BOUND STATES OF THE HYDROGEN ATOM
IN THE PRESENCE OF A MAGNETIC
MONOPOLE FIELD AND AN
AHARONOV-BOHM POTENTIAL

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Abstract

In the present article we analyze the bound states of an electron in a Coulomb field when an Aharonov-Bohm field as well as a magnetic Dirac monopole are present. We solve, via separation of variables, the Schrödinger equation in spherical coordinates and we show how the Hydrogen energy spectrum depends on the Aharonov-Bohm and the magnetic monopole strengths. In passing, the Klein-Gordon equation is solved.

Key-Words: 3.65, 11.10, 4.90

August 1994
CPT-94/P.3066

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After the appearance of the pioneering work of Dirac [1], where the notion of magnetic monopole was introduced, a large body of papers [2, 3, 4, 5] have been published trying to elucidate the theoretical consequences of the existence of this hypothetical particle as well as the experimental verification of its existence. A good theoretical scenario for studying Dirac monopoles is to analyze the motion of electrons in the field generated by magnetic charges. A different approach is to consider the interaction of a monopole with a well studied quantum mechanical system like the Coulomb field, and by analyzing the scattering of electrons or looking at the bound states, to be able to evaluate the possible contribution of magnetic monopole. In this direction, the problem of a relativistic Dirac electron in the presence of a Coulomb field, plus a magnetic monopole and the Aharonov-Bohm potentials have been discussed by Hoang et al. [6] They compute the relativistic Dirac wave function and also obtain the bound states of the problem. Among the particularities of the energy spectrum, Hoang et al find that there are quantum states for which the Aharonov-Bohm contribution is absent. They solve the problem with the help of complex coordinates, representing the group of transformations in two-dimensional complex space, and introducing a vector-parameter notation of the SU(2) group.

Looking at the result reported in Ref [6] a natural question arises: is the strange behavior of the energy spectrum particular to relativistic spin $1/2$ particles, described by the Dirac equation, or is it also observed when we deal with non relativistic spinless Schrödinger particles, or spin zero relativistic particles.

It is the purpose of the present letter to show that there are states of the spectrum of the non relativistic (Schrödinger) hydrogen atom in the presence of a Dirac monopole and a AB potential where the Aharonov Bohm contribution is absent. An analogous result is obtained when we work with the Klein Gordon equation. Consequently, it is not the spin that is responsible for the anomalous behavior on the energy spectrum. Throughout the article we equate the speed of the light $c$ and the Planck constant $\hbar$ to unity.

The Hamiltonian of a (non relativistic) point charge $e$ of reduced mass $m$ in the potentials $V(r)$ and $A$ can be described by means of the Schrödinger equation

$$\left[\frac{1}{2m}(-i\nabla - eA)^2 + V(r)\right] \Psi = H \Psi = i\partial_t \Psi \quad (1)$$
where, in the present problem, \( V(r) \) is the Coulomb scalar potential
\[
V(r) = -\frac{e^2 Z}{r} \tag{2}
\]
\( \vec{A} \) is the sum of the vector potential \( \vec{A}_g \) associated with the Dirac magnetic monopole [1]
\[
\vec{A}_g = g \frac{(1 - \cos \vartheta)}{r \sin \vartheta} \hat{e}_\varphi \tag{3}
\]
where the charge \( g \) of the monopole satisfies the quantization condition
\[
eg g = \frac{n}{2} \tag{4}
\]
and the Aharonov-Bohm potential [7] \( \vec{A}_{AB} \)
\[
\vec{A}_{AB} = \frac{F}{2\pi r \sin \vartheta} \hat{e}_\varphi \tag{5}
\]
Since the potentials (2), (3) and (5) do not depend on the azimuthal angle \( \varphi \) nor on the time, we have that the wave function \( \Psi(\vec{r}, t) \), expressed in spherical coordinates, can be written as follows
\[
\Psi = \Phi(r, \vartheta)e^{i(k_c \varphi - E t)} \tag{6}
\]
where \( \Phi(r, \vartheta) \) satisfies the partial differential equation
\[
\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \vartheta^2} + \frac{\cot \vartheta}{r^2} \frac{\partial \Phi}{\partial \vartheta} - \frac{(k_c - [F/2\pi + g(1 - \cos \vartheta)])^2 \Phi}{r^2 \sin^2 \vartheta} + 2m(E + \frac{e^2 Z}{r})\Phi = 0 \tag{7}
\]
eq (7) can be reduced to two ordinary differential equations with the help of
\[
\Phi = R(r)\Theta(\vartheta) \tag{8}
\]
then, substituting (8) into (7) we arrive at
\[
\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + 2m(E + \frac{e^2 Z}{r})R - \frac{\lambda^2}{r^2}R = 0 \tag{9}
\]
\[
\frac{d^2 \Theta}{d\vartheta^2} + \cot \vartheta \frac{d\Theta}{d\vartheta} - \frac{(k_c - [F/2\pi + g(1 - \cos \vartheta)])^2 \Theta}{\sin^2 \vartheta} + \lambda^2 \Theta = 0 \tag{10}
\]
where $\lambda^2$ is a constant of separation. It is not difficult to see that eq. (9) is the same one we obtain in solving the problem of an electron in a Coulomb field [8]. This is a consequence of the form of the vector potentials $\vec{A}_z$ and $\vec{A}_{AB}$, whose contribution appears in the equation (10), governing the dependence of the wave function on the angle $\vartheta$. After introducing the variable $x$,

