic Oscillator states), which can be scaled. If the first function of the set gives (under scaling) a good estimate for the lowest eigenvalue of H , we suppose that the next function in the set gives also a good estimate for the next eigenvalue and so on. We find in a number of examples that this is so. More precisely the percentage error grows rather slowly with energy, which makes variational estimates useful at low excitation in many systems. We discuss the error and limitations in more detail in Section II on an example. An interesting feature of these estimates is that their asymptotic limit (at large n) may be obtained semiclassically, and these asymptotic estimates are also reasonably accurate at low n . The semiclassical estimate is also obtained variationally.

II. An example: linear potential in one dimension

Consider the bound states of a Schrödinger equation with a linear potential: (We set $\hbar = 1$ throughout)

$$H\psi_n(x) = \left[-\frac{d^2}{dx^2} + x \right] \psi_n(x) = E_n \psi_n(x) \quad (\text{II-1})$$

where $0 \leq x < \infty$, with the boundary conditions $\psi(0) = \psi(\infty) = 0$. This is a well known quantum problem (motion under constant force) [2,3]. The eigenvalues E_n of (1) are known to be the zeros of the Airy function, while the functions $\psi_n(x)$ are segments of the same function. The function and its zeros are tabulated (see e.g., ref. [4]) and we shall quote a few eigenvalues in Table II-1. To start and set up notation, we estimate the ground state energy E_0 with the variational method, and use a harmonic oscillator trial function. We must satisfy the boundary condition $\psi(0) = 0$ and so we take odd harmonic oscillator states. To be more definite, and to match up eqn (1) with an unperturbed Hamiltonian $H^{(0)}(\omega)$, we consider the extension of eqn (1) over the whole line $-\infty < x < \infty$, with potential $V(x) = |x|$, and take an unperturbed Hamiltonian

$$H^{(0)}(\omega) = -\frac{d^2}{dx^2} + \omega^2 x^2 \quad (\text{II-2})$$

which is double that of the standard harmonic oscillator with frequency ω . As trial function for the ground state E_0 of (II-1) we take the first excited state of (II-2) which is the lowest state with a node at the origin:

$$\psi_1^{(0)}(\omega; x) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{2\sqrt{\omega} x}{\sqrt{2}} e^{-\omega x^2/2} \quad (\text{II-3})$$

We compute the expectation value of H , extended over the whole line (potential $|x|$) with the trial function (II-3) to find

$$E_0(\omega) = \langle \psi_1^{(0)} | H | \psi_1^{(0)} \rangle = \frac{3}{2} \omega + \frac{2}{\sqrt{\pi \omega}} \quad (\text{II-4})$$

where the first term comes from the kinetic energy and the second from the potential energy $|x|$. We minimize E_0 with respect to ω to find

$$\bar{\omega} = \left(\frac{4}{9\pi}\right)^{1/3} \simeq 0.52106 \quad (\text{II-5})$$

and compute the energy at the minimum

$$E_0^v(\bar{\omega}) = \frac{9}{2} \bar{\omega} = \frac{9}{2} \left(\frac{4}{9\pi}\right)^{1/3} \simeq 2.3448 \quad (\text{II-6})$$

in quite good agreement with the first zero of the Airy function, the exact energy $E_0 \simeq 2.3381$. We thus have a good approximation for the lowest energy (6) and wavefunction $\psi_1^{(0)}(\bar{\omega}, x)$. It is reasonable to suppose that if ψ_1 gives such a good estimate for E_0 , then ψ_3 will also be good for E_1 . We take the next eigenfunction of $H^{(0)}(\omega)$ which satisfies the boundary condition $\psi_{(0)} = 0$

$$\psi_3^{(0)}(\omega; x) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{3}} [2(\sqrt{\omega} x)^3 - 3\sqrt{\omega} x] e^{-\omega x^2/2} \quad (\text{II-7})$$

(Actually it is simpler to leave out the factors of $\sqrt{\omega}$ and insert them into the expectation values at the end). The expectation value of the Hamiltonian H with the trial function (II-7) is

$$E_1(\omega) = \frac{7}{2} \omega + \frac{3}{\sqrt{\pi} \omega} \quad (\text{II-8})$$

We minimize E_1 with respect to ω :

$$\bar{\omega} = \left(\frac{9}{49\pi} \right)^{1/3} \approx 0.3881 \quad (\text{II-9})$$

and insert $\bar{\omega}$ to find the minimum of E_1 :

$$E_1^V(\bar{\omega}) = \frac{7}{2} \bar{\omega} = \frac{7}{2} \left(\frac{9}{49\pi} \right)^{1/3} \approx 4.0752 \quad (\text{II-10})$$

which (as conjectured) is in quite good agreement with the second zero of the Airy function (the exact energy) $E_1 \approx 4.0879$, though a little (0.31%) below it. But note that $\bar{\omega}_2 \neq \bar{\omega}_1$ and therefore the trial functions are not orthogonal. It is possible to evaluate the energy for the n 'th odd state of the oscillator:

$$E_n(\omega) = \left(2n + \frac{3}{2} \right) \omega + \frac{(2n+1)!}{2^{2n-1} (n!)^2 \sqrt{\pi} \omega}$$

to obtain the variational estimates

$$E_n^V(\bar{\omega}) = \frac{3}{2} (4n+3)^{1/3} \left[\frac{(2n+1)!}{2^{2n-1} (n!)^2 \sqrt{\pi}} \right]^{2/3} \quad (\text{II-11})$$

The variational estimates $E_n^V(\bar{\omega})$ are given in Table II-1 for the

first 6 states and compared to the exact values for the same states. We see first of all from Table II-1 that the differences $\Delta E_n = E_n^V - E_n^{\text{exact}}$ grow slowly with n (in absolute magnitude), at about the same rate as the energy itself, so that the percentage error δE_n stays constant between $n=0$ and $n=5$. We note that for excited states ΔE_n is negative.

The asymptotic limit of E_n^V , as detailed below is:

$$\lim_{n \rightarrow \infty} E_n^V = 6 \left(\frac{n+3/4}{\pi} \right)^{2/3} \simeq 2.7972 (n+3/4)^{2/3} \quad (\text{II-12})$$

which is about 0.485% below the correct asymptotic formula given by the semiclassical WKB result

$$E_n^{\text{WKB}} = \left[\frac{3\pi}{2} (n+3/4) \right]^{2/3} \simeq 2.8108 (n+3/4)^{2/3} \quad (\text{II-13})$$

Therefore the variational energies are about 0.5% too low at high n . This implies that for very large n there are many states in between the predicted E_n^V and E_n^{exact} . The critical value of n , where the prediction E_n^V first coincides with the wrong state is at $n \simeq 137$, which means that the variational approximation is reasonable (in this case) for $n < 20$ say. It is nice that the asymptotic form $(n + 3/4)^{2/3}$ is obtained correctly.

Equation (II-12) can be obtained directly by taking the limit of large n , in equation (II-11), with Stirling's formula for factorials. A nicer and more direct way, which avoids entirely equation (II-11) is to carry out the variational estimate in the

quasi-classical limit. Then, for the harmonic oscillator $x(t) = A \sin \omega t$, and the energy is $E = A^2 \omega^2 / 2 \approx 2(n + \frac{1}{2}) \omega$ which allows the determination of A , from which we can compute $\langle |x| \rangle = 2A/\pi = (4/\pi) \sqrt{\frac{n+1/2}{\omega}}$. Therefore in this limit, the expectation value of the Hamiltonian H , Eqn. (II-1), is:

$$\langle H \rangle = (n + \frac{1}{2}) \omega + \frac{4}{\pi} \sqrt{\frac{n + \frac{1}{2}}{\omega}} \quad (\text{II-14})$$

A variational calculation now gives $\bar{\omega}_n$ and equation (II-12), where we must remember that n is an odd integer.

Even though the variational energies are reasonably accurate we should discuss the composition of the variational states $\psi_{2n+}^{(0)}(\bar{\omega})$ in terms of the true eigenstates of the Hamiltonian (II-1). A simple way to discuss this is by computing the expectation value of the square of the Hamiltonian H^2 with the variational oscillator wavefunctions and estimate the difference $\langle H^2 \rangle - \langle H \rangle^2$. The smaller the difference, the closer one is to the correct eigenfunctions. The results are shown in Table (II-2), where it is seen that the ratio $\langle H^2 \rangle / \langle H \rangle^2$ grows slowly with n . Asymptotically we find that $\langle H^2 \rangle - E_n^2 \approx 0.011 E_n^2$. This suggests that the decomposition of the variational wavefunction in terms of true eigenfunctions is narrow. One can compute numerically the actual overlap integrals between the variational wavefunctions (at $\omega_n = \bar{\omega}_n$) with the appropriate normalized Airy functions. The squares of these amplitudes are the probabilities given in Table II-3. We see from Table II-3 that whereas in the ground state the probability of overlap with the

true ground state is higher than 99%, for higher excitations this probability diminishes slowly. For example in the third excited variational state, the probability to be in the correct ($n=3$) state is about 80%, but the distribution in the 'wrong' states is symmetric, $P_2 \approx P_4 \approx 7.8\%$, $P_1 \approx P_5 \approx 1.75\%$ so that the 'average' properties prevail. The table of probabilities may be checked by computing the expectation values of H , H^2 which agree with the results in tables II-1, II-2. We note that the energy and its square are obtained much more accurately than the wavefunction itself, though other operators, like x^2 whose expectation value is given in Table II-2 are also obtained relatively accurately, with error less than 2%. The probabilities in Table II-3 strongly support the contention that the variational energy estimates are appropriate to their corresponding states, i.e., E_3^v is indeed the energy of the 4th state (we start with $n=0$), etc. But we can only compute probabilities in simple examples when we know the exact wavefunctions (as in this case). We assume however that the conclusion holds for other systems with confining potentials of power law type. We have also compared visually a few wavefunctions, and the trial wavefunctions look similar to the actual wavefunctions, especially as regards their range.

In the discussion above we compared our energy estimate with the known exact energies, to evaluate the error. But suppose we did not know the exact eigenvalues. How could we evaluate the error of the variational estimates? A simple way, is to consider the variational energies the first order energies in a perturbation

expansion with respect to the unperturbed hamiltonian $H^{(0)}(\omega_0)$ at a fixed value of $\bar{\omega}_0$, for the given energy level. We then evaluate the second order contribution to the energy in perturbation theory and this gives an estimate of the error committed in the first order. We shall illustrate this computation for the ground state E_0 and the first excited state E_1 . For the ground state:

$$\Delta E_0^{(2)} = - \sum_{n=1, \dots} \frac{|\langle 2n+1 | V | 1 \rangle|^2}{2n \bar{\omega}}, \quad \text{where } V = |x| - \omega^2 x^2$$

$$\text{and } \langle 3 | V | 1 \rangle = \frac{0.4607}{\sqrt{\omega}} - 1.2247 \omega,$$

$$\langle 5 | V | 1 \rangle = - \frac{0.103}{\sqrt{\omega}},$$

$$\langle 7 | V | 1 \rangle = \frac{0.0477}{\sqrt{\omega}}$$

$$\text{so } \Delta E_0^{(2)} = - \frac{0}{2\bar{\omega}} - \frac{(0.1427)^2}{4\bar{\omega}} - \frac{(0.061)^2}{6\bar{\omega}} - \dots = -0.011$$