$$x = \cos \vartheta$$

we have that eq. (10) takes the form

$$(1 - x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d \Theta}{dx} - \left[ \frac{(m + qx)^2}{1 - x^2} + \lambda^2 \right] \Theta = 0$$

where we have introduced the parameters $q$ and $m$,

$$q = -ge, \quad m = \frac{Fe}{2\pi} + g e - k \varphi$$

In order to solve eq. (12) we make the ansatz

$$\Theta(x) = (1 - x)^{(m + q)/2}(1 + x)^{(q-m)/2}W(x)$$

then, substituting (14) into (12) we arrive at,

$$(1 - y)y \frac{d^2 W}{dy^2} + [m + q + 1 - 2y(1 + q)] \frac{dW}{dy} + (\lambda^2 - q)W = 0$$

where we have introduced the new variable $y$, related to $x$ as follows,

$$(1 - x)/2 = y$$

the solution of eq. (15) can be written with the help of the Gauss hypergeometric function [9]

$$W(y) = \, _2F_1(a, b, c, y)$$

where, the parameters $a$, $b$ and $c$ take the form

$$a = q + \frac{1}{2} - \sqrt{q^2 + \frac{1}{4} + \lambda^2}, \quad b = q + \frac{1}{2} + \sqrt{q^2 + \frac{1}{4} + \lambda^2}, \quad c = m + q + 1$$

Since the wave function $\Psi$, solution of the Schrödinger equation (1), should be normalizable, we have that the function $\Theta(\vartheta)$ satisfies

$$\int_0^\pi \Theta(\vartheta)\Theta(\vartheta) \sin \vartheta d\vartheta = \int_{-1}^1 \Theta(x)\Theta(x) dx = 1$$  \hspace{1cm} (19)$$

The condition (19) imposes some restrictions on the values of the parameters given in (18). In fact, we have that the Gauss hypergeometric functions $\, _2F_1(a, b, c, y)$ reduce to polynomials when $a$ or $b$ is a negative integer. In effect, we have that the Jacobi polynomials are related to the Gauss $\, _2F_1(a, b, c, y)$ functions as follows [9]

$$P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \, _2F_1(-n, n + \alpha + \beta + 1, \alpha + 1, 1 - x)$$  \hspace{1cm} (20)$$

Since the normalization condition for the Jacobi Polynomials [10]