We notice that the second order correction to the ground state from the first excited state vanishes at $\omega = \bar{\omega} = 0.521$. This is a general result for harmonic oscillator trial functions, for all Hamiltonians. The estimate $\Delta E_0^{(2)}$ is of the right order to account for the error in the ground state: The corrected value $(2.3448 - 0.011) \approx 2.3338$ is now too low. We repeat this estimate for the first excited state, (where now $\omega = \bar{\omega}_1 \approx 0.388$)

$$\Delta E_1^{(2)} = - \sum' \frac{|\langle 2n+1 | V | 3 \rangle|^2}{(2n-2)\bar{\omega}}$$

$$\langle 5|V|3 \rangle = \frac{0.6308}{\sqrt{\omega}} - 2.2361\omega$$

$$\langle 7|V|3 \rangle = \frac{0.1363}{\sqrt{\omega}}$$

And we have

$$\begin{aligned} \Delta E_1^{(2)} &= \frac{(0.2644)^2}{2\bar{\omega}} - \frac{(0.1451)^2}{2\bar{\omega}} - \frac{(0.2188)^2}{4\bar{\omega}} - \frac{(0.099)^2}{6\bar{\omega}} - \dots \\ &\simeq +0.09 - 0.0271 - 0.03088 - 0.0042 \simeq 0.0279 \end{aligned}$$

This again overestimates the error in the first excited state, but it is a good internal indicator of the error committed by the variational estimate. The smallness of the ratio $(\Delta E_1^{(2)}/E_1(\bar{\omega}))$ also confirms that the unperturbed Hamiltonian $H^{(0)}(\bar{\omega})$ is a stable starting point for perturbation theory and therefore the first order estimate is reliable.

IIIa. Discussion of the method and a three dimensional example

As noted in the Introduction, the Variational Principle underlies the procedure followed here. This principle states that the eigenvalues of the operator H are extrema of an integral, under suitable variations. But there are many questions beyond this general statement.

One question which may be asked immediately (here in its simplest form) is why the second function $\psi_2^{(0)}$ in an "arbitrary" orthonormal set estimates the second eigenvalue of the Hamiltonian H whose true eigenfunction is ψ_2 ? It is clear that if $\psi_2^{(0)}$ were orthogonal on the true ground state ψ_1 , then the variational

estimate $E_2(\bar{\omega})$ would be an upper bound to the true energy E_2 . But even though $\psi_2^{(0)}$ is only orthogonal to $\psi_1^{(0)}$ the mini-max theorem [6,3] asserts that the true energy E_2 is an upper bound over all the minima of the expectation values of H when we vary $\psi_1^{(0)}$ over all Hilbert space. The upper bound occurs when $\psi_1^{(0)} = \psi_1$. So even the expectation values $E_2^{(0)}(\bar{\omega})$ have extremal properties, and in general they should be below the actual eigenvalue E_2 . To satisfy the conditions of the mini-max theorem one should vary $\psi_2^{(0)}$ more widely than we do here (by just scaling), for example by taking arbitrary linear combinations with $\psi_3^{(0)}, \psi_4^{(0)}$, etc., but even the abbreviated procedure we follow can give reasonable estimates. Why the errors are so small is a matter of chance in having a good orthonormal set. In the example of the previous section it was shown that for low excitation (small n) the best variational wavefunctions have a small "contamination" from neighbouring states and moreover the contamination is symmetrical so that the expectation values do not deviate a lot from those of the right state. Our experience so far, is that the errors for excited states are quite similar with those for ground states. A special feature of all the problems we discuss is that the potentials are confining power law type, which have no intrinsic scale, just a characteristic exponent. In these cases the variational procedure for any level n tries to match the correct size of the wavefunction at the appropriate n , by choice of the scaling parameter. In the end there are no guarantees regarding a good choice of an orthonormal set, one has to try. As explained in the previous section, we can always get an internal

estimate of errors, by computing second order corrections, but even this step is more laborious. Another obvious point is that by virtue of a minimum at $\bar{\omega}$ the actual eigenvalues $E_i(\bar{\omega})$ are not sensitive to small changes in $\bar{\omega}$.

A separate issue concerns the quasi-classical calculation where we applied the variational method in the classical domain to obtain the asymptotic limit of the variational results. This suggests that there is a variational principle for $\langle H \rangle$ in Classical Mechanics. We find that this is the case, for variations of this expectation value for closed (periodic) motion which maintain a constant action. The period will vary, while the action is kept constant. The actual energy is an extremum of $\langle H \rangle$. This is the variational principle actually used by Schrödinger to derive his famous equation. This variational principle is a modification of Maupertuis' principle under the constraints of periodic motion with constant action. A derivation is given in the Appendix. A version of this classical principle was first stated by Van Vleck in 1923 [7].

We turn now to the literature, where our knowledge is undoubtedly incomplete. In a very vague way the procedure we use is related to the so called "Principle of Minimum Sensitivity" of Stevenson [8] which was enunciated in another context. The frequency ω is an artifact of computation not intrinsic to the Hamiltonian H . If we compute in perturbation theory to high order $E_n(\omega)$ the answer should not depend on the starting value of ω . So by choosing a value of $\bar{\omega}$ such that $E_n'(\bar{\omega}) = 0$, we guarantee that

already in first order we are insensitive to this value, and are making a good choice.

Closer to what we are doing, we are aware of the work of P.M. Mathews and collaborators [9] on quartic oscillators, where they use a harmonic oscillator basis and choose the frequency ω_n separately for each state n not variationally but requiring instead a cancellation of matrix elements connecting the state n to the nearest neighbours $n\pm 2$. This is of course more laborious (since it requires the computation of off-diagonal matrix elements) than our procedure, but the idea of using different ω_n for different states is the same.

We are ignorant of other relevant literature, and apologise for omissions.

It is also interesting to speculate when the variational method fails for excited states. We have encountered one simple case which provides an indication. We outline it here briefly: we use a three dimensional harmonic oscillator wavefunction to estimate the energy of the first excited 2S state of the Coulomb problem: we find variationally $E_2 \approx -0.126$ very near the correct value $E_2 = -1/8$, in atomic units. As we know the exact Coulomb wavefunctions we can compute the overlap integrals. We find that the trial state has a 1% probability in the correct 2S state, and therefore the good energy estimate arises by cancellation. Moreover, an estimate of second order corrections (as in the previous section) shows that the result E_2^v is unstable with respect to these corrections which are large. We suspect that the

variational method fails for potentials r^p with p negative because these potentials are more singular at the origin ($r=0$) than the trial Hamiltonian. As a result the spectrum of states, changes sharply at $p = 0$, from purely discrete, to a mixture of discrete spectrum and a continuum. Because of this we have not used the variational method for $p < 0$. However for $p > 0$, changing the number of particles, or the number of dimensions of space does not affect the discrete nature of the spectrum, and so we use the variational method for excited states, provided we take a harmonic oscillator with the right number of degrees of freedom, in the appropriate space, as a reference Hamiltonian. Finally, it is plausible to suppose that the method works best for $p = 2 - \epsilon$ if one uses harmonic oscillator trial functions.

IIIb. A three dimensional example

The eigenvalues for the one-dimensional motion with linear potential are also appropriate to the S-wave bound states of the system in three dimensions [4]. Therefore we discuss the three dimensional extension of the problem (II-1). The wave equation is:

$$[-\nabla^2 + r] \psi_{n\ell m}(\vec{r}) = E_{n\ell} \psi_{n\ell m}(\vec{r}) \quad (\text{III-1})$$

We shall estimate $E_{n\ell}$ by taking expectation values of the Hamiltonian $(-\nabla^2 + r)$ with eigenfunctions of the Harmonic Oscillator $(-\nabla^2 + \omega^2 r^2)$ in three dimensions. The energy of the oscillator state is $2(2n + \ell + 3/2)\omega$, and the expectation value of the kinetic

energy is $(2n + \ell + 3/2)\omega$, by the virial theorem. We have to compute the expectation value of r with the oscillator eigenfunctions $\psi_{n\ell}^{(0)}$. We don't have a general expression in a compact form, but for $n=0$ we obtain [4]:

$$E^V(0, \ell) = \frac{3}{2} (2\ell + 3)^{1/3} \left[\frac{\Gamma(2+\ell)}{\Gamma(\frac{3}{2}+\ell)} \right]^{2/3} \quad (\text{III-2})$$

for $n=1$ we find

$$E^V(1, \ell) = \frac{3}{2} (2\ell + 7)^{1/3} \left[\left(\ell + \frac{9}{4}\right) \frac{\Gamma(2+\ell)}{\Gamma(\frac{3}{2}+\ell)} \right]^{2/3} \quad (\text{III-3})$$

and for $n=2$ we find

$$E^V(2, \ell) = \frac{3}{2} (2\ell + 11)^{1/3} \left[\frac{\ell^2 + \frac{11}{2}\ell + \frac{225}{32}}{(\ell + \frac{3}{2})(\ell + \frac{5}{2})} \cdot \frac{\Gamma(2+\ell)}{\Gamma(\frac{3}{2}+\ell)} \right]^{2/3} \quad (\text{III-4})$$

These formulae are compared to the 'exact' eigenvalues in Table III-1. The 'exact' eigenvalues were obtained by a numerical integration of the radial equation. We see from Table III-1 that the variational estimates are good approximations of the exact values, and the approximation seems to improve at higher ℓ . We have also computed (numerically) the WKB estimates to these energies, and found their errors to be larger than the variational ones; for example at $n=2$, $\ell=3$ the WKB estimate we find is 7.4834, which has an error of 0.28% much larger than the variational estimate 7.5007 (which has an error of 0.05%). We have also compared, for the first two S-wave states the expectation values of

the wavefunction at the origin $|\psi(0)|^2$ (computed with the variational wavefunctions) to the true values, which can be computed exactly. With our Hamiltonian (III-1), the quantity $|\psi(0)|^2$ should be $(4\pi)^{-1} \approx 0.0795$ (for all S-states), whereas with the Harmonic oscillator wavefunctions we found for the ground state $|\psi_{1,0}(0)|^2 = \frac{2}{3\pi^2} \approx 0.0675$ and for the first excitation $|\psi_{2,0}(0)|^2 = \frac{9}{16\pi^2} \approx 0.0651$ an error of 15-18% quite consistent with the small error in the energy, which is quadratic relative to the error in the wavefunction $\psi(\vec{r})$. We have also obtained an asymptotic formula for the energy levels of the Hamiltonian (III-1), by carrying out the variational calculation in the semiclassical domain. We find in the limit of small ℓ , ($n \gg \ell$),

$$\langle r \rangle_{n\ell} = \frac{1}{\pi\sqrt{\omega}} \left[4\sqrt{n+\frac{3}{4}} + \frac{\ell}{\sqrt{n+\frac{3}{4}}} \right] \quad (\text{III-5})$$

and

$$E_{n\ell} = 6 \left(n + \frac{\ell}{2} + \frac{3}{4} \right)^{1/3} \left[\frac{1}{\pi} \sqrt{n+\frac{3}{4}} + \frac{\ell}{4\pi\sqrt{n+\frac{3}{4}}} \right]^{2/3} \quad (\text{III-6})$$

where the expectation value (III-5) is appropriate to the harmonic oscillator ($\frac{1}{2}p^2 + \frac{1}{2}\omega^2 r^2$) and (III-6) corresponds to the Hamiltonian (III-1). This estimate should be valid for $\ell \ll n$ when the centrifugal potential is a small perturbation, but the formula (III-6) works reasonably (error $\approx 1\%$) even at low n , as seen from Table III-1. For $\ell = 0$, equation (III-6) reduces to equation (II-12).

IV Three body estimates

We apply, in this section, the variational method to a few examples of three body systems with pairwise interactions. All particles have unit mass, and the Hamiltonian H has the form:

$$H = \frac{1}{2} \sum_i^3 \vec{p}_i^2 + \frac{1}{2} \sum_{i < j} r_{ij}^p \quad (\text{IV-1})$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$, and the power $p > 0$. We will compare with the numerical results presented by Richard [4] for $p=1$; for $p=0.1$ we use ref. [4,10]. Upon change to the familiar Jacobi coordinates $\vec{\rho}$ and $\vec{\lambda}$ and elimination of the centre-of-mass coordinate, the internal Hamiltonian depends only on $\vec{\rho}, \vec{\lambda}$ ($\sqrt{2}\vec{\rho} = \vec{r}_1 - \vec{r}_2$; $\sqrt{6}\vec{\lambda} = \vec{r}_1 + \vec{r}_2 - 2\vec{r}_3$). We shall use as reference zero order Hamiltonian $H^{(0)}(\omega)$ a 3 particle harmonic oscillator:

$$H^{(0)}(\omega) = \frac{1}{2} (\vec{p}_{\rho}^2 + \vec{p}_{\lambda}^2) + \frac{1}{2} \omega^2 (\vec{\rho}^2 + \vec{\lambda}^2) \quad (\text{IV-2})$$

which corresponds (for $\omega^2 = 3$) to the power $p=2$ in eqn. (IV-1). The energy eigenstates of $H^{(0)}(\omega)$ are $E_N = (N+3)\omega$, and except for the ground state they are degenerate. The states may be classified by the angular momentum eigenvalues L, L_z and permutational symmetry under exchange of 1,2,3, the 'original' particles. At low excitation these quantum numbers suffice to label the states, but at higher N there are more degeneracies [4,12]. We shall be interested in the ground state, the lowest P and D waves ($L = 1, 2$) and the first excited $L=0$ state which is totally symmetric under

permutations. Let us start to apply the variational principle for the ground state, where the procedure (and result) are well known.

For totally symmetric (and antisymmetric) states ψ^S we can rewrite the expectation value of the potential energy

$$\langle V \rangle_S = \langle \psi^S | \frac{1}{2} \sum_{ij} r_{ij}^p | \psi^S \rangle = \frac{3}{2} \langle \psi^S | r_{12}^p | \psi^S \rangle \quad (\text{IV-3})$$

in terms of the expectation value of a single pair, which in turn can be expressed in the Jacobi coordinate $\vec{\rho}$

$$\sqrt{2} \vec{\rho} = \vec{r}_1 - \vec{r}_2 \quad (\text{IV-4})$$

$$\langle V \rangle_S = \frac{3}{2} \cdot 2^{p/2} \langle \psi^S | \rho^p | \psi^S \rangle \quad (\text{IV-5})$$

where $\rho = |\vec{\rho}|$. Therefore the expectation value of the Hamiltonian H requires only the computation of the expectation value of one coordinate, raised to the p^* power (for Symmetric and Antisymmetric states); the kinetic energy is the same as for harmonic forces, half the total energy. Recalling that the ground state of $H^{(0)}$ is

$$\psi_{00}^{(0)} = \frac{1}{\pi^{3/2}} e^{-(\rho^2 + \lambda^2)/2} \quad (\text{IV-6})$$

(where we set $\omega = 1$; we can re-instate ω in the final answer using dimensional arguments), we have

$$\begin{aligned}
\langle \psi_{00} | V | \psi_{00} \rangle &= \frac{3}{2\pi^3} \cdot 2^{p/2} \int d^3\rho \, d^3\lambda \, e^{-(\rho^2 + \lambda^2)} \rho^p \\
&= \frac{3}{\sqrt{\pi}} \cdot 2^{p/2} \Gamma\left(\frac{3+p}{2}\right)
\end{aligned} \tag{IV-7}$$

and therefore the expectation value of H in the ground state of $H^{(0)}(\omega)$ is

$$E_{00}(\omega) = \frac{3}{2} \omega + \frac{3}{\sqrt{\pi}} \left(\frac{2}{\omega}\right)^{p/2} \Gamma\left(\frac{3+p}{2}\right) \tag{IV-8}$$

where we re-introduced the frequency ω . A variational calculation now gives the estimate

$$E_{00}^V = 3 \cdot \frac{2+p}{p} \cdot \left[\frac{p}{2\sqrt{\pi}} \Gamma\left(\frac{3+p}{2}\right) \right]^{\frac{2}{2+p}} \tag{IV-9}$$

It may be checked that for $p=2$ (harmonic oscillator) formula (IV-9) is exact (with $\omega = \sqrt{3}$), while for a linear potential, $p=1$, it gives $E_0 \approx 3.87114$ which is 0.2% above the true energy (see ref. [4] which contains both the variational estimate, and the exact value at $p=1$). For $p=0.1$ formula (IV-9) gives $E_0 = 1.88278$ which is 0.14% above the true ground state [4,10]. So we have evidence that $H^{(0)}$ is a good reference Hamiltonian.

Consider the first excited state of the Hamiltonian $H^{(0)}$ (of the same symmetry as the ground state, this is sometimes called $[56, 0^+]'$). The eigenfunction at $E = 5\omega$ is

$$\psi_{20}^{(0)} = \frac{1}{\sqrt{3\pi^3}} (\rho^2 + \lambda^2 - 3) e^{-(\rho^2 + \lambda^2)/2} \tag{IV-10}$$

and repeat the similar calculation to find:

$$E_{20}^V = 5 \cdot \frac{2+p}{p} \left[\frac{p(12+2p+p^2)}{40\sqrt{\pi}} \Gamma\left(\frac{3+p}{2}\right) \right]^{\frac{2}{2+p}} \quad (\text{IV-11})$$

Again this formula is exact for $p=2$ ($E_2 \approx 5\sqrt{3}$), while for $p=1$, the linear potential, formula (IV-11) gives $E_2^V \approx 5.32593$ which is about 0.1% too high compared to the (precise) value quoted in [4], $E_2 \approx 5.3207$. For $p \approx 0.1$ equation (IV-11) is 0.05% too low compared to the value quoted in the literature [1,10]. It is reasonable to suppose that the variational estimates (IV-11), (IV-9) have an error smaller than 0.5% for any p between 0 and 3 and so they should be useful for quark model estimates.

We can go to higher states in this fashion, though the labour involved increases for higher states. We have computed the next state $N=4$ which is totally symmetric and $L=0$ for the special case $p=1$, and the result is in Table (IV-1). We have used the wavefunction:

$$\psi_{40}^{(6)} = \frac{1}{\sqrt{24}\pi^3} \left[(\rho^2 + \lambda^2)^2 - 8(\rho^2 + \lambda^2) + 12 \right] e^{-(\rho^2 + \lambda^2)/2} \quad (\text{IV-12})$$

The error for the estimate of energy $E \approx 6.6108$ is about 0.2% compared to the value quoted in ref. [4].