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) dx = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{2n+\alpha+\beta+1 \Gamma(n+1) \Gamma(n+\alpha+\beta+1)} \delta_{nm}$$  \hspace{1cm} (21)$$
as well the regularity\footnote{This condition is related to the Hermitean character of the operator \(10\) which permits us to consider the constant of separation $\lambda^2$ as a real quantity. The normalizability of $\Theta(x)$ could be guaranteed by imposing the weaker conditions $\alpha > -1$, and $\beta > -1$} of the function $\Theta(x)$ in $x = \pm 1$ require that $\alpha > 0$ and $\beta > 0$, and we are interested in obtaining the most general solution of $\Theta(\vartheta)$, we are going to consider another solutions of eq. (15) which are also expressed in terms of Gauss functions

$$W_2 = (1-x)^{-\alpha-\beta-1} \, _2F_1(c-a, c-b, c, x), \quad W_3 = x^1 \, _2F_1(a-c+1, b-c+1, 2-c, x)$$

$$W_4 = x^{1-\gamma}(1-x)^{-\alpha-\beta-1} \, _2F_1(1-a, 1-b, 2-c, x)$$  \hspace{1cm} (22)$$

which after imposing the condition of normalizability, reduce to Jacobi polynomials. Therefore, the solution of eq. (12) reads

$$c_0(1-x)^{m+1/2}(1+x)^{b-m+1/2} P_n^{(a+b-m)}(x)$$  \hspace{1cm} (23)$$
where \( c_0 \) is a constant of normalization which can be obtained in a straightforward way from (21), and the integer \( n \) reads

\[
   n = \sqrt{q^2 + \frac{1}{4} + \lambda^2 - \frac{1}{2} - \frac{1}{2}(|m + q| + |q - m|)} \quad (24)
\]

From (24) we have that when the inequality

\[
   |m| < |q| \quad (25)
\]

is satisfied, \( n \) takes the form

\[
   n = \sqrt{q^2 + \frac{1}{4} + \lambda^2 - |q| - \frac{1}{2}} \quad (26)
\]

Analogously, we have that for

\[
   |m| > |q| \quad (27)
\]

we obtain

\[
   n = \sqrt{q^2 + \frac{1}{4} + \lambda^2 - |m| - \frac{1}{2}} \quad (28)
\]

The energy spectrum of the hydrogen atom can be obtained from the expression [8]

\[
   E = -\frac{mZ^2e^4}{2(N + l + 1)^2} \quad (29)
\]

where \( N \) takes integer values

\[
   N = 0, 1, 2, ... \quad (30)
\]

and the parameter \( l \) is related to \( \lambda \) as follows

\[
   l = \sqrt{\frac{1}{4} + \lambda^2 - \frac{1}{2}} \quad (31)
\]

Then, we have that when condition (25) fulfills the energy spectrum reduces to

\[
   E = -\frac{mZ^2e^4}{2 \left( N + \sqrt{(n + 1/2 + |q|)^2 - q^2 + 1/2} \right)^2} \quad (32)
\]
Analogously, we have that when $|m| > |q|$, the energy takes the form

$$E = -\frac{mZ^2e^4}{2\left(N + \sqrt{(n + 1/2 + |m|)^2 - q^2 + 1/2}\right)^2}$$  \hspace{1cm} (33)

Here some comments are in order. The presence of the Dirac monopole introduces substantial modifications in the energy spectrum relative to the pure Aharonov-Bohm + Coulomb case [11, 12]. In fact when the monopole is absent, the expression (25) is never valid and eq. (33) reduces to

$$E = -\frac{mZ^2e^4}{2(N + n + 1/2 + k_\varphi + 1)^2}$$  \hspace{1cm} (34)

which is the energy spectrum of a particle in a Coulomb field plus Aharonov-Bohm potential [11], valid for any value of the parameter $m$. A completely unexpected situation arises when the inequality $|m| < |q|$ takes place. In this case we have

$$\left|\frac{Fe}{2\pi} + ge - k_\varphi\right| < |ge|$$  \hspace{1cm} (35)

and the energy spectrum is given by eq. (32) that does not depend on the Aharonov-Bohm potential. This remarkable property was already observed by Hoang et al [6] when the problem is tackled in the framework of a relativistic spin $1/2$ particle.