We consider now the lowest $L=1$ states of mixed symmetry. Here the zero order wavefunctions have the form:

$$\psi_{11}^p = \frac{1}{\pi^{3/2}} (\rho_x + i\rho_y) e^{-(\rho^2 + \lambda^2)/2}$$

$$\psi_{11}^\lambda = \frac{1}{\pi^{3/2}} (\lambda_x + i\lambda_y) e^{-(\rho^2 + \lambda^2)/2} \quad (\text{IV-13})$$

and the expectation value of the potential energy has the form

$$\langle \psi_{11}^p | V | \psi_{11}^p \rangle = \frac{3 \cdot 2^{p/2}}{4} [\langle \psi_{11}^p | \rho^p | \psi_{11}^p \rangle + \langle \psi_{11}^\lambda | \rho^p | \psi_{11}^\lambda \rangle] \quad (\text{IV-14})$$

which leads to the variational estimate for the energy

$$E_{11}^V = 4 \cdot \frac{2+p}{p} \left[\frac{6 \cdot (6+p)}{16 \sqrt{\pi}} \Gamma\left(\frac{3+p}{2}\right) \right]^{\frac{2}{2+p}} \quad (\text{IV-15})$$

We have also looked at the lowest totally symmetric D-state ($n=0$, $L=2$). A harmonic oscillator state is

$$\psi_{22}^S = \frac{1}{2\pi^{3/2}} (\rho_+^2 + \lambda_+^2) e^{-(\rho^2 + \lambda^2)/2} \quad (\text{IV-16})$$

and the expectation value of ρ^p is

$$\langle \rho^p \rangle_{22} = \frac{1}{\sqrt{\pi}} \left[\frac{1}{15} (30 + 8p + p^2) \right] \Gamma\left(\frac{3+p}{2}\right) \quad (\text{IV-17})$$

which can be used to obtain a variational estimate for the symmetric D-state:

$$E_{22}^V = 5 \cdot \left(\frac{2+p}{p} \right) \left[\frac{p}{100 \sqrt{\pi}} (30 + 8p + p^2) \Gamma\left(\frac{3+p}{2}\right) \right]^{\frac{2}{2+p}} \quad (\text{IV-18})$$

We turn now to the asymptotic behaviour (at large N) of the symmetric S-wave mode, using the classical limit of the 3 particle harmonic oscillator. The corresponding classical motion is a "breathing" mode of the three particles in the plane, with the three particles forming equilateral triangles at all times. If we denote the position of the particles by $\vec{r}_i = \vec{A}_i \cos \omega t$, the three vectors $\vec{A}_i$ sum to zero and the interparticle distances are $|\vec{r}_{12}(t)| = \sqrt{3}A \cos \omega t$. Denoting the energy of the system by $E = (N+3)\omega$, we can compute A in terms of the energy E : $A = \sqrt{2(N+3)/3\omega}$, and therefore the expectation value:

$$\langle r_{12}^p \rangle = (\sqrt{3} A)^p \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p+2}{2})} \quad (\text{IV-19})$$

Therefore we can compute the expectation value of the Hamiltonian:

$$\langle H \rangle = \frac{1}{2} (N+3) \omega + \frac{3}{2} \langle r_{12}^p \rangle \quad (\text{IV-20})$$

and obtain the semi-classical asymptotic estimate:

$$E_{N0} = \frac{1}{2} \left(\frac{p+2}{p} \right) (N+3)^{\frac{2p}{2+p}} \left[\frac{3p}{\sqrt{\pi} 2^{\frac{2-p}{2}}} \frac{\Gamma(\frac{1+p}{2})}{\Gamma(\frac{2+p}{2})} \right]^{\frac{2}{2+p}} \quad (\text{IV-21})$$

where N is even. For $p=1$, this formula is much simpler

$$E_{N0}(p=1) = 3 \left[\frac{3}{2\pi} (N+3) \right]^{\frac{2}{3}} \simeq 1.83266 (N+3)^{\frac{2}{3}} \quad (\text{IV-22})$$

Formula (IV-21) is compared to more precise values at $N=0,2,4$

in Table IV-1, which contains also the previous variational estimates of this section. It can be noted that the asymptotic estimates (IV-21) have errors of the order of 2% or less. The formulae (IV-21) indicate that the large N behaviour of the 3-body symmetric states of $L=0$ is similar to that of the single particle in a linear potential. The behaviour $N^{2/3}$ follows from WKB like arguments in a hyperspherical formalism, but the next-to-leading constant (3 in $(N+3)$) is specific to this harmonic oscillator argument. In semiclassical context this constant is called the "Maslov" index. Therefore equation (IV-21) suggests that the correct Maslov index is +3 for this three particle problem in three dimensions. There is additional numerical evidence that the form of equation (IV-21) is correct. Reference [10] gives, for $p=0.1$, the energies of the first three states of angular momentum zero which are totally symmetric under permutations, for motion in 5-dimensions. Analogy with harmonic oscillators suggests that we replace in formula (IV-21) $(N+3)$ by $(N+5)$ to obtain the appropriate semiclassical estimates. This replacement does in fact give the energies of the first 3 states to within -1.9%, -1.3% and -0.85% respectively. The relevant formula in 5 dimensions (appropriate to the normalization of reference [10], which has Er_i^p rather than $2^{-1}Er_i^p$, as we have here, following reference [4]) is $E_N = 3.2394.(N+5)^{0.095238}$, where N is even: 0, 2, 4, ...

Application to Baryon Spectrum

It is interesting to apply these formulae to experimental data. One possibility, suggested by Høgaasen and Richard [11] in 1983 is

to form the ratio $(E_{1M} - E_{00})/(E_{20} - E_{00})$, which we denote here by $R_1(p)$, and determine empirically the exponent p from the baryon spectrum. These authors determined the function $R_1(p)$ numerically using the hyperspherical formalism (see e.g. ref. [4]). Using the variational estimates (IV-9), (V-10) and (IV-15) we can evaluate this ratio to find:

$$R_1(p) = \frac{E_{1M}^V - E_{00}^V}{E_{20}^V - E_{00}^V} = \frac{4 \cdot \left(\frac{6+p}{8}\right)^{\frac{2}{2+p}} - 3}{5 \left(\frac{12+12p+p^2}{20}\right)^{\frac{2}{2+p}} - 3} \quad (\text{IV-23})$$

which is valid for $p > 0$. The formula (IV-23) agrees with the graph of reference [11] at the level of 1%, even though they have been obtained by different methods. As noted in reference [11] the data on baryon resonances, appropriate to the ratio $R_1(p)$ does not fit confining potentials. If we switch-off hyperfine interactions, we would infer from the data that $E_{00} \approx 1135$ MeV, $E_{20} \approx 1600$ MeV and $E_{1M} \approx 1610$ MeV (for these numbers see ref. [13]). This would correspond to $R_1(p) \approx 1$, whereas in the range $0 \leq p \leq 2$ the ratio $R_1(p)$ has values between 0.435 and 0.5. So we have a problem. One possibility which suggests itself is that perhaps the different permutational symmetry of the negative parity state makes the formula (IV-23) inapplicable to real baryons, where the true potential is not a simple function of r_i , but it is a "string potential from a junction". To explore this possibility, we have computed the ratio $R_2(p)$ which involves only states which are totally symmetric under permutations:

$$R_2(p) = \frac{E_{22}^V - E_{20}^V}{E_{22}^V - E_{00}^V} \quad (\text{IV-24})$$

and can be evaluated variationally, using the formulae (IV-18), (IV-9) and (IV-10) to obtain

$$R_2(p) = \frac{5 \left[\frac{1}{50} (30 + 8p + p^2) \right]^{\frac{2}{2+p}} - 5 \left[\frac{1}{20} (12 + 2p + p^2) \right]^{\frac{2}{2+p}}}{5 \left[\frac{1}{50} (30 + 8p + p^2) \right]^{\frac{2}{2+p}} - 3} \quad (\text{IV-25})$$

The ratio $R_2(p)$ vanishes for $p=2$, (when $E_{22} = E_{20}$) and rises slowly when p decreases below 2, up to $R_2 \approx 0.19$ for $p \approx 0$. But resonance data on baryons [13] shows $E_{22} \approx 1850$ MeV so that the empirical value of $R_2 \approx 0.35$ which again is far too large to be accommodated by a confining potential with $0 \leq p \leq 2$. We have here the same problem for R_2 as that found by Hogaasen and Richard for the ratio R_1 . The permutational symmetry is not the reason for the discrepancy. We do not know the resolution of these two problems. We have tried to shift the position of the S wave excited state which is much broader than the other resonances, to find a common value of p for R_1 and R_2 , but failed to find a convincing solution. Perhaps the P_{11} resonance is not a 3 quark state, but as suggested by Zhenping Li [14], a q^3 -gluon configuration. But if that is the case where is the 2S state?

V Summary and Conclusions

In Summary, we have shown how to extend the variational method from the ground state to excited states of systems with confining

potentials. The error is sufficiently small to make this method practical at low excitation, especially in baryon spectroscopy. At high excitation the method gives the correct asymptotic form of the energy, which can be connected to a classical variational principle for the average Hamiltonian. Another advantage of the variational approximation is that the solutions satisfy the virial theorem. The main disadvantage of the method is that the wavefunctions of different states are not mutually orthogonal.

We have applied the variational method to an "inverse problem" in baryon spectroscopy, confirming and extending a negative result by Hogaasen and Richard.

It is clear that in principle the method is applicable to other types of problems, N-body problems in general, with confining potentials (and discrete spectra). It would also be of interest to study problems in quantum chaos.

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Appendix A: Variational Principles in Classical and Quantum Mechanics

We discuss the relation between variational principles in quantum and classical mechanics, and our approximate variational estimates for excited states.

For simplicity we consider a system with one degree of freedom and periodic motion. In the classical limit (ie for large quantum number n) the quantum averaging over a stationary state ψ_n corresponds to classical averaging over time:

$$\langle H \rangle \equiv \langle \psi_n | \hat{H} | \psi_n \rangle \underset{n \text{ large}}{\simeq} \langle H \rangle_T \equiv \frac{1}{T} \int_0^T dt H(q(t), p(t)) \quad (\text{A1})$$

where $q(t), p(t)$ are the solution of the classical equations of motion for the system, and T is the period for this solution. The classical analog of the system being in the state ψ_n during quantum averaging is the constraint that the solution $q(t), p(t)$ has a fixed value for the "abbreviated" action J :

$$J(T) = \int_0^T p(t) \dot{q}(t) dt = \oint p dq = \text{constant} \quad (\text{A2})$$

According to the quantum variational principle, the eigenstates of the operator $\hat{H}$ are the extrema of the functional $\langle H \rangle$ in (A1):

$$\delta \langle H \rangle \equiv \delta \langle \psi_n | \hat{H} | \psi_n \rangle = 0 \quad (\text{A3})$$

with constraint

$$\langle \psi_n | \psi_n \rangle = 1$$

where the subscript n denotes the N^{th} [or $(n+1)$ st] solution of (A3).

The classical analog of these equations is:

$$\delta \langle H \rangle_T = 0 \quad (\text{A4})$$

under the constraint

$$J = \oint p dq = \text{constant}$$

where the extremum of $\langle H \rangle_T$ is with respect to arbitrary trial periodic functions $q(t)$, $p(t)$, with possibly different period T , but the same value of the 'abbreviated' action J in (A4). Our approximate variational procedure chooses a specific class of trial functions, solutions of a harmonic oscillator of frequency ω . The classical variational principle (A4) was stated by Van Vleck [7] just before the invention of quantum mechanics. We derive (A4) classically from the variation of Hamilton's action:

$$S(T) = \int_0^T dt (p \dot{q} - H(q, p)) \quad (\text{A5})$$

constrained to periodic functions $q(t)$:

$$q(0) = q(T) \quad (\text{A6})$$

This constraint equation (A6) may be rewritten as a condition on the variation $\delta q(T)$:

$$\delta q(T + \delta T) = q(T) - q(T + \delta T)$$

or for small variations

$$\delta q(T) = - \dot{q}(T) \delta T \quad (A7)$$

We can now compute δS the variation of S_m , eqn. (A5):

$$\delta S(T) = -H(T) \delta T + \int_0^T dt \left[\delta p \left(\dot{q} - \frac{\partial H}{\partial p} \right) - \delta q \left(\dot{p} + \frac{\partial H}{\partial q} \right) \right] \quad (A8)$$