It is straightforward to extend the results obtained for the Schrödinger case to a spin-zero relativistic particle associated with the Klein-Gordon equation

$$\left[(-i\nabla - e\vec{A})^2 + m^2\right]\Psi = \hbar^2\Psi = \left[i\partial_t - eV(r)\right]^2\Psi$$  \hspace{1cm} (36)

which can be solved in spherical coordinates in a similar way to the Schrödinger equation. Writing the wave function $\Psi$ in the form

$$\Psi(\vec{r}, t) = \frac{1}{r}R(r)\Theta(\vartheta)e^{i(k_\varphi\varphi - Et)}$$  \hspace{1cm} (37)

and substituting (37) into (36) we arrive at

$$\left(\frac{d^2}{dr^2} - \frac{\lambda^2 - Z^2e^4}{r^2} + \frac{2Ze^2E}{r} - m^2 + E^2\right)R(r) = 0$$  \hspace{1cm} (38)
and the differential equation for $\Theta(\vartheta)$ is just the same one obtained for the Schrödinger case (10). The energy spectrum can be obtained after solving eq. (38) and requiring the normalizability of the wave function. The values of $E$ are

$$E = \frac{m}{\sqrt{1 + \frac{Z^2 e^4}{E^2}}}$$

(39)

where

$$L = N + \frac{1}{2} + \sqrt{\lambda^2 + \frac{1}{4} - Z^2 e^4}, \quad N = 0, 1, 2,...$$

(40)

The above expression (40) for $| m | < | q |$ reduces to

$$L = N + \frac{1}{2} + \sqrt{(n+|q| + \frac{1}{2})^2 - q^2 - Z^2 e^4}$$

(41)

Substituting (41) into (39), and expanding in $Ze^2$ we find

$$E' = E - m = -\frac{mZ^2 e^4}{2(N + \lambda + \frac{1}{2} + \sqrt{(n+|q| + \frac{1}{2})^2 - q^2 - Z^2 e^4})^2}$$

(42)

and therefore the energy spectrum (39) does not depend on the Aharonov-Bohm field when $| m | < | q |$. The energy spectrum for $| m | > | q |$ can be obtained after replacing $| q |$ with $| m |$ in eq. (41) and (42). It is not difficult to see that eqs. (42) and (32) are identical. Analogous result we have when $| m | > | q |$. This shows that the result obtained for a non-relativistic charged particle is also valid when we consider a relativistic spinless Klein Gordon particle.

Regarding the energy spectra associated with the Schrödinger and Klein-Gordon equations, it would be interesting to compare them with the one obtained when we deal with a Dirac particle. In this case we have that the energy spectrum reads [6]

$$E = m \left(1 + \frac{Z^2 e^4}{[N + (\chi^2 - Z^2 e^4)^{1/2}]^2}\right)^{-1/2}$$

(43)

where

$$\chi^2 = (J + \frac{1}{2})^2 - q^2,$$

(44)
Then, in the non relativistic limit, after making the expansion of the energy spectrum \( E \) (43) in powers of \( Z^2 \epsilon \) we obtain

\[
E' = E - m = -\frac{m Z^2 \epsilon^4}{2(N + \chi)^2} \tag{45}
\]

a result which slightly differs from (32) and (33) when the magnetic monopole contribution is present. In fact, for \( J + 1/2 = n+ | m | +1 \), we have that \( E' \) (45) reads

\[
E' = -\frac{m Z^2 \epsilon^4}{2(N + \sqrt{(n+ | m | +1)^2 - q^2})^2} \tag{46}
\]

An analogous result can be obtained for \( J + 1/2 = n+ | q | +1 \), after making the substitution \( q \) for \( m \) in (46). It is noticeable that such a discrepancy disappears when \( q = 0 \). In fact, in this case the energy spectrum (46) reduces to the expression (34). Also we have that for \( (J + \frac{1}{2})^2 \gg q^2 \), the relativistic Dirac spectrum \( E' \) and the Schrödinger one take the same form. Then we have that the presence of the Dirac monopole makes a difference. Since the Klein-Gordon and the Schrödinger equations are associated with spin zero particles, we can affirm that the coupling between the spin and the Dirac monopole strength is responsible for the difference between the energy spectra (46) and (33).

Acknowledgments

The author wishes to express his indebtedness to the Centre de Physique Théorique for the suitable conditions of work. Also the author wishes to acknowledge the CONICIT of Venezuela and the Vollmer Foundation for financial support.

References