If we introduce the average action $\langle S \rangle$:

$$\langle S \rangle = \frac{1}{T} S(T) = \frac{1}{T} J - \langle H \rangle_T \quad (A9)$$

and consider only variations δp , δq , δT which maintain J constant, we may rewrite equation (A8) in terms of a variation of $\langle H \rangle_T$ the average energy:

$$\delta \langle H \rangle_T = \frac{\delta T}{T} [H(T) - \langle H \rangle_T] - \frac{1}{T} \int_0^T dt \left[\delta p \left(\dot{q} - \frac{\partial H}{\partial p} \right) - \delta q \left(\dot{p} + \frac{\partial H}{\partial q} \right) \right] \quad (A10)$$

It is seen from (A10) that if $p(t)$, $q(t)$ satisfy the classical equations of motion, then equation (A4) follows, since $H(T) = \langle H \rangle_T$.

Conversely, if the l.h.s of (A10) vanishes, for arbitrary δT , δq , δp , we have to satisfy Hamilton's equations, and conservation of energy at $t=T$ ($H(T) = \langle H \rangle_T$). In other words, the variational principle for the average energy is equivalent to Hamilton's principle on the class of trial functions with fixed J . From a classical point of view, the least action principle is more natural since the energy is conserved, and can be eliminated. But from a quantum point of view, it is the value of the action which we know, and it is natural to have a variational principle at fixed action, for the average energy.

Table II-1
Linear Potential in One Dimension

n	2n + 1	$\bar{\omega}_{2n+1}$	$E_n^v(\bar{\omega})$	E_n^{exact}	ΔE_n	$\delta E_n\%$	$\Delta E_n / (E_{n+1} - E_n) \%$
0	1	0.52106	2.3448	2.3381	0.0067	0.29%	+0.39%
1	3	0.3881	4.0752	4.0879	-0.0127	-0.31%	-0.89%
2	5	0.3332	5.4978	5.5206	-0.0228	-0.41%	-1.81%
3	7	0.3003	6.7565	6.7867	-0.0302	-0.44%	-2.62%
4	9	0.2775	7.9076	7.9441	-0.0365	-0.46%	-3.40%
5	11	0.2603	8.9804	9.0226	-0.0462	-0.47%	-4.54%

Table II-2
Variational expectation values of operators

n	$\langle H^2 \rangle$	$E_n^2(\bar{\omega})$	$\langle H^2 \rangle / E_n^2(\bar{\omega})$	$\langle x^2 \rangle$	$\langle x^2 \rangle_{\text{exact}}$	$\delta \langle x^2 \rangle$
0	5.5259	5.4981	1.0050	2.8787	2.9156	-1.27%
1	16.7630	16.6073	1.0094	9.0183	8.9125	+1.19%
2	30.5419	30.2258	1.0104	16.5066	16.2544	+1.55%
3	46.1442	46.0593	1.0108	24.9750	24.5649	+1.67%
4	63.2207	62.5301	1.0110	34.239	33.658	+1.73%
5	81.5416	80.6476	1.0111	44.179	43.417	+1.75%

Table II-3
Probabilities of overlap with true eigenstates (in %)

n	P_0	P_1	P_2	P_3	P_4	P_5	ΣP_n
0	99.82	0.017	0.101	0.041	0.013	0.004	99.996
1	2.52	96.00	0.811	0.556	0.092	0.016	99.995
2	0.74	5.13	89.50	3.43	1.09	0.101	99.39
3	0.235	1.79	7.82	80.75	7.78	1.65	100.03
4	0.09	0.62	3.15	10.22	70.35	13.69	98.12
5	0.04	0.25	1.20	4.72	12.06	58.95	77.22

Table III-1
Variational energies for linear potential in three dimensions

$n\ell$	$E_{n\ell}^v$	$E_{n\ell}^{\text{exact}}$	$\Delta E^v/E\%$	$E_{n\ell}$ (III-6)
00	2.3448	2.3381	0.29%	2.3090
01	3.3678	3.3612	0.20%	3.3164
02	4.2544	4.2482	0.15%	4.3051
03	5.0569	5.0509	0.12%	5.2863
10	4.0752	4.0879	-0.31%	4.0620
11	4.8797	4.8844	-0.10%	4.8282
12	5.6297	5.6297	-0.00%	5.5839
13	6.3350	6.3321	+0.05%	6.3332
20	5.4978	5.5206	-0.41%	5.4904
21	6.1928	6.2076	-0.24%	6.1515
22	6.8603	6.8689	-0.12%	6.8057
23	7.5007	7.5046	-0.05%	7.4550

Table IV-1
Energy Levels E_{NL} of Three-Body Systems

L = 0									
	E_{00}^V (IV-9)	E_{00} exact	E_{00}^{asy} (IV-21)	E_{20}^V (IV-11)	E_{20} exact	E_{20}^{asy} (IV-21)	E_{40}^V	E_{40} exact	E_{40}^{asy} (IV-21)
p=2	5.1961	5.1961	5.1961	8.6603	8.6603	8.6603	-	12.1244	12.1244
p=1	3.8711	3.8631	3.8121	5.3259	5.3207	5.3587	6.6108	6.5953	6.7063
p=0.1	1.8828	1.8802	1.8587	1.9613	1.9589	1.9513	-	2.0143	2.0149

$L \neq 0$								
	E_{01}^V (IV-15)	E_{01} exact	E_{02}^V (IV-18)	E_{02} exact				
p=2	6.9282	6.9282	8.6603	8.6603				
p=1	4.7219	-	5.4670	-				
p=0.1	1.9390	1.9362	1.9787	-				

*the exact values are taken from reference [4]. A misprint is corrected for E_{20} at $p = 0.1$.